On the Riemann Hypothesis

The conjecture "The non-trivial zeros of Riemann's zeta have all multiplicity 1" is true! Further mathematical connections with some sectors of string theory.

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Abstract

In this work the authors reproduce and deepen the themes of RH already presented in [25] [26], explaining formulas and showing different "special features" that are usually introduced with the theorem of prime numbers and useful to investigate further ways. One of the major results of this paper, through all the steps outlined, is that the conjecture on zeros of the Riemann's zeta is true and demonstrable with some analytical steps and a theoretical remark (see. [30]).
In the **Chapter 1** (Remark A) and in the conclusion of **Chapter 3** (Remark B), we have described the mathematical aspects concerning the proof of the conjecture "The nontrivial zeros of Riemann's zeta have all multiplicity 1". In the **Chapter 2,** we have described why $\psi(x)$ is an equivalent RH. In the **Chapter 3**, we have described the mathematical aspects concerning the "Theorem free Region from nontrivial zeros". In the **Chapter 4**, we have described also some mathematical arguments concerning the zeta strings and the p-adic and adelic strings. In conclusion, in the **Chapter 5**, we have showed the possible mathematical connections between some equations regarding the **Chapter 4** and some equations of the Riemann Hypothesis here presented.

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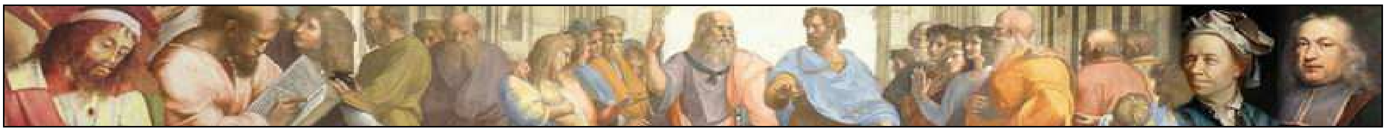

Introduction

Riemann is today linked at two unresolved conjectures. The conjecture “The non-trivial zeros of Riemann’s zeta have all multiplicity 1” is one of these.

The authors show that this conjecture is true. The proof is based on two theoretical remarks.

1. Remark A

We saw that Riemann [25] defined ζ(s) as a function of complex variable s. The first step of Riemann was to extend (or to *analytically continue*) ζ(s) to all of C \ {1} This can be accomplished by noticing that s=σ+it and $n^{-s}=s\int_n^\infty x^{-s-1}dx$ then:

$$\begin{aligned}\varsigma(s)&=\sum_{n=1}^{\infty}\frac{1}{n^s}=\sum_{n=1}^{\infty}\left(s\int_n^\infty\frac{dx}{x^{s+1}}\right)=s\sum_{n=1}^{\infty}\int_n^\infty\frac{dx}{x^{s+1}}\\ &=s\int_1^\infty\left(\sum_{n\le x}1\right)\frac{dx}{x^{s+1}}=s\int_1^\infty\frac{[x]}{x^{s+1}}dx=s\int_1^\infty\frac{x-\{x\}}{x^{s+1}}dx \qquad (1) \qquad (^1)\\ &=\frac{s}{s-1}-s\int_1^\infty\frac{\{x\}}{x^{s+1}}dx,\ \sigma>1\end{aligned}$$

Since {x}∈[0,1), it follows that the last integral converges for σ>0 and defines a continuation of ζ(s) to the half-plane σ=Re(s)>0. We can extend ζ(s) to a holomorphic function on all C \ {1}, in fact from the last integral s=1 is a simple pole with residue 1. We note that for s real and s>0 the integral in (1) is always positive real. Then from (1) ζ(s)<0, s∈(0.1) and ζ(s)>0, s∈(1.∞).

A popular expression of Euler is:

$$\varsigma(s)=\prod_{p=prime}(1-p^{-s})^{-1}\ .$$

From this expression we can obtain

$$\ln\varsigma(s)=-\sum_{p=prime}\ln(1-p^{-s})=\sum_{p=primes}\sum_{k=1}^{\infty}\frac{p^{-ks}}{k} \qquad (2)$$

In (2) we have applied the integration of Newton linked to expression:

$$\frac{1}{1-x}=1+x+x^2+x^3+...$$

If the previous expression is integrated for x and you change the sign to bring the term 1-x to numerator, then we obtain:

[1] [x] is the greatest integer ≤ x or floor of x; {x}=x-[x] is the fractional part of x.

$$-\log(1-x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + ...$$

Now, we introduce the ***von Mangoldt's function*** *(also called lambda function)*:

$$\Lambda(n) = \begin{cases} \log p, & if\ n = p^k,\ \ p\ \ prime,\ k \geq 1 \\ 0, & \text{otherwise} \end{cases} \qquad (3)$$

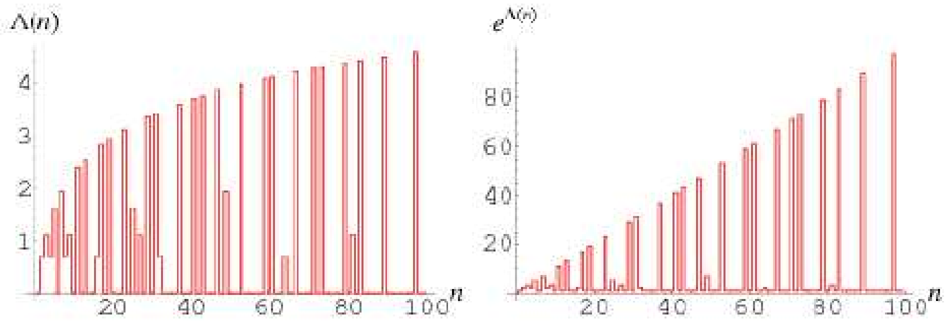

Figure 1 – von Mangoldt's function

From (2) we have:

$$p^{-ks} = \begin{cases} n^{-s}, & if\ n = p^k \\ 0, & \text{otherwise} \end{cases}$$

and if we use the rules of logarithm: n = p^k , k = $\log_p$ n = log n/log p then:

$$\frac{1}{k} = \begin{cases} \frac{\text{logp}}{\text{logn}} = \frac{\Lambda(n)}{\log n}, & when\ \ n = p^k \\ 0, & \text{otherwise} \end{cases}$$

Further the (2) becomes:

$$\ln \varsigma(s) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n\ \ n^s} = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n} n^{-s} \qquad (4)$$

The (4) is very interesting because consents to pass from "a multiplicative problem" to "an additive problem", even if we are started from the Euler's product.

Consequently if we do the derivative of (4) then we obtain:

$$\frac{\varsigma'(s)}{\varsigma(s)} = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} \qquad (5)$$

We note that for the von Mongoldt's function we have also:

$\log n = \sum_{d|n} \Lambda(d)$ where $d|n$ are divisor of n

<u>Example</u>

n=12

We remember that $12=2^2*3$ and that the divisors of 12 are: 1, 2, 3, 4, 6, 12, then:
$\log 12=\Lambda(1)+\Lambda(2)+\Lambda(3)+\Lambda(2^2)+\Lambda(2*3)+\Lambda(2^2*3)$

From (3) is:
log 12= 0 + log 2 + log 3 + log 2 + 0 + 0= log(2*3*2) = log 12

Pafnuty Lvovich Chebyshev introduced two functions:

$$\theta(x)=\sum_{p\le x}\log p \qquad \text{1st Chebyshev's function}$$

$$\psi(x)=\sum_{n\le x}\Lambda(n) \qquad \text{2nd Chebyshev's function}$$

This functions are very important in the proofs linked to prime numbers, because they are simple to use.
Another equivalent formulas for θ(x) is:

$$\theta(x)=\sum_{p\le x}\log p=\sum_{k=1}^{\pi(x)}\log p_k = \ln\left|\prod_{k=1}^{\pi(x)} p_k\right|$$

Hardy and Wright (1979) showed that:

$$\lim_{x\to\infty}\frac{x}{\theta(x)}=1$$

or:

$$\theta(x)\approx x$$

From here we have the Figure 2.

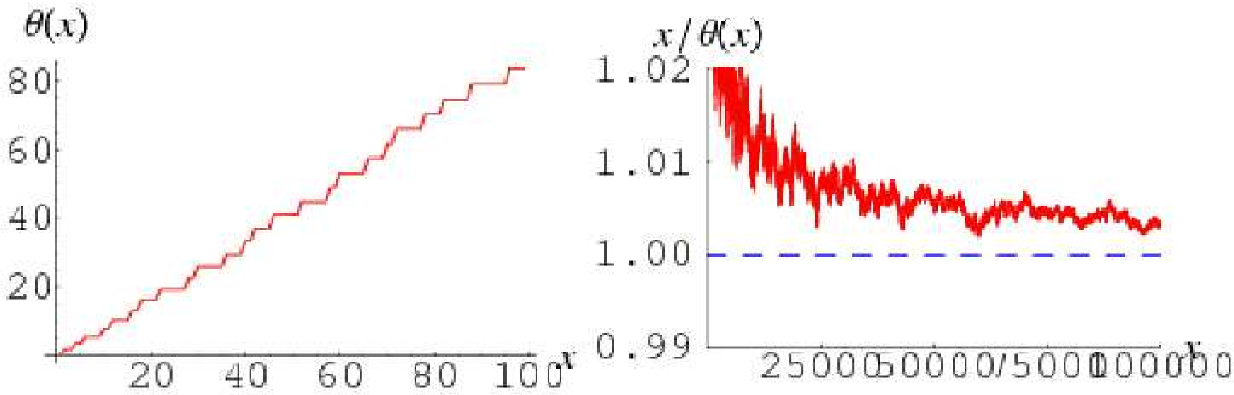

Figure 2 - θ(x) Chebyshev's function

We also can write ψ(x):

$$\psi(x)=\sum_{\substack{p\le x\\ p^k\le x}}\log p=\sum_{k=1}^{\pi(x)}\Lambda(k) \qquad (6)$$

In the previous formula the sums runs over all prime numbers p and positive integers k such that $p^k \leq x$ and therefore, potentially, includes some primes multiple times.

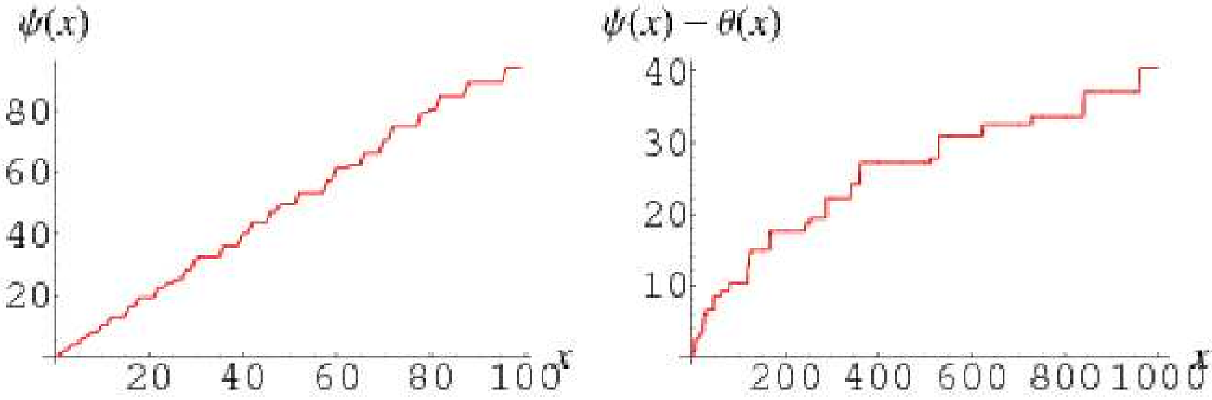

Figure 3 - ψ(x) Chebyshev's function

A simple and nice formula for ψ(x) is:

$$\psi(x) = \ln|lcm(1,2,3,4,...x)| \quad ([2])$$

or

$$lcm(1,2,3,4,...x) = e^{\psi(x)}$$

<u>Example</u>
x=10
$lcm(1,2,3,4,5,6,7,8,9,10)=5*7*2^3*3^2=2520$

$\psi(10)=\ln 2520 = \ln 5*7*2^3*3^2 = \ln 5 + \ln 7 + 3\ln 2 + 2\ln 3$

Now an *equivalent PNT or an equivalent RH* is:

$$\psi(x) \approx x \qquad (7)$$

Finally ψ(x) and θ(x) are linked as follow:

$$\psi(x) = \sum_{k=1}^{\infty} \theta(x^{\frac{1}{k}})$$

The previous formula has got a finite number of terms, because $\theta(x^{1/2})=0$ for $n>\log_2 x$.

The functions ψ(x) and θ(x) are in some ways more natural then prime counting function π(x), because they deal with multiplication of primes. In a multiplicative problem they are better.

It can be obtained a link from ζ(s) and ψ(x) inverting (5); in fact, starting from (5), *the Fourier inversion formula* implies for each a>1:

$$\psi(x) = \lim_{T\to\infty} \frac{1}{2\pi i} \int_{a-iT}^{a+iT} \left(-\frac{\zeta'(s)}{\zeta(s)}\right) x^s \frac{ds}{s}, \; x > 0 \qquad (8)$$

[2] For italian readers the term *lcm* (least common multiple) is equivalent to the term *mcm*

A link between ψ(x) and the nontrivial zeros (with multiplicities) of Riemann zeta function is the so-called *explicit formula (Riemann-von Mangoldt)*:

$$\psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - (\ln 2\pi) - \frac{1}{2}\ln(1 - x^{-2}) \quad (9)$$

For x>1 and x not prime number or prime power and ρ a nontrivial zero.

The (9) gives a very precise description of the error in the approssimations (7), and, more important, it relations the estimation of this error to the location of the nontrivial zeros.

We note that de la Vallèe-Poussin showed that the term-by-term integration of both sides of (9) is a valid operation for x>1:

$$\psi_1(x) = \int_0^x \psi(t)dt = \frac{x^2}{2} - \sum_{\rho} \frac{x^{\rho+1}}{\rho(\rho+1)} - \sum_{n} \frac{x^{-2n+1}}{2n(2n-1)} - x\log(2\pi) + const \quad (10)$$

It is clear that, as $x \to \infty$, the last three terms on the right hand side of the (10) are all $o(x^2)$.

2. Why ψ(x) is an equivalent RH

Now we can show that *ζ(1+it)≠0* or that *there aren't nontrivial zeros on the line σ=1.*
If we remember that s=σ+it, taking the real part, from (4) is:

$$\mathrm{Re}(\ln \zeta(s)) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n \ n^{\sigma}} \cdot \cos(t \log n)$$

By trigonometric identity $3 + 4\cos t + \cos 2t = 2(1 + \cos t)^2 \geq 0$ then we obtain:

$$3\,\mathrm{Re}(\ln \zeta(s)) + 4\,\mathrm{Re}(\ln \zeta(\sigma + it)) + \mathrm{Re}(\ln \zeta(\sigma + 2it)) \geq 0$$

so exponentiating, this gives:

$$|\zeta(\sigma)|^3 |\zeta(\sigma + it)|^4 |\zeta(\sigma + 2it)| \geq 1 \qquad (11)$$

As we have seen in (1), ζ(s) has got a simple pole in s=1 with residue 1. This is equivalent to says:

$$\lim_{s \to 1}(s-1)\zeta(s) = 1$$

Now we suppose that ζ(s) has one zero of order m≥1 at $s_0 = 1 + it_0$ then it is equivalent to:

$$\lim_{s \to s_0}(s - s_0)^{-m}\zeta(s) = c$$

for some $c \in C \setminus \{0\}$. Taking $s = \sigma + it_0$ and σ>1 then we can rewrite (11) as:

$$|\varsigma(\sigma)|^3|\sigma-1|^3\cdot\frac{|\varsigma(\sigma+it_0)|^4}{|s-s_0|^{4m}}\cdot|\varsigma(\sigma+2it_0)|\geq\frac{|\sigma-1|^3}{|s-s_0|^{4m}}$$
$$=\frac{|\sigma-1|^3}{|\sigma-1|^{4m}}=\frac{1}{|\sigma-1|^{4m-1}}$$

Letting $\sigma\to1^+$, and taking account the two limits above, we obtain that there is a pole of order 4m-3≥1 at s=1+2it$_0$. This is impossible, then $\zeta(1+it)\neq0$ for $t\in R\setminus\{0\}$ is true. Therefore if ρ is a nontrivial zero of ζ(s), then Re(ρ)<1, $|x^\rho-1|<1$ and the infinite sum $\sum_\rho\frac{1}{\rho(\rho+1)}$ in (10) converges absolutely. This implies that $\sum_\rho\frac{x^{\rho-1}}{\rho(\rho+1)}$ converges uniformly in x and:

$$\lim_{x\to\infty}\sum_\rho\frac{x^{\rho-1}}{\rho(\rho+1)}=\sum_\rho\lim_{x\to\infty}\frac{x^{\rho-1}}{\rho(\rho+1)}=\sum_\rho 0=0$$

So also the second term of (10) is bounded by $o(x^2)$. Therefore we can conclude that $\psi_1(x)\sim x^2/2$.

In general if two functions are asymptotic one can’t conclude that their derivative are asymptotyc; but we know that the derivative $\psi=\psi_1'$ is a monotonic non-decreasing function, then we can conclude that (7) is true or **ψ(x) ~ x is an equivalent RH**.

We know that de la Vallèe-Poussin, have shoved that for the Prime Numbers Theorem (PNT) is:

$$\psi(x)=x+O(xe^{-a\sqrt{\ln x}})\quad(12)$$

Riemann have showed also for the PNT that:

$$\psi(x)=x+O(\sqrt{x}\ln x^2)\quad(13)$$

The (13) in [26] is the **RH10**.

Furthermore, there are some interesting formulas

The *reflection formula of Euler*:

$$\Gamma(s)\Gamma(1-s)=\frac{\pi}{\sin\pi s},\quad s\notin Z\quad(14)$$

The *Legendre’s formula*:

$$\Gamma(s)\Gamma(s+\frac{1}{2})=\sqrt{\pi}2^{1-2s}\Gamma(2s)\quad(15)$$

Riemann, have obtained the following more symmetrical form:

$$\xi(s)=\frac{1}{2}s(s-1)\zeta(s)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}=(s-1)\zeta(s)\Gamma\left(\frac{s}{2}+1\right)\pi^{-\frac{s}{2}}\quad(16)$$

where

$$\Gamma(s) := \int_0^\infty e^{-t} t^{s-1} dt = \int_0^\infty \frac{t^s}{e^{-t}} \frac{dt}{t}$$

All the zeros concerning the gamma function for s=-2,-4,… are defined "trivial zeros" while the other are the "nontrivial zeros" of the Riemann zeta function. The eq. (16) is important because introduces the notion of symmetry.

A second form for the integral of the gamma function is the *Mellin transform* $1/e^{t}$ (or the *Laplace transform*).

Riemann wrote that the function ξ **satisfies the following properties**:

a) $\xi(s) = \xi(1-s)$
b) ξ is a whole function and $\xi(\bar{s}) = \overline{\xi(s)}$
c) $\xi(\frac{1}{2} + it) \in R$
d) If $\xi(s)=0$, then $0 \leq \sigma \leq 1$
e) $\xi(0) = \xi(1) = 1/2$
f) $\xi(s)>0$ for each $s \in R$

The property a) derives from (14)(15)(16). The b) derives from (14) since that ξ is olomorphic for $\sigma \geq 1$, because the simple pole of ζ at 1 is removed by the factor s-1, and there aren't other poles for $\sigma>1$. Furthermore the (a) implicates ξ olomorphic on all C. The second part of (b) derives from (14) and for every meromorphic function the zeros are real or are introduced in conjugated couples.

Combining (a)(b) we get (c) and similarly (d) but before we note that $\xi(s) \neq 0$ for $\sigma>1$.

Furthermore, we note that for the (1) we have $\zeta(0) = -1/2$. The (e) for $\xi(0)$ derives from $\Gamma(1) = 1$, and from (a) for $\xi(1)$.

To try (f) we note that from (13) we have $\Gamma(s) > 0$ for each $s \in R$ and from (16) and (1) then we obtain (f) for $s>0$ and $s \neq 0,1$.

Corollary *The zeros of the ξ function are identical to the nontrivial zeros of the ζ function.*

Riemann in his notes referred to $\xi(1/2+iu)$, with u complex variable. The fact that all the zeros of this function are real is equivalent to the fact that all the zeros haves $Re(s)= \sigma=1/2$; thence the corollary is equivalent to the Riemann Hypothesis (RH).

Riemann Hypothesis *For every nontrivial zero $s=\sigma+it$ in ζ, then $\sigma=1/2$ (or equivalently all the nontrivial zeros of ζ(s) lies on the critical line).*

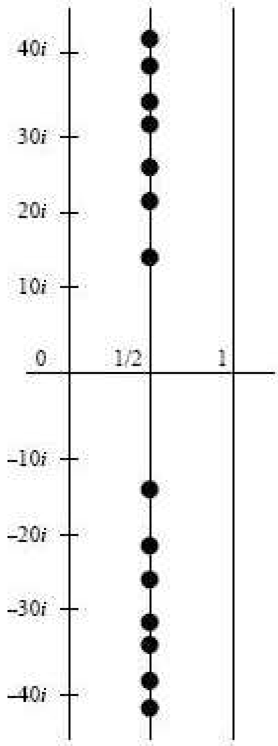

Figure 4 – critical strip and critical line σ=1/2

Conjecture on the multiplicity of the nontrivial zeros *All the nontrivial zeros are simple, or with multiplicity 1.*

We define *f a whole function of the finite order* if:

$$\log |f(s)| = O(|s|^A) \text{ for some } A > 0. \quad (17)$$

The order of f is the lower limit of all the A, for which (17) is true.

Theorem 1 (Weierstrass)

Let $\{c_n\}$ an infinite sequence of complex numbers, such that $0<|c_1|\leq|c_2|\leq\ldots$, and we assume that his limit point is ∞. Then exists a whole function f(s) with zeros (with prescribed multiplicity) in correspondence of these complex numbers.

Theorem 2 (Weierstrass)

Every whole function g(s) of order ≤1, that doesn't have zeros in C, can be written as $g(s) = e^{a+bs}$, where a and b are constants, while every whole function f(s) of order ≤1, which has N≤∞ zeros to c1,c2,c3,…≠0, can be written in the form

$$f(s) = e^{a+bs}\prod_{n=1}^{\infty}\left[\left(1-\frac{s}{c_n}\right)e^{\frac{s}{c_n}}\right]$$

where a and b are constants and the product converges absolutely (if N = ∞) for all the s ∈ C.

Weierstrass tried that:

$$\frac{1}{\Gamma(s)} = se^{\gamma s}\prod_{n=1}^{\infty}\left[\left(1+\frac{s}{n}\right)e^{-\frac{s}{n}}\right] \qquad (18)$$

By (18) Hadamard and de La Valèe Poussin reached to the proof of the Prime Numbers Theorem (PNT) and to show at least two of the Riemann hypothesis (that the zeros are infinite and that can not be on σ=1), because ξ(s) is a whole function of order 1.

The (18) show also that Γ(s) doesn't have zeros in C. Furthermore, since the functional equation in terms of gamma: $\Gamma(s+1) = s\Gamma(s)$ is true then Γ(s) has simple poles in s=0,-1,-2,… with residue 1. In fact is:

$$\lim_{s\to 0^+}\frac{1}{\Gamma(s)} = \lim_{s\to 0^+}\frac{s}{\Gamma(s+1)} = 0$$

where Γ(1)=1.

Product formula for ξ(s)

From the **Hadamard's factorization Theorem** we know that the right hand side of (16) can be rewritten as follow:

$$\xi(s) = e^{A+Bs}\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{s/\rho} \qquad (19)$$

where ρ are the nontrivial zeros in the critical strip while A and B two constants. If we use the right hand side of (16) in (19) we obtain that:

$$\varsigma(s) = \frac{e^{A+Bs}}{(s-1)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}}\prod_{\rho}\left(\left(1-\frac{s}{\rho}\right)e^{s/\rho}\right) \qquad (20)$$

If we pass the (20) to the logarithms we obtain that :

$$\log\varsigma(s) = A + Bs - \log(s-1) - \log\Gamma\left(\frac{s}{2}\right) + \frac{s}{2}\log\pi + \sum_{\rho}\left(\log\left(1-\frac{s}{\rho}\right)+\frac{1}{\rho}\right)$$

Deriving we obtain:

$$\frac{\varsigma'(s)}{\varsigma(s)} = B - \frac{1}{s-1} - \frac{1}{2}\frac{\Gamma'(s)}{\Gamma(s)} + \frac{1}{2}\log\pi + \sum_{\rho}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right) \qquad (21)$$

The (21) show simply a pole at s=1 and nontrivial zeros at s=ρ while the trivial zeros are contained in the gamma function.

Here it is possible to find the values of A and B for s=0. In fact from the closed forms studied in [26] concerning to the *Basilea's problem*, we have that

$$\varsigma(0) = -\frac{1}{2}$$

$$\frac{\varsigma'(0)}{\varsigma(0)} = \log 2\pi$$

Because $\Gamma'(0)=-\gamma$ (Eulero's constant) and $\Gamma(0)=1$, thence from (18) is:

$$B=\log 2\pi-1-\frac{\gamma}{2}-\frac{1}{2}\log\pi=\frac{1}{2}\log 4\pi-1-\frac{\gamma}{2}$$

For s=0, $\varsigma(0)=-\frac{1}{2}$ we obtain that:

$$A=\log(\frac{1}{2})$$

All this shows that starting from the right hand side of (16) we obtain the (19).

3. Theorem free Region from nontrivial zeros

There exists a positive constant c such that $\varsigma(s)$ *doesn't have zeros in the region*

$$\sigma\geq 1\text{-}\frac{\mathrm{c}}{\mathrm{logt}},\ \mathrm{t}\geq 2 \qquad (22)$$

Proof

In (5) we have that $-\frac{\varsigma'(s)}{\varsigma(s)}=\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}$; now we use the real part:

$$-\mathrm{Re}(\frac{\varsigma'(s)}{\varsigma(s)})=\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}\cos(t\log n),\ \sigma>1 \quad (23)$$

For the eqs. (4) and (5) and that s=σ+it, we take the real part and from the (4) we obtain

$$\mathrm{Re}(\ln\varsigma(s))=\sum_{n=2}^{\infty}\frac{\Lambda(n)}{\log n\ \ n^{\sigma}}\cdot\cos(t\log n)$$

If we use the trigonometric identity 3 + 4 cos t + cos 2t = 2(1 + cos t)^2≥0, thence we have:

$$3\,\mathrm{Re}(\ln\varsigma(s))+4\,\mathrm{Re}(\ln\varsigma(\sigma+\mathrm{it}))+\mathrm{Re}(\ln\varsigma(\sigma+2\mathrm{it}))\geq 0$$

From the (16) we obtain that:

$$3(-\frac{\varsigma'(s)}{\varsigma(s)})+4(-\mathrm{Re}(\frac{\varsigma'(\sigma+\mathrm{it})}{\varsigma(\sigma+\mathrm{it})}))+(-\mathrm{Re}(\frac{\varsigma'(\sigma+\mathrm{i2t})}{\varsigma(\sigma+\mathrm{i2t})})\geq 0 \quad (24)$$

Now we examine every term of (24).

First term $3(-\frac{\varsigma'(s)}{\varsigma(s)})$

For the first term a simple pole in s=1 implicates that

$$-\frac{\varsigma'(s)}{\varsigma(s)}<\frac{1}{\sigma-1}+A,\ \mathrm{A}>0\ \ 1\leq\sigma\leq 2$$

Second and third term of (24)

Form the (24) we note that the zeros to the left of σ=1 to height t and 2t surely influence.

If t≥2 and 1≤σ≤2 then for the second term of (21) Γ < A2 logt.

In conclusion, we have that

$$-\mathrm{Re}\left(\frac{\varsigma'(s)}{\varsigma(s)}\right) < A3\log t - \sum_{\rho}\mathrm{Re}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right) \qquad (25)$$

Now we appraise the second to the right. We consider $\rho=\beta+i\gamma$ with β real and ρ nontrivial zero, thence

$$\mathrm{Re}\left(\frac{1}{s-\rho}\right)=\mathrm{Re}\left(\frac{1}{\sigma-\beta+i(t-\gamma)}\right)=\mathrm{Re}\left(\frac{\sigma-\beta-i(t-\gamma)}{|s-\rho|^2}\right)=\frac{\sigma-\beta}{|s-\rho|^2}$$

If we consider that 0<β<1 and 1<σ≤2, we have that $\mathrm{Re}\left(\frac{1}{s-\rho}\right)\geq 0$

Likewise is: $\mathrm{Re}\left(\frac{1}{\rho}\right)=\frac{\beta}{|\rho|^2}\geq 0$

Thence, for t≥2 and 1≤σ≤2 the (23):

$$-\mathrm{Re}\left(\frac{\varsigma'(\sigma+2it)}{\varsigma(\sigma+2it)}\right) < A3\log t < A4\log t$$

Also fixing t≥2 and $\rho=\beta+i\gamma$ and considering only the first term of the summation of (25) we have that:

$$-\mathrm{Re}\left(\frac{\varsigma'(\sigma+it)}{\varsigma(\sigma+it)}\right) < A3\log t - \frac{1}{\sigma-\beta} \qquad (26)$$

If we replace in (24) , the eqs. (25) and (26) we obtain that:

$$\frac{3}{\sigma-1}+3A+(4A3+A1)\log t-\frac{4}{\sigma-\beta}>0$$

If 1<σ≤2 then is:

$$\frac{4}{\sigma-\beta}<\frac{3}{\sigma-1}+A5\log t$$

Now we fix that $\sigma=1+\frac{\delta}{\log t}$ with δ very small and such not to be violated 1<σ≤2

$$\frac{4}{1+\frac{\delta}{\log t}-\beta}<\frac{3}{1+\frac{\delta}{\log t}-1}+A5\log t=\frac{3+A5\delta}{\delta}\log t$$

$$\frac{4\delta}{(3+A5\delta)\log t}<1+\frac{\delta}{\log t}-\beta$$

$$\beta<1+\frac{\delta}{\log t}-\frac{4\delta}{(3+A5\delta)\log t}$$

Thence, in conclusion, we have that:

$$\beta < 1 - \frac{\delta}{\log t}\left(\frac{1-\delta A5}{3 + A5\delta}\right)$$

or:

$$\beta < 1 - \frac{c}{\log t} \quad \text{with } c>0 \qquad (27)$$

Consequently the nontrivial zeros in the critical strip respect the (27) i.e. the (22), that represents a Theorem, is true.

The "Theorem free Region from nontrivial zeros" with the (12) and the error term that it represents brings us to the conclusion that setting a=c :

$$\psi(x) = x + O(xe^{-c\sqrt{\ln x}})$$

The (22) gives the error term:

$$\frac{\mathrm{c}}{\mathrm{logt}} \geq 1\text{-}\sigma, \ \mathrm{t} \geq 2$$

For demonstrate (22) we haven't used the Riemann hypothesis (RH). If we use the RH, i.e. σ=1/2 we have that c=logt/2; i.e. c cannot be a constant but must depend from t (imaginary height of the zero). This means that the area of search of the zeros (symmetrical) widens to farr to increase of T but also always remaining inside the critical strip excluded the confinements.

Let f a meromorphic function in a domain and inside a contour C simply closed and positively directed, such that f is analytical and not nothing on C. Then, we have:

$$\frac{1}{2\pi}\Delta_C \arg(f(z)) = Z - P$$

where Z is the number of the zeros and P is the number of poles, with multiplicity, inside C.

With this technique is possible to show [27] inside a rectangular contour C, positively directed and of vertex -1,2, 2+iT, -1+iT that considering $N(T) = \frac{1}{2\pi}\Delta_C \arg(\xi(s))$, then is:

$$N(\gamma_n) = n \approx \frac{\gamma_n}{2\pi}\log\frac{\gamma_n}{2\pi} \qquad (28)$$

Here γ_n is the height of the nth zero. For the above expression, we have:

$$\log n \approx \log\gamma_n + \log\log\gamma_n - \log 2\pi \approx \log\gamma_n$$

Thence the (28) can be rewritten as:

$$n \approx \frac{\gamma_n}{2\pi}\log n \qquad (29)$$

Thence for n→∞ is:

$$\gamma_n \approx \frac{2\pi n}{\log n} \qquad (30)$$

The eq.(30) gives a value of the height of the nth zero. For example $\gamma_{1000} \approx \frac{2\pi 1000}{\log 1000} \approx 909.6 \ldots$

With the theory of the whole functions and the Weiestrass and Hadamard Theorems, we have understood that the function ζ has *endless nontrivial zeros in the critical strip*; furthermore we know that their density is express through the *Riemann-von Mangoldt formula* N(T):

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T) \quad (31)$$

where T is the imaginary part of the nontrivial zero and this is defined "height".

The eq. (31) can be obtained by the proof of the Theorem: *If T is the height of the zeros, then* $N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + O(1/T)$ *where* $S(T) = \frac{1}{\pi}\arg\zeta(\frac{1}{2} + iT)$ *is defined by the continuous variation on the line from 2 to 2+iT to ½+iT.*

The proof of it always exploits the "Principle of the matter". A following demonstrable Theorem affirms that: $S(T) = O(\log T)$. Thence, we obtain the eq. (31).

From the interpretation of (31) is understood that for $T\to\infty$ the nontrivial zeros grow thick.

We have the following functional equation:

$$\zeta(s) = \left[2^s \pi^{s-1}\Gamma(1-s)\sin\frac{\pi s}{2}\right]\zeta(1-s) \quad (32)$$

The Riemann hypothesis (RH) that all the nontrivial zeros of ζ(s) are on the critical line of the strip, i.e. Re(s)=1/2 is equivalent to:

$$|\zeta(\frac{1}{2} + \Delta + it)| = 0 \;\Rightarrow\; \Delta = 0 \qquad \text{equivalent RH}$$

A possible proof can be made using the functional equation (32).

For $0 \le \Delta \le ½$ and for all $t \ge 2\pi+1$

$$|\zeta(\frac{1}{2} - \Delta + it)| \ge |\zeta(\frac{1}{2} + \Delta + it)| \qquad (33)$$

or if is possible that for $0 < \Delta \le ½$ and for all $t \ge 2\pi+1$

$$|\zeta(\frac{1}{2} - \Delta + it)| > |\zeta(\frac{1}{2} + \Delta + it)| \qquad (34)$$

from here should achieve the equivalent RH. This is because in (29) the nontrivial zeros or lies on Re(s)=1/2 or are in couple s= ½ + Δ +it for 0 < Δ ≤ ½.

Derivative's Theorem

If in the range 0 ≤ Δ ≤ ½ and for all t≥2π+1 we have that

$$|\zeta'(\frac{1}{2}-\Delta+it)| \geq |\zeta'(\frac{1}{2}+\Delta+it)| \qquad (35)$$

or equivalently

$$|\frac{\partial}{\partial\Delta}\zeta(\frac{1}{2}-\Delta+it)| \geq |\frac{\partial}{\partial\Delta}\zeta(\frac{1}{2}+\Delta+it)| \qquad (36)$$

then the RH is true.

This detail is important. The Theorem is shown in [28]. Such theorem allows to get a lot of information that the derivative must respect one so that it reaches the RH.

Now we take the symmetrical functional equation as follow:

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s) \qquad (37)$$

where s=σ+ib

If we take the absolute value of both sides of (37) is:

$$|\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)||\zeta(s)|=|\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)||\zeta(1-s)| \qquad (38)$$

Is clear that for all the complex numbers is:

$$|\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)|>0 \qquad |\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)|>0 \qquad (39)$$

If is true that

$$|\zeta(s)|-|\zeta(1-s)|\neq 0 \qquad (40)$$

it is evident that at least one of two terms of (40) would be different from zero; but from (37) we have that if one of the terms is positive it is also the other. Thence, any "nontrivial zero" can be had in a zone where the (40) is true. Thence **the (40) is false** for the concept of zero as solution of an equation!

Thence, the nontrivial zeros lies in a region where is true that :

$$|\zeta(s)|=|\zeta(1-s)| \qquad (41)$$

or, the nontrivial zeros have a vaule σ that respects (41).

Now we hypothesize for absurd that exists more critical lines for different values of σ, on which could be present nontrivial zeros.

It is evident that every critical line intersects the real axle only in one point, for which the number of critical lines is equal to the number of intersections with the real axle; thence, the value of σ to the intersection can be obtained solving the (41) with s=σ; thence, the (41) becomes:

$$|\zeta(\sigma)| = |\zeta(1-\sigma)| \quad (42)$$

Rather than to exploit the analytical continuation of the Riemann zeta function for Re(s)>0 we can use the alternating zeta function [29]:

$$\zeta(\sigma) = \frac{1}{1-2^{1-\sigma}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{\sigma}} \quad (43)$$

We note that the prime derivative of (43) is:

$$\zeta'(\sigma) = \left[\frac{-2^{1-\sigma}\ln 2}{(1-2^{1-\sigma})^2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{\sigma}} - \frac{1}{1-2^{1-\sigma}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{\sigma}} \ln n\right] < 0$$

that is valid for all the values σ in the critical strip.

Because the prime derivative is negative, then the function is tightly decreasing and the zeta is consequently an injective function; i.e. for every value of the domain corresponds a value of the co-domain (at least one), thence the (41) is true only for σ=1/2.

Obviously contrarily if

$$\left|\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\right| \neq \left|\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\right| \Rightarrow |\zeta(s)| \neq |\zeta(1-s)|$$

here we haven't nontrivial zeros, while we have nontrivial zeros (solutions of an equation) with

$$\zeta(s) = 0 \Rightarrow \left|\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\right| = \left|\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\right| \Rightarrow |\zeta(s)| = |\zeta(1-s)|$$

Now for to obtain the values of σ for all the hypothesize critical lines, it could be used also the following expression:

$$\pi^{-\frac{\sigma}{2}}\Gamma\left(\frac{\sigma}{2}\right) = \pi^{-\frac{1-\sigma}{2}}\Gamma\left(\frac{1-\sigma}{2}\right) \quad (44)$$

We examine separately the prime derivatives in comparison to σ of (44):

$$\frac{d\pi^{-\frac{\sigma}{2}}}{d\sigma} = -\frac{1}{2}\pi^{-\frac{\sigma}{2}}\ln\pi < 0$$

Also here emerges the injectivity on σ in the critical strip.

While is:

$$\frac{d\Gamma\left(\frac{\sigma}{2}\right)}{d\sigma} = -\frac{1}{2}\Gamma\left(\frac{\sigma}{2}\right)[\frac{1}{\sigma/2} + \gamma + \sum_{n=1}^{\infty}(\frac{1}{n+\sigma/2} - \frac{1}{n})] < 0$$

Also here the gamma function is injective on σ.

If we write:

$$g(\sigma) = \pi^{-\frac{\sigma}{2}}\Gamma\left(\frac{\sigma}{2}\right) = g(1-\sigma) = \pi^{-\frac{1-\sigma}{2}}\Gamma\left(\frac{1-\sigma}{2}\right) \quad (45)$$

also the derivative of g(σ) is negative and g is injective and this can happen only for σ=1/2 for which exists only one critical line in the critical strip; but such conclusion is also what the *RH affirms*.

Remark B

If we have a polynomial of any degree, with real or complex variables, the search for roots is possible do it by various methods, for example:

• iterative method
• Newton's Method
• Method of Sturm's and related Theorem
• etc.

Hereafter we use only the method of Newton (see [30]).

We remember that if f(z) is a function and α is a root such that f(α) = 0 then it is possible to express for meromorphic functions $f : C \to \overline{C}$ a **Newton's map** N_f (z) as follows:

$$N_f(z) = z - \frac{f(z)}{f'(z)} \qquad (46)$$

Now we know that:

- If α is a simple root (multiplicity 1) of f(z) then f(α) = 0 and N_f (α) = 0, N'_f (α) = α and
  $$N_f(z) - \alpha = O((z-\alpha)^2), \quad z \to \alpha$$
- If α is a root with multiplicity greater than 1 of f(z) then f(α) = 0, N_f (α) = α, $|N'_f(\alpha)|<1$ and
  $$|N_f(z) - \alpha| \le C\,|z-\alpha|, \quad 0 < C < 1,\ z \to \alpha$$

Proof of the conjecture "The non-trivial zeros of Riemann's zeta have all multiplicity 1"

The proof is based on the Newton's Method.

In general, if we were interested at the values of the roots, it would be possible, with several iterations, to start from a value z_0 and to arrive at a n-th term such that $Nf^n(\alpha)=\alpha$; but for the proof we don't need to find the values of the roots but only say something about the multiplicity of these roots.

By Remark A, in the case of the Riemann's zeta the (46) becomes:

$$N_{\varsigma}(z) = z - \frac{\varsigma(z)}{\varsigma'(z)} \qquad (47)$$

The (47) through (5) of Remark A becomes:

$$N_{\varsigma}(z) = z + \frac{1}{\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{z}}} \qquad (48)$$

In the (48) is:

$$N_{\varsigma}(\alpha) \approx \alpha \qquad (49)$$

because

$$\frac{1}{\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{|\alpha|}}} << 1$$

where for the von Mongoldt's function in (48) is defined as in (3); we have a sum of von Mongoldt's functions (so the sum at denominator is not null). In addition, at the left hand side of (48) there are only constants, then we have:

$$N'_{\varsigma}(\alpha) = 0 \qquad (50)$$

By Remark B with (49) and (50) we conclude the conjecture "The non-trivial zeros of the Riemann's zeta have all multiplicity 1" is true!.

4. On some equations concerning p-adic strings, p-adic and adelic zeta functions, zeta strings and zeta nonlocal scalar fields.

Now we describe some mathematical arguments concerning the p-adic and adelic strings and the zeta strings (see [31] [32] [33] [34] [35]).

Like in the ordinary string theory, the starting point of p-adic strings is a construction of the corresponding scattering amplitudes. Recall that the ordinary crossing symmetric Veneziano amplitude can be presented in the following forms:

$$A_{\infty}(a,b) = g^{2}\int_{R} |x|_{\infty}^{a-1} |1-x|_{\infty}^{b-1} dx = g^{2}\left[\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} + \frac{\Gamma(b)\Gamma(c)}{\Gamma(b+c)} + \frac{\Gamma(c)\Gamma(a)}{\Gamma(c+a)}\right] = g^{2}\frac{\zeta(1-a)}{\zeta(a)}\frac{\zeta(1-b)}{\zeta(b)}\frac{\zeta(1-c)}{\zeta(c)} =$$
$$= g^{2}\int DX \exp\left(-\frac{i}{2\pi}\int d^{2}\sigma\partial^{\alpha}X_{\mu}\partial_{\alpha}X^{\mu}\right)\prod_{j=1}^{4}\int d^{2}\sigma_{j}\exp\left(ik_{\mu}^{(j)}X^{\mu}\right), \quad (51-54)$$

where $\hbar = 1$, $T = 1/\pi$, and $a = -\alpha(s) = -1 - \frac{s}{2}$, $b = -\alpha(t)$, $c = -\alpha(u)$ with the condition $s + t + u = -8$, i.e. $a + b + c = 1$.
The p-adic generalization of the above expression

$$A_\infty(a,b) = g^2 \int_R |x|_\infty^{a-1} |1 - x|_\infty^{b-1} dx ,$$

is:

$$A_p(a,b) = g_p^2 \int_{Q_p} |x|_p^{a-1} |1 - x|_p^{b-1} dx , \quad (55)$$

where $|...|_p$ denotes p-adic absolute value. In this case only string world-sheet parameter x is treated as p-adic variable, and all other quantities have their usual (real) valuation.
Now, we remember that the Gauss integrals satisfy adelic product formula

$$\int_R \chi_\infty\left(ax^2 + bx\right) d_\infty x \prod_{p \in P} \int_{Q_p} \chi_p\left(ax^2 + bx\right) d_p x = 1 , \quad a \in Q^\times , \quad b \in Q , \quad (56)$$

what follows from

$$\int_{Q_v} \chi_v\left(ax^2 + bx\right) d_v x = \lambda_v(a) |2a|_v^{-\frac{1}{2}} \chi_v\left(-\frac{b^2}{4a}\right), \quad v = \infty, 2, ..., p... . \quad (57)$$

These Gauss integrals apply in evaluation of the Feynman path integrals

$$K_v(x'', t''; x', t') = \int_{x',t'}^{x'',t''} \chi_v\left(-\frac{1}{h} \int_{t'}^{t''} L(\dot{q}, q, t) dt\right) D_v q , \quad (58)$$

for kernels $K_v(x'', t''; x', t')$ of the evolution operator in adelic quantum mechanics for quadratic Lagrangians. In the case of Lagrangian

$$L(\dot{q}, q) = \frac{1}{2}\left(-\frac{\dot{q}^2}{4} - \lambda q + 1\right),$$

for the de Sitter cosmological model one obtains

$$K_\infty(x'', T; x', 0) \prod_{p \in P} K_p(x'', T; x', 0) = 1 , \quad x'', x', \lambda \in Q , \ T \in Q^\times , \quad (59)$$

where

$$K_v(x'', T; x', 0) = \lambda_v(-8T) |4T|_v^{-\frac{1}{2}} \chi_v\left(-\frac{\lambda^2 T^3}{24} + [\lambda(x'' + x') - 2]\frac{T}{4} + \frac{(x'' - x')^2}{8T}\right). \quad (60)$$

Also here we have the number 24 that correspond to the Ramanujan function that has 24 "modes", i.e., the physical vibrations of a bosonic string. Hence, we obtain the following mathematical connection:

$$K_v(x'',T;x',0)=\lambda_v(-8T)|4T|_v^{-\frac{1}{2}}\chi_v\left(-\frac{\lambda^2T^3}{24}+[\lambda(x''+x')-2]\frac{T}{4}+\frac{(x''-x')^2}{8T}\right)\Rightarrow$$

$$\Rightarrow\frac{4\left[anti\log\frac{\int_0^\infty\frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (60b)$$

The adelic wave function for the simplest ground state has the form

$$\psi_A(x)=\psi_\infty(x)\prod_{p\in P}\Omega(|x|_p)=\begin{cases}\psi_\infty(x),x\in Z\\0,x\in Q\setminus Z\end{cases}, \quad (61)$$

where $\Omega(|x|_p)=1$ if $|x|_p\le 1$ and $\Omega(|x|_p)=0$ if $|x|_p>1$. Since this wave function is non-zero only in integer points it can be interpreted as discreteness of the space due to p-adic effects in adelic approach. The Gel'fand-Graev-Tate gamma and beta functions are:

$$\Gamma_\infty(a)=\int_R|x|_\infty^{a-1}\chi_\infty(x)d_\infty x=\frac{\zeta(1-a)}{\zeta(a)},\quad \Gamma_p(a)=\int_{Q_p}|x|_p^{a-1}\chi_p(x)d_px=\frac{1-p^{a-1}}{1-p^{-a}}, \quad (62)$$

$$B_\infty(a,b)=\int_R|x|_\infty^{a-1}|1-x|_\infty^{b-1}d_\infty x=\Gamma_\infty(a)\Gamma_\infty(b)\Gamma_\infty(c), \quad (63)$$

$$B_p(a,b)=\int_{Q_p}|x|_p^{a-1}|1-x|_p^{b-1}d_px=\Gamma_p(a)\Gamma_p(b)\Gamma_p(c), \quad (64)$$

where $a,b,c\in C$ with condition $a+b+c=1$ and $\zeta(a)$ is the Riemann zeta function. With a regularization of the product of p-adic gamma functions one has adelic products:

$$\Gamma_\infty(u)\prod_{p\in P}\Gamma_p(u)=1,\quad B_\infty(a,b)\prod_{p\in P}B_p(a,b)=1,\quad u\neq 0,1,\quad u=a,b,c, \quad (65)$$

where $a+b+c=1$. We note that $B_\infty(a,b)$ and $B_p(a,b)$ are the crossing symmetric standard and p-adic Veneziano amplitudes for scattering of two open tachyon strings. Introducing real, p-adic and adelic zeta functions as

$$\zeta_\infty(a)=\int_R\exp(-\pi x^2)|x|_\infty^{a-1}d_\infty x=\pi^{-\frac{a}{2}}\Gamma\left(\frac{a}{2}\right), \quad (66)$$

$$\zeta_p(a)=\frac{1}{1-p^{-1}}\int_{Q_p}\Omega(|x|_p)|x|_p^{a-1}d_px=\frac{1}{1-p^{-a}},\quad \mathrm{Re}\,a>1, \quad (67)$$

$$\zeta_A(a)=\zeta_\infty(a)\prod_{p\in P}\zeta_p(a)=\zeta_\infty(a)\zeta(a), \quad (68)$$

one obtains

$$\zeta_A(1-a)=\zeta_A(a), \quad (69)$$

where $\zeta_A(a)$ can be called adelic zeta function. We have also that

$$\zeta_A(a)=\zeta_\infty(a)\prod_{p\in P}\zeta_p(a)=\zeta_\infty(a)\zeta(a)=\int_R \exp\left(-\pi x^2\right)|x|_\infty^{a-1}d_\infty x\cdot\frac{1}{1-p^{-1}}\int_{Q_p}\Omega\left(|x|_p\right)|x|_p^{a-1}d_p x. \quad (69b)$$

Let us note that $\exp\left(-\pi x^2\right)$ and $\Omega\left(|x|_p\right)$ are analogous functions in real and p-adic cases. Adelic harmonic oscillator has connection with the Riemann zeta function. The simplest vacuum state of the adelic harmonic oscillator is the following Schwartz-Bruhat function:

$$\psi_A(x)=2^{\frac{1}{4}}e^{-\pi x_\infty^2}\prod_{p\in P}\Omega\left(|x_p|_p\right), \quad (70)$$

whose the Fourier transform

$$\psi_A(k)=\int\chi_A(kx)\psi_A(x)=2^{\frac{1}{4}}e^{-\pi k_\infty^2}\prod_{p\in P}\Omega\left(|k_p|_p\right) \quad (71)$$

has the same form as $\psi_A(x)$. The Mellin transform of $\psi_A(x)$ is

$$\Phi_A(a)=\int\psi_A(x)|x|^a d_A^\times x=\int_R\psi_\infty(x)|x|^{a-1}d_\infty x\prod_{p\in P}\frac{1}{1-p^{-1}}\int_{Q_p}\Omega\left(|x|_p\right)|x|^{a-1}d_p x=\sqrt{2}\Gamma\left(\frac{a}{2}\right)\pi^{-\frac{a}{2}}\zeta(a) \quad (72)$$

and the same for $\psi_A(k)$. Then according to the Tate formula one obtains (69).
The exact tree-level Lagrangian for effective scalar field φ which describes open p-adic string tachyon is

$$\mathcal{L}_p=\frac{1}{g^2}\frac{p^2}{p-1}\left[-\frac{1}{2}\varphi p^{-\frac{\Box}{2}}\varphi+\frac{1}{p+1}\varphi^{p+1}\right], \quad (73)$$

where p is any prime number, $\Box=-\partial_t^2+\nabla^2$ is the D-dimensional d'Alambertian and we adopt metric with signature $(-+\ldots+)$. Now, we want to show a model which incorporates the p-adic string Lagrangians in a restricted adelic way. Let us take the following Lagrangian

$$L=\sum_{n\geq 1}C_n\mathcal{L}_n=\sum_{n\geq 1}\frac{n-1}{n^2}\mathcal{L}_n=\frac{1}{g^2}\left[-\frac{1}{2}\phi\sum_{n\geq 1}n^{-\frac{\Box}{2}}\phi+\sum_{n\geq 1}\frac{1}{n+1}\phi^{n+1}\right]. \quad (74)$$

Recall that the Riemann zeta function is defined as

$$\zeta(s)=\sum_{n\geq 1}\frac{1}{n^s}=\prod_p\frac{1}{1-p^{-s}}, \quad s=\sigma+i\tau, \quad \sigma>1. \quad (75)$$

Employing usual expansion for the logarithmic function and definition (75) we can rewrite (74) in the form

$$L = -\frac{1}{g^2}\left[\frac{1}{2}\phi\zeta\left(\frac{\Box}{2}\right)\phi + \phi + \ln(1-\phi)\right], \quad (76)$$

where $|\phi| < 1$. $\zeta\left(\frac{\Box}{2}\right)$ acts as pseudodifferential operator in the following way:

$$\zeta\left(\frac{\Box}{2}\right)\phi(x) = \frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk\,, \quad -k^2 = k_0^2 - \vec{k}^2 > 2+\varepsilon\,, \quad (77)$$

where $\tilde{\phi}(k) = \int e^{(-ikx)}\phi(x)dx$ is the Fourier transform of $\phi(x)$.

Dynamics of this field ϕ is encoded in the (pseudo)differential form of the Riemann zeta function. **When the d'Alambertian is an argument of the Riemann zeta function we shall call such string a "zeta string".** Consequently, the above ϕ is an open scalar zeta string. The equation of motion for the zeta string ϕ is

$$\zeta\left(\frac{\Box}{2}\right)\phi = \frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon} e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk = \frac{\phi}{1-\phi} \quad (78)$$

which has an evident solution $\phi = 0$.

For the case of time dependent spatially homogeneous solutions, we have the following equation of motion

$$\zeta\left(\frac{-\partial_t^2}{2}\right)\phi(t) = \frac{1}{(2\pi)}\int_{|k_0|>\sqrt{2}+\varepsilon} e^{-ik_0 t}\zeta\left(\frac{k_0^2}{2}\right)\tilde{\phi}(k_0)dk_0 = \frac{\phi(t)}{1-\phi(t)}. \quad (79)$$

With regard the open and closed scalar zeta strings, the equations of motion are

$$\zeta\left(\frac{\Box}{2}\right)\phi = \frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk = \sum_{n\geq 1}\theta^{\frac{n(n-1)}{2}}\phi^n\,, \quad (80)$$

$$\zeta\left(\frac{\Box}{4}\right)\theta = \frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{4}\right)\tilde{\theta}(k)dk = \sum_{n\geq 1}\left[\theta^{n^2} + \frac{n(n-1)}{2(n+1)}\theta^{\frac{n(n-1)}{2}-1}\left(\phi^{n+1}-1\right)\right], \quad (81)$$

and one can easily see trivial solution $\phi = \theta = 0$.

The exact tree-level Lagrangian of effective scalar field φ, which describes open p-adic string tachyon, is:

$$\mathcal{L}_p = \frac{m_p^D}{g_p^2}\frac{p^2}{p-1}\left[-\frac{1}{2}\varphi p^{-\frac{\Box}{2m_p^2}}\varphi + \frac{1}{p+1}\varphi^{p+1}\right], \quad (82)$$

where p is any prime number, $\Box = -\partial_t^2 + \nabla^2$ is the D-dimensional d'Alambertian and we adopt metric with signature $(-+...+)$, as above. Now, we want to introduce a model which incorporates all the above string Lagrangians (82) with p replaced by $n\in N$. Thence, we take the sum of all Lagrangians $\mathcal{L}_n$ in the form

$$L=\sum_{n=1}^{+\infty}C_n\mathcal{L}_n=\sum_{n=1}^{+\infty}C_n\frac{m_n^D}{g_n^2}\frac{n^2}{n-1}\left[-\frac{1}{2}\phi n^{-\frac{\Box}{2m_n^2}}\phi+\frac{1}{n+1}\phi^{n+1}\right], \quad (83)$$

whose explicit realization depends on particular choice of coefficients C_n, masses m_n and coupling constants g_n.

Now, we consider the following case

$$C_n=\frac{n-1}{n^{2+h}}, \quad (84)$$

where h is a real number. The corresponding Lagrangian reads

$$L_h=\frac{m^D}{g^2}\left[-\frac{1}{2}\phi\sum_{n=1}^{+\infty}n^{-\frac{\Box}{2m^2}-h}\phi+\sum_{n=1}^{+\infty}\frac{n^{-h}}{n+1}\phi^{n+1}\right] \quad (85)$$

and it depends on parameter h. According to the Euler product formula one can write

$$\sum_{n=1}^{+\infty}n^{-\frac{\Box}{2m^2}-h}=\prod_p\frac{1}{1-p^{-\frac{\Box}{2m^2}-h}}. \quad (86)$$

Recall that standard definition of the Riemann zeta function is

$$\zeta(s)=\sum_{n=1}^{+\infty}\frac{1}{n^s}=\prod_p\frac{1}{1-p^{-s}}, \quad s=\sigma+i\tau, \quad \sigma>1, \quad (87)$$

which has analytic continuation to the entire complex s plane, excluding the point $s=1$, where it has a simple pole with residue 1. Employing definition (87) we can rewrite (85) in the form

$$L_h=\frac{m^D}{g^2}\left[-\frac{1}{2}\phi\zeta\left(\frac{\Box}{2m^2}+h\right)\phi+\sum_{n=1}^{+\infty}\frac{n^{-h}}{n+1}\phi^{n+1}\right]. \quad (88)$$

Here $\zeta\left(\frac{\Box}{2m^2}+h\right)$ acts as a pseudodifferential operator

$$\zeta\left(\frac{\Box}{2m^2}+h\right)\phi(x)=\frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{2m^2}+h\right)\tilde{\phi}(k)dk, \quad (89)$$

where $\tilde{\phi}(k)=\int e^{(-ikx)}\phi(x)dx$ is the Fourier transform of $\phi(x)$.

We consider Lagrangian (88) with analytic continuations of the zeta function and the power series $\sum\frac{n^{-h}}{n+1}\phi^{n+1}$, i.e.

$$L_h = \frac{m^D}{g^2}\left[-\frac{1}{2}\phi\zeta\left(\frac{\Box}{2m^2}+h\right)\phi + AC\sum_{n=1}^{+\infty}\frac{n^{-h}}{n+1}\phi^{n+1}\right], \quad (90)$$

where AC denotes analytic continuation.
Potential of the above zeta scalar field (90) is equal to $-L_h$ at $\Box = 0$, i.e.

$$V_h(\phi) = \frac{m^D}{g^2}\left(\frac{\phi^2}{2}\zeta(h) - AC\sum_{n=1}^{+\infty}\frac{n^{-h}}{n+1}\phi^{n+1}\right), \quad (91)$$

where $h \neq 1$ since $\zeta(1) = \infty$. The term with ζ -function vanishes at $h = -2, -4, -6, \ldots$. The equation of motion in differential and integral form is

$$\zeta\left(\frac{\Box}{2m^2}+h\right)\phi = AC\sum_{n=1}^{+\infty}n^{-h}\varphi^n, \quad (92)$$

$$\frac{1}{(2\pi)^D}\int_{R^D} e^{ixk}\zeta\left(-\frac{k^2}{2m^2}+h\right)\tilde{\phi}(k)dk = AC\sum_{n=1}^{+\infty}n^{-h}\phi^n, \quad (93)$$

respectively.
Now, we consider five values of h, which seem to be the most interesting, regarding the Lagrangian (90): $h = 0$, $h = \pm 1$, and $h = \pm 2$. For $h = -2$, the corresponding equation of motion now read:

$$\zeta\left(\frac{\Box}{2m^2}-2\right)\phi = \frac{1}{(2\pi)^D}\int_{R^D} e^{ixk}\zeta\left(-\frac{k^2}{2m^2}-2\right)\tilde{\phi}(k)dk = \frac{\phi(\phi+1)}{(1-\phi)^3}. \quad (94)$$

This equation has two trivial solutions: $\phi(x) = 0$ and $\phi(x) = -1$. Solution $\phi(x) = -1$ can be also shown taking $\tilde{\phi}(k) = -\delta(k)(2\pi)^D$ and $\zeta(-2) = 0$ in (94).
For $h = -1$, the corresponding equation of motion is:

$$\zeta\left(\frac{\Box}{2m^2}-1\right)\phi = \frac{1}{(2\pi)^D}\int_{R^D} e^{ixk}\zeta\left(-\frac{k^2}{2m^2}-1\right)\tilde{\phi}(k)dk = \frac{\phi}{(1-\phi)^2}. \quad (95)$$

where $\zeta(-1) = -\frac{1}{12}$.
The equation of motion (95) has a constant trivial solution only for $\phi(x) = 0$.
For $h = 0$, the equation of motion is

$$\zeta\left(\frac{\Box}{2m^2}\right)\phi = \frac{1}{(2\pi)^D}\int_{R^D} e^{ixk}\zeta\left(-\frac{k^2}{2m^2}\right)\tilde{\phi}(k)dk = \frac{\phi}{1-\phi}. \quad (96)$$

It has two solutions: $\phi = 0$ and $\phi = 3$. The solution $\phi = 3$ follows from the Taylor expansion of the Riemann zeta function operator

$$\zeta\left(\frac{\Box}{2m^2}\right) = \zeta(0) + \sum_{n\geq 1}\frac{\zeta^{(n)}(0)}{n!}\left(\frac{\Box}{2m^2}\right)^n, \quad (97)$$

as well as from $\tilde{\phi}(k) = (2\pi)^D 3\delta(k)$.
For $h = 1$, the equation of motion is:

$$\frac{1}{(2\pi)^D}\int_{R^D} e^{ixk}\zeta\left(-\frac{k^2}{2m^2}+1\right)\tilde{\phi}(k)dk = -\frac{1}{2}\ln(1-\phi)^2, \quad (98)$$

where $\zeta(1) = \infty$ gives $V_1(\phi) = \infty$.
In conclusion, for $h = 2$, we have the following equation of motion:

$$\frac{1}{(2\pi)^D}\int_{R^D} e^{ixk}\zeta\left(-\frac{k^2}{2m^2}+2\right)\tilde{\phi}(k)dk = -\int_0^{\phi}\frac{\ln(1-w)^2}{2w}dw. \quad (99)$$

Since holds equality

$$-\int_0^1\frac{\ln(1-w)}{w}dw = \sum_{n=1}^{\infty}\frac{1}{n^2} = \zeta(2)$$

one has trivial solution $\phi = 1$ in (99).

Now, we want to analyze the following case: $C_n = \frac{n^2-1}{n^2}$. In this case, from the Lagrangian (83), we obtain:

$$L = \frac{m^D}{g^2}\left[-\frac{1}{2}\phi\left\{\zeta\left(\frac{\square}{2m^2}-1\right)+\zeta\left(\frac{\square}{2m^2}\right)\right\}\phi+\frac{\phi^2}{1-\phi}\right]. \quad (100)$$

The corresponding potential is:

$$V(\phi) = -\frac{m^D}{g}\frac{31-7\phi}{24(1-\phi)}\phi^2. \quad (101)$$

We note that 7 and 31 are prime natural numbers, i.e. $6n \pm 1$ with $n = 1$ and 5, with 1 and 5 that are Fibonacci's numbers. Furthermore, the number 24 is related to the Ramanujan function that has 24 "modes" that correspond to the physical vibrations of a bosonic string. Thence, we obtain:

$$V(\phi) = -\frac{m^D}{g}\frac{31-7\phi}{24(1-\phi)}\phi^2 \Rightarrow \frac{4\left[anti\log\frac{\int_0^{\infty}\frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (101b)$$

The equation of motion is:

$$\left[\zeta\left(\frac{\square}{2m^2}-1\right)+\zeta\left(\frac{\square}{2m^2}\right)\right]\phi = \frac{\phi\left[(\phi-1)^2+1\right]}{(\phi-1)^2}. \quad (102)$$

Its weak field approximation is:

$$\left[\zeta\left(\frac{\Box}{2m^2}-1\right)+\zeta\left(\frac{\Box}{2m^2}\right)-2\right]\phi=0\,,\quad (103)$$

which implies condition on the mass spectrum

$$\zeta\left(\frac{M^2}{2m^2}-1\right)+\zeta\left(\frac{M^2}{2m^2}\right)=2\,.\quad (104)$$

From (104) it follows one solution for $M^2>0$ at $M^2\approx 2.79m^2$ and many tachyon solutions when $M^2<-38m^2$.

We note that the number 2.79 is connected with $\phi=\frac{\sqrt{5}-1}{2}$ and $\Phi=\frac{\sqrt{5}+1}{2}$, i.e. the "aurea" section and the "aurea" ratio. Indeed, we have that:

$$\left(\frac{\sqrt{5}+1}{2}\right)^2+\frac{1}{2^2}\left(\frac{\sqrt{5}-1}{2}\right)=2{,}772542\cong 2{,}78\,.$$

Furthermore, we have also that:

$$(\Phi)^{14/7}+(\Phi)^{-25/7}=2{,}618033989+0{,}179314566=2{,}79734$$

With regard the extension by ordinary Lagrangian, we have the Lagrangian, potential, equation of motion and mass spectrum condition that, when $C_n=\frac{n^2-1}{n^2}$, are:

$$L=\frac{m^D}{g^2}\left[\frac{\phi}{2}\left\{\frac{\Box}{m^2}-\zeta\left(\frac{\Box}{2m^2}-1\right)-\zeta\left(\frac{\Box}{2m^2}\right)-1\right\}\phi+\frac{\phi^2}{2}\ln\phi^2+\frac{\phi^2}{1-\phi}\right],\quad (105)$$

$$V(\phi)=\frac{m^D}{g^2}\frac{\phi^2}{2}\left[\zeta(-1)+\zeta(0)+1-\ln\phi^2-\frac{1}{1-\phi}\right],\quad (106)$$

$$\left[\zeta\left(\frac{\Box}{2m^2}-1\right)+\zeta\left(\frac{\Box}{2m^2}\right)-\frac{\Box}{m^2}+1\right]\phi=\phi\ln\phi^2+\phi+\frac{2\phi-\phi^2}{(1-\phi)^2}\,,\quad (107)$$

$$\zeta\left(\frac{M^2}{2m^2}-1\right)+\zeta\left(\frac{M^2}{2m^2}\right)=\frac{M^2}{m^2}\,.\quad (108)$$

In addition to many tachyon solutions, equation (108) has two solutions with positive mass: $M^2\approx 2.67m^2$ and $M^2\approx 4.66m^2$.

We note also here, that the numbers 2.67 and 4.66 are related to the "aureo" numbers. Indeed, we have that:

$$\left(\frac{\sqrt{5}+1}{2}\right)^2+\frac{1}{2\cdot 5}\left(\frac{\sqrt{5}-1}{2}\right)\cong 2.6798\,,$$

$$\left(\frac{\sqrt{5}+1}{2}\right)^2+\left(\frac{\sqrt{5}+1}{2}\right)+\frac{1}{2^2}\left(\frac{\sqrt{5}+1}{2}\right)\cong 4.64057\,.$$

Furthermore, we have also that:

$$(\Phi)^{14/7} + (\Phi)^{-41/7} = 2{,}618033989 + 0{,}059693843 = 2{,}6777278;$$
$$(\Phi)^{22/7} + (\Phi)^{-30/7} = 4{,}537517342 + 0{,}1271565635 = 4{,}6646738.$$

Now, we describe the case of $C_n = \mu(n)\frac{n-1}{n^2}$. Here $\mu(n)$ is the Mobius function, which is defined for all positive integers and has values 1, 0, – 1 depending on factorization of n into prime numbers p. It is defined as follows:

$$\mu(n) = \begin{cases} 0, & n = p^2 m \\ (-1)^k, & n = p_1 p_2 \dots p_k,\, p_i \neq p_j \\ 1, & n = 1, (k = 0). \end{cases} \quad (109)$$

The corresponding Lagrangian is

$$L_\mu = C_0 \mathcal{L}_0 + \frac{m^D}{g^2}\left[-\frac{1}{2}\phi \sum_{n=1}^{+\infty} \frac{\mu(n)}{n^{\frac{\Box}{2m^2}}}\phi + \sum_{n=1}^{+\infty}\frac{\mu(n)}{n+1}\phi^{n+1}\right] \quad (110)$$

Recall that the inverse Riemann zeta function can be defined by

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{+\infty}\frac{\mu(n)}{n^s}, \quad s = \sigma + it, \quad \sigma > 1. \quad (111)$$

Now (110) can be rewritten as

$$L_\mu = C_0 \mathcal{L}_0 + \frac{m^D}{g^2}\left[-\frac{1}{2}\phi \frac{1}{\zeta\left(\frac{\Box}{2m^2}\right)}\phi + \int_0^\infty \mathcal{M}(\phi)d\phi\right], \quad (112)$$

where $\mathcal{M}(\phi) = \sum_{n=1}^{+\infty}\mu(n)\phi^n = \phi - \phi^2 - \phi^3 - \phi^5 + \phi^6 - \phi^7 + \phi^{10} - \phi^{11} - \dots$ The corresponding potential, equation of motion and mass spectrum formula, respectively, are:

$$V_\mu(\phi) = -L_\mu(\Box = 0) = \frac{m^D}{g^2}\left[\frac{C_0}{2}\phi^2\left(1 - \ln\phi^2\right) - \phi^2 - \int_0^\phi \mathcal{M}(\phi)d\phi\right], \quad (113)$$

$$\frac{1}{\zeta\left(\frac{\Box}{2m^2}\right)}\phi - \mathcal{M}(\phi) - C_0\frac{\Box}{m^2}\phi - 2C_0\phi\ln\phi = 0, \quad (114)$$

$$\frac{1}{\zeta\left(\frac{M^2}{2m^2}\right)} - C_0\frac{M^2}{m^2} + 2C_0 - 1 = 0, \quad |\phi| << 1, \quad (115)$$

where usual relativistic kinematic relation $k^2 = -k_0^2 + \vec{k}^2 = -M^2$ is used.
Now, we take the pure numbers concerning the eqs. (104) and (108). They are: 2.79, 2.67 and 4.66. We note that all the numbers are related with $\Phi = \frac{\sqrt{5}+1}{2}$, thence with the aurea ratio, by the following expressions:

$$2{,}79 \cong (\Phi)^{15/7}; \qquad 2{,}67 \cong (\Phi)^{13/7} + (\Phi)^{-21/7}; \qquad 4{,}66 \cong (\Phi)^{22/7} + (\Phi)^{-30/7}. \quad (116)$$

5. Mathematical connections

Now we describe some possible mathematical connections between some equations concerning the p-adic, adelic and zeta strings and some equations concerning the Riemann Hypothesis.
With regard the **Chapter 1**, we have the following interesting connections, between the eqs (1), (8) and (10) and the eqs. (55), (69b) and (78) of **Chapter 4**:

$$\varsigma(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} \left(s \int_n^{\infty} \frac{dx}{x^{s+1}} \right) = s \sum_{n=1}^{\infty} \int_n^{\infty} \frac{dx}{x^{s+1}}$$
$$= s \int_1^{\infty} \left(\sum_{n \le x} 1 \right) \frac{dx}{x^{s+1}} = s \int_1^{\infty} \frac{[x]}{x^{s+1}} dx = s \int_1^{\infty} \frac{x - \{x\}}{x^{s+1}} dx \Rightarrow$$
$$= \frac{s}{s-1} - s \int_1^{\infty} \frac{\{x\}}{x^{s+1}} dx, \ \sigma > 1$$
$$\Rightarrow A_p(a,b) = g_p^2 \int_{Q_p} |x|_p^{a-1} |1-x|_p^{b-1} dx \Rightarrow$$
$$\Rightarrow \zeta_A(a) = \zeta_\infty(a) \prod_{p \in P} \zeta_p(a) = \zeta_\infty(a)\zeta(a) = \int_R \exp(-\pi x^2) |x|_\infty^{a-1} d_\infty x \cdot \frac{1}{1-p^{-1}} \int_{Q_p} \Omega(|x|_p) |x|_p^{a-1} d_p x. \quad (117)$$

$$\varsigma(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} \left(s \int_n^{\infty} \frac{dx}{x^{s+1}} \right) = s \sum_{n=1}^{\infty} \int_n^{\infty} \frac{dx}{x^{s+1}}$$
$$= s \int_1^{\infty} \left(\sum_{n \le x} 1 \right) \frac{dx}{x^{s+1}} = s \int_1^{\infty} \frac{[x]}{x^{s+1}} dx = s \int_1^{\infty} \frac{x - \{x\}}{x^{s+1}} dx \rightarrow$$
$$= \frac{s}{s-1} - s \int_1^{\infty} \frac{\{x\}}{x^{s+1}} dx, \ \sigma > 1$$
$$\rightarrow \zeta\left(\frac{\Box}{2}\right)\phi = \frac{1}{(2\pi)^D} \int_{k_0^2 - \vec{k}^2 > 2+\varepsilon} e^{ixk} \zeta\left(-\frac{k^2}{2}\right) \tilde{\phi}(k) dk = \frac{\phi}{1-\phi}, \quad (118)$$

$$\psi(x) = \lim_{T\to\infty} \frac{1}{2\pi i}\int_{a-iT}^{a+iT}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)x^s \frac{ds}{s}, \; x > 0 \rightarrow$$

$$\Rightarrow A_p(a,b) = g_p^2 \int_{Q_p} |x|_p^{a-1} |1-x|_p^{b-1} dx \Rightarrow$$

$$\Rightarrow \zeta_A(a) = \zeta_\infty(a)\prod_{p\in P}\zeta_p(a) = \zeta_\infty(a)\zeta(a) = \int_R \exp\left(-\pi x^2\right)|x|_\infty^{a-1} d_\infty x \cdot \frac{1}{1-p^{-1}}\int_{Q_p}\Omega\left(|x|_p\right)|x|_p^{a-1} d_p x . \quad (119)$$

$$\psi(x) = \lim_{T\to\infty} \frac{1}{2\pi i}\int_{a-iT}^{a+iT}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)x^s \frac{ds}{s}, \; x > 0 \rightarrow$$

$$\rightarrow \zeta\left(\frac{\square}{2}\right)\phi = \frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon} e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk = \frac{\phi}{1-\phi}, \quad (120)$$

$$\psi_1(x) = \int_0^x \psi(t)dt = \frac{x^2}{2} - \sum_\rho \frac{x^{\rho+1}}{\rho(\rho+1)} - \sum_n \frac{x^{-2n+1}}{2n(2n-1)} - x\log(2\pi) + const \rightarrow$$

$$\Rightarrow A_p(a,b) = g_p^2 \int_{Q_p} |x|_p^{a-1} |1-x|_p^{b-1} dx \Rightarrow$$

$$\Rightarrow \zeta_A(a) = \zeta_\infty(a)\prod_{p\in P}\zeta_p(a) = \zeta_\infty(a)\zeta(a) = \int_R \exp\left(-\pi x^2\right)|x|_\infty^{a-1} d_\infty x \cdot \frac{1}{1-p^{-1}}\int_{Q_p}\Omega\left(|x|_p\right)|x|_p^{a-1} d_p x . \quad (121)$$

$$\psi_1(x) = \int_0^x \psi(t)dt = \frac{x^2}{2} - \sum_\rho \frac{x^{\rho+1}}{\rho(\rho+1)} - \sum_n \frac{x^{-2n+1}}{2n(2n-1)} - x\log(2\pi) + const \rightarrow$$

$$\rightarrow \zeta\left(\frac{\square}{2}\right)\phi = \frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon} e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk = \frac{\phi}{1-\phi}. \quad (122)$$

With regard the **Chapter 2**, we have the following interesting connections, between the eq. (16) and eqs (55), (69b) and (78) of **Chapter 4**:

$$\xi(s) = \frac{1}{2}s(s-1)\zeta(s)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}} = (s-1)\zeta(s)\Gamma\left(\frac{s}{2}+1\right)\pi^{-\frac{s}{2}} \quad \rightarrow$$

$$\Gamma(s) := \int_0^\infty e^{-t}t^{s-1}dt = \int_0^\infty \frac{t^s}{e^{-t}}\frac{dt}{t}$$

$$\Rightarrow A_p(a,b) = g_p^2 \int_{Q_p} |x|_p^{a-1} |1-x|_p^{b-1} dx \Rightarrow$$

$$\Rightarrow \zeta_A(a) = \zeta_\infty(a)\prod_{p\in P}\zeta_p(a) = \zeta_\infty(a)\zeta(a) = \int_R \exp\left(-\pi x^2\right)|x|_\infty^{a-1} d_\infty x \cdot \frac{1}{1-p^{-1}}\int_{Q_p}\Omega\left(|x|_p\right)|x|_p^{a-1} d_p x \rightarrow$$

$$\rightarrow \zeta\left(\frac{\square}{2}\right)\phi = \frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon} e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk = \frac{\phi}{1-\phi}. \quad (123)$$

With regard the **Chapter 3**, we have the following interesting connections, between the eqs. (24) and (45) and the eqs. (78) and (112) of **Chapter 4**:

$$3\left(-\frac{\varsigma'(s)}{\varsigma(s)}\right)+4\left(-\operatorname{Re}\left(\frac{\varsigma'(\sigma+\mathrm{it})}{\varsigma(\sigma+\mathrm{it})}\right)\right)+\left(-\operatorname{Re}\left(\frac{\varsigma'(\sigma+\mathrm{i2t})}{\varsigma(\sigma+\mathrm{i2t})}\right)\right)\geq 0\rightarrow$$

$$\rightarrow\zeta\left(\frac{\square}{2}\right)\phi=\frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon}e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk=\frac{\phi}{1-\phi}\rightarrow$$

$$\rightarrow L_\mu=C_0\mathcal{L}_0+\frac{m^D}{g^2}\left[-\frac{1}{2}\phi\frac{1}{\zeta\left(\frac{\square}{2m^2}\right)}\phi+\int_0^\infty\mathcal{M}(\phi)d\phi\right],\quad(124)$$

$$g(\sigma)=\pi^{-\frac{\sigma}{2}}\Gamma\left(\frac{\sigma}{2}\right)=g(1-\sigma)=\pi^{-\frac{1-\sigma}{2}}\Gamma\left(\frac{1-\sigma}{2}\right)\rightarrow$$

$$\rightarrow\zeta\left(\frac{\square}{2}\right)\phi=\frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon}e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk=\frac{\phi}{1-\phi}\rightarrow$$

$$\rightarrow L_\mu=C_0\mathcal{L}_0+\frac{m^D}{g^2}\left[-\frac{1}{2}\phi\frac{1}{\zeta\left(\frac{\square}{2m^2}\right)}\phi+\int_0^\infty\mathcal{M}(\phi)d\phi\right],\quad(125)$$

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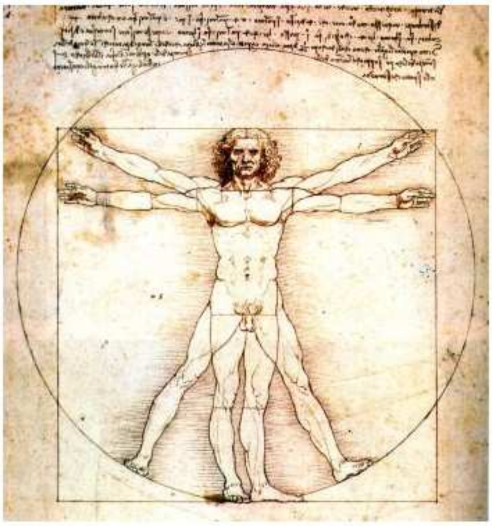

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