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DYNAMIC GAMES AND GROWTH CYCLES IN UNIONISED ECONOMIES

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Abstract

In this paper we integrate the dynamic models formulated by the microeconomic theory of trade unions and the differential games approach. We demonstrate that the results of the dynamic monopoly union model, elaborated by Kidd and Oswald (1987) and Jones and McKenna (1994), can be obtained as solutions of a Stackelberg differential game between firms and unions under particular assumptions on union's membership dynamic. We also develop a Nash differential game whose solutions imply a cycle in wage share of product and employment rate which resembles Goodwin's (1967) cyclical growth path. This result is obtained by use of the Hopf theorem on local bifurcations, under particular hypotheses on the membership dynamics.

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Dynamic Games and Growth Cycles in Unionised Economies

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1. Introduction

The dynamic extensions of the static models produced by the so-called microeconomic theory of trade unions (Kidd and Oswald, 1987; Gottfries and Horn, 1987; Jones, 1987; Jones and McKenna, 1994; Chiarini, 1996a) retain many essential elements of their static counterparts. In particular, trade unions keep on being depicted as an institutional device set up by a group of workers in order to turn income distribution to their favour by use of a bargain mechanism (usually a Nash, or a form of monopolistic solution). However, when examined from other points of view, the dynamic approach to trade unions enriches the static versions. For example, the law of motion describing the environment in which the agents act enables to endogenize union membership, a variable usually treated as a parameter in static models. Furthermore, the equilibrium solutions of the dynamic models are generally characterised by smaller efficiency losses than those obtained in static models.

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This differences notwithstanding, the most unsatisfactory aspect of trade unions models remains untouched, as the conflictual and strategic relationships between unions and firms still lack a systematic analysis. Most of the traditional dynamic models do not incorporate firms' behaviour in an explicit way: firms reaction to union's wage policy are rather considered as an environmental constrain faced by workers' associations[1]. Moreover, it is not really clear whether the firms' reaction function (usually modelled as a standard labour demand curve) can be obtained in the same way as that of the union, i.e., assuming perfect information and rational behaviour in the intertemporal version.

Since trade union models should be conceived as a simplified representation of the distributive conflict taking place in noncompetitive labour markets, then it is realistic to consider that, in a particular labour market, firms will try to coordinate in some way their actions in order to respond to union's wage policy. In this case, firms' behaviour should not be taken so passive as in standard dynamic models: the possibility for the firms to condition union's actions would probably yields different results.

The lack of a thorough analysis of the strategic interplay between agents suggests to reformulate the dynamic monopoly union (DMU) models by employing the analytical apparatus provided by noncooperative game theory. This is our main motivation to move away from the standard DMU versions and to construct a series of dynamic games in continuous time between a union and a group of firms. This exercise will produce two main results.

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[1] See, e.g., Kidd an Oswald (1987) or Jones and McKenna (1994).

First, we will show that it is possible to formulate a Stackelberg differential game (with open loop information structure, and the union envisaged as the leader) which generates the same solutions as those of the standard DMU. This result heavily depends on the form of the differential equation representing the dynamic of union membership: if it is assumed to be linear in the wage, the employment and the level of the same membership, the correspondence exists; if a nonlinearity is instead inserted in the equation of motion, the same result can no longer be obtained.

Second, it will be possible to develop a Nash (open loop) differential game which yields economically significant solutions (time paths for wage, employment and membership). These solutions define an attractor which is not an isolated steady state equilibrium point. We will demonstrate in fact that, for positive values of the discount rate (common to the firms and the union), a limit cycle emerges as a solution of the differential game, provided that the nonlinear part of the membership dynamics implies a strong "cumulativeness" in the unionisation process. Recalling Goodwin's (1967) analysis, growth cycles in the wage share of output and in the employment rate can hence be explained, if the process of membership formation exhibit the right features, in terms of strategies rationally chosen by capitalists (firms) and workers (union) in the intertemporal conflict over income distribution.

A similar topic is tackled by Mehrling (1986), which examine the possible equilibria arising form the conflictual interaction of capitalists and workers treated as homogeneous classes. In that contribution four different (steady state) equilibria are derived, each corresponding to different assumptions about the degree to which each class is organised to promote its own interests. If a class is organised while the other is composed by agents who acts in an atomistic way, then the resulting steady state solution can be thought as a hierarchical equilibrium in which the first class of agents acts as a leader and the second class (the unorganised one) acts as a follower. Our approach differs form Mehrling's in some important aspects; first, we will not consider explicitly capital accumulation in our games (the level of capital will be taken as given); second, the dynamic games developed in the subsequent sections derive directly form the dynamic monopoly union models, therefore we analyse the strategic interaction between capitalists (firms) and workers (unions) by adopting the guidelines offered by the microeconomic theory trade union behaviour. Furthermore, we will consider a labour market (or an entire economy) in which agents are strongly organised to promote their own interests.

The paper is organised as follows. In section 2 and 3 we focus on Stackelberg differential games in order to highlight the hypotheses which produce results equal to or different from those obtained in traditional dynamic models. In section 4 we turn our attention to Nash differential games and in section 5 we discuss a particular differential game whose Nash equilibrium strategies imply cyclical paths for wages, employment and membership. To conclude, in section 6, we compare our results with those generated by Goodwin's (1967) model.

2. The monopoly union model as a differential game

In the DMU models, the trade union chooses the time path for wages which maximises the discounted flow of its utility under the constraint represented by the dynamics (a differential equation) of union membership m whereas, in every period, firms determine employment n as a function of the value of wages ruling in the same period, w (Kidd and Oswald, 1987; Jones and McKenna, 1994; Chiarini, 1996a). Such a passive behaviour raises the question of whether it is indeed rational for the firms (which must optimise a discounted flow of profits in the same time span as that faced by unions) to adopt this kind of strategy.

Dynamic games between trade unions and firms propose themselves as suitable approaches to tackle this issue. In particular they could be formulated with the aim of singling out the hypotheses on information and the sequential structure of agents' actions that could allow dynamic games to produce the same results as those generated by the DMU model. And yet the dynamics of membership and the associated problems are not explicitly dealt with by the existing attempts to transform trade unions models into dynamic games (Espinosa and Rhee, 1988; Mulder and Van Der Ploeg, 1987; Van Der Ploeg, 1987). Keeping at distance from the current literature, we will try to integrate these two approaches by reformulating the DMU model in terms of a differential game and by explicitly considering the dynamics of union membership.

The essential elements of such a game can be summarised into four items. First, the aim of each players is to maximise, in every instant, the sum of his payoffs; the control variables are $w(t)$ for the trade union and $n(t)$

for the firms. Their strategies are therefore explicitly modelled. Second, for the sake of simplicity, we will assume that the informative structure is of an open loop type: for every instant of time which has been reached, $t \in [0,\infty)$, each player knows only the initial state of the system m_0, the instant t , and the time path of his own control variable. Third, the open loop informative structure simplifies the determination of the strategies' space, which in this case coincides with the set of feasible actions, i.e. with the possible functions $w(t)$ and $n(t)$. Fourth, the structure of interaction between the agents in the DMU model suggests to rely on the notion of Stackelberg equilibrium (in global strategies), and to conceive, in every instant, the union as the leader and firms as followers. In fact, in the DMU scheme there is a high level of market power held by the union (which acts as a monopolist): it has the complete control on the wage level; in this setting appears then reasonable to assume that the union has also a hierarchical predominance in the structure of the interaction with the firm, and so can enjoy of the advantage of "moving first". The union will decide the wage level taking account of the reactions of the firm to his wage policy, in terms of employment levels, while the firm will formulate his decisions taking as given the wage policy adopted by the union. Practically, we analyse a modification of the original DMU scheme, in which firms are allowed to respond strategically to the wage policy set by the union[2].

If $w(t)$ is taken as given, the problem of the follower can be treated as a standard problem of optimal control:

[2] The noncooperative differential game approach can offer useful insights also in the case of the dynamic Nash bargaining, in which the union and the firm determine wage and employment time paths in a cooperative way (see Marchetti 1998b).

$$\max_{n(t)} J_f = \int_0^{\infty} e^{-\delta t}\left[pf(n) - wn\right]dt \qquad (1)$$

$$\text{s.t.} \quad \dot{m} = n - m \qquad (2)$$

$$m(0) = m_0$$

where m is the membership level, $f(n)$ - with $f' > 0, f'' < 0$ - is a single-input (labour) production function and δ is a discount rate. The form of equation (2) implies the post entry closed shop hypothesis: in the labour market there exists an automatic mechanism which ensures that all employed workers are union members; whoever loses his/her job leaves the union in the following period. It is interesting to note that, in the case of the post entry closed shop hypothesis, m can only diminish or remain constant; in this case, the strategy space of the firm is $n \in [0, m]$, for the closed shop implies that in each instant the constraint $n \leq m$ must hold. Denoting with λ_f the costate variable of the follower, the Hamiltonian of this problem is:

$$H_f = e^{-\delta t}\left\{\left[pf(n) - wn\right]\right\} + \lambda_f (n - m)$$

from which we obtain the necessary condition for a maximum:

$$e^{-\delta t}\left[pf'(n) - w\right] + \lambda_f = 0 \qquad (3)$$

$$\dot{m} = n - m$$

$$\dot{\lambda}_f = \lambda_f \qquad (4)$$

$$\lim_{t\to\infty} \lambda_f(t) = 0 \tag{5}$$

The unique continuous function $\lambda_f(t)$ which satisfies both (4) and (5) is the solution $\lambda_f(t) = 0 \;\; \forall t \in [0,\infty)$. In this case the firms decisional rule assumes a very simple form; in fact, by substituting this result into (3), we get:

$$[pf'(n) - w] = 0$$

or:

$$pf'(n) = w \qquad \forall t \in [0,\infty)$$

This means that the firms behaves in the same way in the dynamic and in the static situation: in both cases it determines n in such a way as to equalise wage and marginal productivity of labour (firms formulate a normal labour demand curve).

The trade union's problem is of a different nature: acting as a leader, in determining its optimal strategy, the union will have to consider the optimal reactions of the follower - as defined by (3)-(5) - in every instant of time. We assume that the union has an instantaneous utility function of utilitarian type $U(t) = nu(w) + (m-n)u(b)|$, with $u' > 0, u'' < 0$ and where b represents the reservation income. The union's control problem is thus[3]:

[3] In this case we need to carry the firm’s costate variable trough the leader’s optimisation problem; see Bašar Olsder (1995).

$$\max_{w(t)} J_u = \int_0^\infty e^{-\delta t}\{nu(w) + (m-n)u(b)\}dt \tag{6}$$

$$\text{s.t.} \quad e^{-\delta t}[pf'(n) - w] + \lambda_f = 0 \tag{3}$$

$$\dot{m} = n - m$$

$$\dot{\lambda}_f = \lambda_f \tag{4}$$

$$\lim_{t\to\infty} \lambda_f(t) = 0 \tag{5}$$

As (3)-(5) imply that $pf'(n) = w$, this optimisation problem coincides with the monopoly union problem discussed in Kidd and Oswald (1987):

$$\max_{w(t)} J_u = \int_0^\infty e^{-\delta t}\{nu(w) + (m-n)u(b)\}dt$$

$$\text{s.t.} \quad pf'(n) = w$$

$$\dot{m} = n - m$$

The dynamical system, which can now be obtained in the standard manner, exhibits an unstable steady state equilibrium point (a saddle point):

$$\dot{n} = \frac{1}{d\beta(n)/dn}\left[(1+\delta)\beta(n) - \delta u(b)\right]$$

$$\dot{m} = n - m \tag{7}$$

$$\beta(n) = [nu'f'' + u]$$

The equilibrium values at the steady state are $m^*=n^*$, $\beta(n^*) = \frac{\delta}{1+\delta}u(b)$ and $w^* = pf'\left[\beta^{-1}\left(\frac{\delta}{1+\delta}\right)u(b)\right]$. The result we reach is that, if we assume the informative structure is open loop and take (2) as equation of motion, the DMU may indeed be interpreted as a Stackelberg differential game between firms and the trade union.

Since equilibrium solutions in Stackelberg games can be time inconsistent, the question arises of whether the optimal strategy representing the solution of system (7) is in fact time consistent. One method of ascertaining time inconsistency is to set the multiplier of the constraint (4) in the Hamiltonian of the union's optimisation problem equal to zero, to solve the problem of the leader, and to check whether the strategies calculated in this way are incompatible with the necessary optimum conditions (for both players) of the original game. Applying this method to our case, it is trivial to verify that the strategies thus obtained are the same; the equilibrium strategies representing the solutions to system (7) are hence time consistent.

3. Nonlinear equation of motion and the DMU

In the previous section, we have shown that the DMU can be rationalized as a differential game. We now wish to prove that this correspondence is not robust with respect to changes in the hypotheses made by the DMU in its basic version. In particular, if equation (2) is modified, for instance by adding other variables - as, for example, in Jones and McKenna (1994) - the results of the differential game no longer coincide with those of the modified DMU[4].

To show this we formulate a Stackelberg differential game which retain the open loop information structure and the follower role for the firms, but which incorporates a more general equation of motion: $\dot{m} = \theta n + \phi(m) + kw$. The parameter $\theta > 0$ represents the positive effects on the membership's rate of growth due to a greater employed labour force. Parameter $k>0$ implies that workers, in their decisions on whether to join or not the union, take count of the latter's achievements in wage claims: if union gains high wages, more workers will decide to join the union. This behaviour can be seen as a phenomenon opposite to the free riding: the more the workers' organisation succeed in pursuing the members' (wage) objectives, the more this same organisation will collect consent among potential participants. Function $\phi(m)$ describes how the absolute level of membership affects the rate of growth of the same membership. These assumptions allows to drop the closed shop hypothesis[5] so that the players' strategy spaces will be different form those assumed in the closed shop model; the union will choose the wage so that $w \in [b, W]$, where W is the

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[4] More precisely, the equation of motion must contain some nonlinearities. In fact, if the differential equation representing the open shop membership dynamics is linear, then the Stackelberg differential game yields exactly the same results as the corresponding DMU. See footnote 6, below.

[5] This equation of motion is an approximation of that derived by Jones and McKenna (1994) from an explicit consideration of the workers' optimal choice with respect to the joining of the union. They assume that $\theta > 0$ and $k > 0$, while ϕ' can be either positive or negative. In this scheme, in order to definitely drop the closed shop hypothesis, we must

maximum achievable level of wage[6], while the admissible strategy space for the firms will be the interval $[0, N]$, where N is the full employment level. It is worth underlining that $N > m$, for we assume an open shop labour market; the unemployment rate can then be defined as $(N\text{-}n)/N$. The firms optimisation problem is:

$$\max_{n(t)} J_f = \int_0^\infty e^{-\delta t}\left[pf(n) - wn\right]dt \tag{8}$$

$$\text{s.t.}\, \dot{m} = \theta n + \phi(m) + kw \tag{9}$$

$$m(0) = m_0$$

and the corresponding Hamiltonian is:

$$H_f = e^{-\delta t}\left\{\left[pf(n) - wn\right]\right\} + \lambda_f(\theta n + \phi(m) + kw)$$

from which the following first order conditions are obtained:

$$e^{-\delta t}\left[pf'(n) - w\right] + \lambda_f \theta = 0 \tag{10}$$

$$\dot{\lambda}_f = -\phi' \lambda_f, \qquad \lim_{t\to\infty} \lambda_f(t) = 0$$

The leader's problem can be written as:

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assume that the wage determined by the union is valid for all the workers of the sector, not only for union members.

$$\max_{w(t)} J_u = \int_0^\infty e^{-\delta t}\{nu(w)+(m-n)u(b)\}dt \tag{11}$$

$$\text{s.t.} \quad \dot{m} = \theta n + \phi(m) + kw$$

$$e^{-\delta t}[pf'(n)-w] + \lambda_f \theta = 0$$

$$\dot{\lambda}_f = -\phi' \lambda_f$$

$$\lim_{t\to\infty} \lambda_f(t) = 0$$

The Hamiltonian in this case is:

$$H_u = e^{-\delta t}\{nu(w)+(m-n)u(b)\} + \lambda_u(\theta n + \phi(m) + kw) + \gamma(-\lambda_f \phi') +$$
$$+\nu\left[e^{-\delta t}(pf'(n)-w) + \lambda_f \theta\right]$$

where γ is a parameter which depends on time and ν is a parameter independent of time. The first order conditions are:

$$e^{-\delta t}[pf'(n)-w] + \lambda_f \theta = 0$$

$$\dot{\lambda}_f = -\phi' \lambda_f$$

$$\dot{\gamma} = \gamma\phi' - \nu\theta$$

$$e^{-\delta t}[nu'(w)-\nu] + k\lambda_u$$

$$\dot{\lambda}_u = -e^{-\delta t}u(b) - \lambda_u \phi' - \gamma\lambda_f \phi''$$

$$\gamma(0) = 0 \qquad m(0) = m_0 \qquad \lim_{t\to\infty}\lambda_f(t) = 0 \qquad \lim_{t\to\infty}\lambda_u(t) = 0$$

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[6] Presumably $W = pf(n)/n$, i.e. at the maximum level of wage the profit is zero.

Since this is a so-called 'two point boundary value problem'[7], the solution can only be found by use of numerical methods; in general, it is however likely that the solution turns out to be different from that of the corresponding DMU model[8]. This fundamental difference between games (8)-(11) and (1)-(6) is essentially due to the form of the membership dynamics (besides the open loop information structure). If these dynamics are expressed by a linear equation (in *n*,*m*,*w*) such as (2), the value of *m* for the firms (represented by λ_f) is zero for every *t*; firms' optimal behaviour would be therefore to adapt itself to the union's wage policy, i.e., to react to *w* according to $pf'(n) = w \;\; \forall t$. Some kind of nonlinearity in (2) is instead sufficient to make $\lambda_f \neq 0$. If λ_f is zero for every *t*, this means that the shadow price of membership is zero for the firms: the latter attribute no importance to the possibility of affecting the union's wage strategy by means of the control of *m*'s time path, exerted via *n*. The linear form of equation (2) implies that the effect of *m* on the growth rate of the same membership is too weak to make it convenient for the firms to condition their employment policy to the control of *m* in order to force the union to lower wages. The latter strategy becomes convenient for the firms only

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[7] See for instance Intriligator (1971).

[8]As mentioned in footnote 3, if the membership dynamics are taken to be linear - as in the case of $\dot{m} = \theta n - \phi m + kw$ - the Stackelberg differential game yields exactly the same results as the corresponding DMU. In this case, the follower's optimal control problem leads to the differential equation in the costate variable: $\dot{\lambda}_f = \phi \lambda_f$, this equation, together with the terminal condition on λ_f, implies $\lambda_f = 0$. The leader's problem, which takes these conditions as constraints, generates the same differential equations as those of a DMU version having $\dot{m} = \theta n - \phi m + kw$ as its equation of motion.

when the effect of m on $\dot{m}$| is strong enough, e.g. when there is some kind of nonlinearity, such as $\phi(\cdot)$, in the membership dynamics

4. Nash games

The switching from equation (2) to equation (9) yields further consequences, as it also enables the formulation of a game characterised by a different decision structure, i.e., a Nash differential game. Relying again on open loop information, the problem can be set up by assuming that, at the initial instant t=0, players choose their strategies for the whole time span, and that they reach their decisions taking their opponent's strategy as given. Although less rich than the Stackelberg one (because the union behaves more myopically), the Nash structure however is much more analytically tractable.

We will adopt a particular specification of the equation (9):

$$\dot{m} = \theta \ln n + \phi(m) + kw \tag{12}$$

which differs form (9) only in the term $\theta \ln n$[9]. We also assume, for simplicity, that the production function is linear in n: $f(n)=An$, $A>0$. In this case firms' problem will be expressed by:

[9] This specification is adopted only for analytical tractability and it doesn't imply conceptual differences with the (9); in fact the sign of the impact of n on membership's variation $\dot{m}$ remains positive.

$$\max_{n(t)} J_u = \int_0^\infty e^{-\delta t}\left(pAn - wn\right)dt$$

$$\text{s.t.} \quad \dot{m} = \theta \ln n + \phi(m) + kw\,, \qquad m(0) = m_0$$

From the Hamiltonian $H_f = e^{-\delta t}(pAn - wn) + \lambda_f(\theta \ln n + \phi(m) + kw)$ we can calculate the first order conditions of the problem:

$$\frac{\partial H_f}{\partial n} = e^{-\delta t}(pA - w) + \lambda_f \frac{\theta}{n}$$

$$-\frac{\partial H_f}{\partial m} = \dot{\lambda}_f = -\phi' \lambda_f\,, \qquad \lim_{t \to \infty} \lambda_f = 0$$

After some manipulation we obtain an equation of the wage change rate:

$$\dot{w} = \frac{1}{n}\left[(pA - w)(\phi' n - \delta n + \dot{n}\right]$$

The problem for the union is:

$$\max_{w(t)} J_u = \int_0^\infty e^{-\delta t}\left\{nu(w) + (m-n)u(b)\right\}dt$$

$$\text{s.t.} \quad \dot{m} = \theta \ln n + \phi(m) + kw \qquad m(0) = m_0$$

its Hamiltonian - $H_u = e^{-rt}\left\{nu(w) + (m-n)u(b)\right\} + \lambda_u(\theta \ln n + \phi(m) + kw)$ - implies:

$$e^{-\delta t}\left[nu'(w)\right]+k\lambda_u=0$$
$$\dot{\lambda}_u=-e^{-\delta t}u(b)-\lambda_u\phi'$$

Differentiating the first of these equations with respect to t and substituting the result in the second equation, we obtain:

$$\dot{n}u'+nu''\dot{w}=-(\phi'-\delta)nu'+ku(b)$$

The dynamical system resulting from the two optimal control problems can hence be expressed as:

$$\begin{aligned}\dot{n}&=\frac{ku(b)}{\eta(w)}-(\phi'-\delta)n\\ \dot{w}&=\frac{1}{n\eta(w)}(pf'-w)ku(b)\\ \dot{m}&=\theta\ln n+\phi(m)+kw\end{aligned}\tag{13}$$

where $\eta(w)=u'+(pA-w)u''$. The steady state values of w, n and m are:

$$w^*=pA;\qquad n^*=\frac{ku(b)}{(\phi'-\delta)u'};\quad \phi(m^*)=-(kw^*+\theta\ln n^*)\tag{14}$$

The second of equations (14) implies that the steady state values of this game are generally different from those of the corresponding DMU version (see footnote 3). Furthermore, (14) implies that during the adjustment process the values of wage and employment can be different from those implied by the labour demand curve. *n* and *w* are certainly set according to the demand of labour when firms and the union reach the steady state.

5. Optimal strategies and wage-employment cycles

In order to study the dynamics of system (13) we have to calculate its linearization in a neighbourhood of the point $E=(n^*,w^*,m^*)$, which is given by:

$$\begin{bmatrix} \dot{n} \\ \dot{w} \\ \dot{m} \end{bmatrix} = \begin{bmatrix} -(\phi' - \delta) & 0 & -n^*\phi'' \\ 0 & -\dfrac{ku(b)}{u'n^*} & 0 \\ \dfrac{\theta}{n^*} & k & \phi' \end{bmatrix} \begin{bmatrix} (n-n^*) \\ (w-w^*) \\ (m-m^*) \end{bmatrix} \qquad (15)$$

from the characteristic equation we get:

$$\left(r + \frac{ku(b)}{u'n^*} \right) \left[(r + (\phi' - \delta))(r - \phi') + \theta\phi'' \right] = 0$$

so as to obtain the eigenvalues r_i, i=1,2,3 as:

$$r_1 = -\frac{ku(b)}{u'n^*} \qquad \text{(<0 in the equilibrium point)}$$

$$r_{2/3} = \frac{\delta \pm \sqrt{\delta^2 - 4[\theta\phi'' - \phi'(\phi' - \delta)]}}{2}$$

Although a casual analysis of such eigenvalues does not allow to draw definite conclusions on the local stability of system (13), or on the qualitative characteristics of the equilibrium point (the signs of $r_{2/3}$ depend on the various coefficients in a complicated way), we can try to establish whether system (13) possesses attractors qualitatively different from a fixed point by applying some results of the local theory of bifurcations (cfr. Lorenz, 1993; Medio, 1993; Arnold, 1988; Guckenheimer and Holmes, 1983). In particular, we will make use of the Hopf theorem to determine whether (13) includes closed orbits (cycles) among its possible solutions, and whether one of these is actually an attractor (a limit cycle).

The Hopf theorem can be easily summarised (Hassard, Kazarinoff e Wan, 1981):

Part I (existence). If: (i) the dynamical system $\dot{x} = F(x, \xi)$, $F \in C^L$, $x \in \Re^n$, where $\xi \in \Re$ is a parameter, has an isolated equilibrium point at $x*$, (ii) for the critical value $\bar{\xi}$, the Jacobian matrix $\partial F(x*, \bar{\xi}) / \partial x$ possesses two pure imaginary eigenvalues $r(\bar{\xi}) = \pm i\omega$, while all other eigenvalues are negative, and (iii) $\left. \frac{d[\mathrm{Re}\, r(\xi)]}{d\xi} \right|_{\xi = \bar{\xi}} > 0$ then for ξ belonging to a neighbourhood of $\bar{\xi}$, the system $\dot{x} = F(x, \xi)$ possesses a periodic solution.

Part II (bifurcation direction and stability): if

$$\xi_2 = -\operatorname{Re} c_1(\bar{\xi})\left[d \operatorname{Re} r(\bar{\xi}) / d\xi\right]^{-1} > 0$$

$$\text{where } c_1(\bar{\xi}) = i(2\operatorname{Im} r(\bar{\xi}))^{-1}\left[g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right] + \frac{g_{21}}{2}$$

then the periodic solution emerges for $\xi > \bar{\xi}$; furthermore if $\beta_2 = 2\operatorname{Re} c_1(\bar{\xi}) < 0$ the periodic solution is stable and is a limit cycle[10].

In the case of system (13) we take δ as a parameter and let $\bar{\delta} = 0$ be the critical value; thus we have:

$$r_{2/3} = \sqrt{-(\theta\phi' - (\phi')^2)}$$

Furthermore we have:

$$\left.\frac{d[\operatorname{Re} r(0)]}{d\delta}\right|_{\delta=0} = \frac{1}{2}$$

Assuming that $\phi'' > 0$ and $\phi' > 0$, the conditions of Part I of Hopf theorem will be satisfied if $\theta > \frac{\phi'^2}{\phi''}$; in this case a periodic solution of (13) exists for

[10] A synthesis of the proof of the theorem and the procedure used to calculate the coefficients *g* can be found in the Appendix.

a δ in a neighbourhood of zero. Condition $\theta > \frac{\phi'^2}{\phi''}$ implies that the second order derivative of $\phi(m)$ must be not only positive but greater than the first derivative of the same function.

As for direction and stability, we must calculate the coefficient $c_1(0)$. By putting $\sqrt{(\theta\phi'' - (\phi')^2)} = \omega$ we obtain (see the Appendix):

$$
\begin{aligned}
g_{11} &= \frac{1}{4}\left[i\left(\frac{\theta\phi''' - 2\phi'\phi''}{\omega} \right)\right] \\
g_{02} &= \frac{1}{4}\left[2\frac{\omega(\omega+1)}{\theta} + i\left(\frac{\theta\phi''' - 2\phi'\phi''}{\omega} - 4\frac{\omega\phi'}{\theta} \right)\right] \\
g_{20} &= \frac{1}{4}\left[2\frac{\omega(\omega+1)}{\theta} + i\left(\frac{\theta\phi''' - 2\phi'\phi''}{\omega} \right)\right] \qquad (16) \\
g_{21} &= \frac{1}{8}\left[\phi'\frac{\left[\phi'^2(\omega-1)+\omega^2\right]}{\theta^2} + i\left(\frac{\phi'^4 + \theta^3\phi^{IV} - \phi'\phi'''(3\theta+1)^2}{\theta^2\omega} + \phi'\frac{\left[\omega(\theta\omega+\phi')+\phi^2\right]}{\theta} \right)\right]
\end{aligned}
$$

with the derivatives of ϕ and the value of ω calculated at point *E*. According to Part II of the Hopf theorem, we have:

$$
\operatorname{Re} c_1(0) = \frac{1}{4}\left(-B(\omega-1)\frac{(\theta+8\omega)}{\theta\omega^2} + \frac{2\phi'\left[(\omega-1)(2\theta+\phi')+\omega^2\right]}{3\theta^2} \right)
$$

$$
B = \theta\phi''' - \phi'\phi''
$$

From the last expression we can see that if:

$$\frac{1}{8}\left[\frac{-3\theta(\omega-1)(\theta+8\omega)B+4\omega^2(\omega-1)\theta\phi'+2\omega^2(\omega-1)\phi'^2+2\omega^{4\phi'}}{3\theta^2\omega^2}\right]<0$$

then the periodic solution is economically significant, as it emerges when $\delta > 0$. In this case, it is also stable, so as to propose itself as a limit cycle attracting all the other trajectories. The previous expression is negative if its numerator is negative, as its denominator $3\theta^2\omega^2$ is positive. The numerator sign depends in a complicated way on the relative size of the parameters; we can however examine an interesting case. Let's consider these two hypotheses: ϕ''' is very small (zero on the limit) and $\omega-1<0$, i.e. $\theta\phi''-\phi'^2<1$. In this case the denominator will be likely negative, as all its terms would be negative, apart form the positive addendum $2\omega^4\phi'$.

The emerging of limit cycle as a consequence of the players' optimal strategies heavily depends on the sign of the derivatives of ϕ :ϕ' and ϕ'' must be positive and $\phi''' \to 0$. Combining the condition for existence with those for stability, a relation between ϕ 's derivatives and parameter θ can be obtained: we have a limit cycle when:

$$\frac{\phi'^2}{\phi''}<\theta<\frac{1+\phi'^2}{\phi''} \tag{17}$$

A possible interpretation of this result implies two distinct observations. First, being $\phi' > 0$ and $\phi'' > 0$, there must be a kind of strong "cumulativeness" of dynamic law (12); function ϕ represents in fact the impact that the level of membership has on its own growth rate, and the first three positive derivatives of ϕ imply that this impact increases with increasing rates (and with constant rate variations, as $\phi''' \to 0$). Second, the constraint imposed by (17) implies that the impact of the membership level (represented by the derivatives of ϕ) on $\dot{m}$ is somehow balanced by the impact of the employment level (represented by θ) on the same rate: these two effects must be partially compensating.

A careful inspection of system (13) offers an economic interpretation of the cyclical process implied by these equations. Starting from a wage level which is lower than marginal productivity and from quite a high membership level, the second of equations (13) makes clear that it will be convenient for the union to raise wages ($\dot{w}$ is positive) in order to gain a higher share of product[11]. The effect on *m* produced by the raise in *w* is described by (12). A growing wage level implies (via the coefficient *k*) an increase in union participation: workers will tend to join the union due to its success in wage claims. The increase in *m* causes a growth of the term $\phi' - \delta$ (since ϕ'>0) which, in its turn, induces firms to reduce employment: the first of equations (13) shows that there exists a certain point at which $\dot{n}$ will be negative for a growing *m*. The decrease of *n* will eventually

[11] The fact that the wage can be different form marginal productivity can induce some confusion, considering that we are dealing with profit-maximizing firms. Anyway, in a dynamic game there can be some reason for firms to fix *n* not to equate marginal productivity to wage. Infact, is the strategic interaction in a dynamic context that pushes

influence equation (12) (via the coefficient θ) inducing negative growth rates of *m*. At this stage, due to the strong cumulativeness of (12) in *m*, the membership will keep diminishing at an accelerated rate, until it reaches a level such that $\phi' - \delta$ is negative. When this happens, the best thing the union can do, in the face of excessively reduced employment and membership, is to restrain its wage claims[12]. When the decrease in *w*, together with $\phi' - \delta$ <0, induces an increase of *n*, this growth in employment persists until $\dot{m}$ in (12) becomes positive; this will cause an increase in $\phi' - \delta$ until it returns to a positive value, thus inducing the union to increase *w* once again.

Equation (12) represents a constraint on the union's possibility of imposing wage claims, as it shows that the value of wages and employment have an influence on union's size, which affects directly union's utility *U*(*t*). The process described above can be exploited by firms who, by manoeuvring employment, can reduce membership until the union is forced to reduce wages; a device able to discipline industrial relations via membership is thus activated. It is worth underlining that the existence of this mechanism is conditioned by the institutional (and/or historical) characteristics of the unionisation phenomenon experienced in a given economy, as expressed by function ϕ and by parameter θ.

In particular, it is important to investigate on the nature of function $\phi(m)$. It represent the impact exerted by the absolute dimension of the union on the time variation of the membership. It then represent the

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firms to act in such a way, in order to inlfuence thet union's wage policy.

[12] In fact, in a neighbourhood of the steady state we have: $(\phi' - \delta) = ku(b)/n\eta(w)$.

institutional characteristics of the autonomous consolidation process of the union’s internal organisation, indicating how the union as an organisation can expand (or maintain) is dimension by means of internal resources. This process, owing to the institutional complexities which characterise the aggregation and the consolidation of the workers’ consent, can manifest strong nonlineraities. For instance, if the internal resources that the union can activate in order to enforce its unity and to gather consent grow strongly with the union’s dimension, then the first and second derivatives of function $\phi(m)$ can be both positive.

Parameter δ also plays a crucial role. Not only does the existence of the limit cycle depend on this parameter, but it also conditions the union's incentive for wage growth: the greater is δ, the sooner will the union be driven to moderate its wage claims and the more convenient it will be for the firms to support the level of employment. When a high value is attached to the future, more importance is attributed by the union to the employment aim than to the wage objective.

Finally, the emergence of the limit cycles depend also on the exitence of a kind of wage effect, represented by parameter *k*. In fact, if *k*=0, the necessary optimality condition for the union’s problem becomes $e^{-\delta t}\left[nu'(w)\right]+k\lambda_{u}=0$, which implies that the union choice is a constant wage level: $w=W$, $\forall t$. If the membership dynamics (12) doesn’t depend on *w*, the union is not sufficiently stimulated to restrain from a high wage policy.

It is important to note that the emerging of a limit cycle as a solution of system (13) heavily depends on the hypotheses made on the informative structure. The form of equations (13) depends on the fact that in the information sets of the players there are only the initial value of the state variable (m_0) and the calendar date. This implies that the players are not able to observe the past actions, once those are effectively implemented. We can also say that there is a kind of precommitment: each player commit himself to follow the strategy which he chooses at the beginning of the game, knowing the initial information on the actions available to his opponent.

Under less restrictive assumptions on the information available to the players, the equilibrium strategies which solves the game of section 4 could be quite different. For instance, if we adopt a *closed loop* informative structure, in which each player knows the past actions undertaken by his opponent, the final result can be much more complex. In this setting each player can learn by observing the past, and then evaluate on these basis the behaviour of the opponent. This could constitute an incentive for the players to adopt policies based, for instance, on cooperation. In this case, firms and unions could achieve final trajectories which pareto-dominate the cycles of system (13)[13]. In particular, under appropriate assumptions on the parameters values, the players could agree to play a strategy which maximise the sum of the two payoff functionals, so to solve the prisoner's dilemma which characterise the union-firm dynamic game.

Unfortunately, the adoption of more complex informative structure bears a high degree of difficulty in the solution procedure of the differential games. With information structures different from the open loop case, differential games gets in general analytically untractable, with the exception of some particular cases. For example, if we adopt a *memoryless perfect state* information structure for the game of section 4, then the players know, in addition to the initial value of *m* and to the calendar date, also the value of the state variable at the present time $m(t)$. This implies that the strategy spaces and the action spaces of the players no more coincide. The strategy of each player is now not only a function of time, but also a function of the state variable, so that system (13) becomes a system of partial differential equations, which can generally be solved only by means of numerical methods.

6. Concluding Remarks

The explicit introduction of the strategic interaction between agents not only notably modifies the outcome of the dynamic models - in the sense that the dynamics of system (7) are qualitatively different from the dynamics of system (13) - but also enables us to establish connections with models and theories belonging to other areas of research.

A model whose outcome is similar to that of system (13) is the well known Goodwin's (1967, 1991) model, where the dynamics is of the predator-prey type: the Lotka-Volterra differential equations, used to represent the distributive conflict between capitalists and workers, generate a cyclical solution in the wage share of income and in the employment rate. In that case, the wage dynamics, which are taken as given, represent a

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[13] For an attempt to investigate the pareto-efficiency of games such those of section 4 see Marchetti (1998).

disciplinary device operating within the labour market, one of the two equations of the model being in fact a Phillips curve.

System (13) also generates a stable cycle in the wage share of products and in the employment rate[14], even if the closed orbit is not necessarily identical to that produced by the Goodwin's model. In fact, Goodwin's model implies a different description of the economic structure: it is assumed that aggregate production is characterised by fixed coefficients in employment and capital, and that there is an exogenous rate of technical progress. Capitalists reinvest all their profits in the capital stock so to enlarge productive capacity and effective demand, while workers spend all their income in consumption. The cyclical behaviour of the system arises because a high level of profits and investment increases the capital stock and employment (because of the Leontiev technology) The increase in employment induces an increase in wages, via a kind of Phillips curve mechanism. The subsequent growth in wage share produces a reduction in profits which worsen capitalists' investment perspectives. This reduces the growth of production and increases the unemployment rate, which in turn restore, via a decrease in wages, favourable profitability conditions for investments.

System (13) implies a different interpretation of the underlying economic structure. The basic scheme of interaction between firms and unions is still grounded in a supply-oriented vision of the economic process, but there is no explicit consideration of the accumulation process, the capital stock being considered as given. Unemployment emerges only as a

[14] Given the hypothesis on the production function, the wage share is equal to w/pA.

consequence of the market power of the agents and of their ability to control the price variables (wage). System (13) then describes an economy in which there isn’t a real economic growth. so that its limit cycles should be properly considered as short term economic oscillation, rather than “growth cycles” in the Goodwin’s sense. In our case, however, the time path of wage and employment represents the equilibrium of a dynamic game. The periodic orbit of system (13) can be seen as an example of the theoretical possibility that a problem of optimal control possesses a cyclical solution, rather than the more common saddle point solution. Benhabib and Nishimura (1979) showed this possibility for a multisectoral model of optimal growth; in our case, however, the periodic orbit is not simply the result of the optimal program of a single agent which acts in a parametrical environment, but is the consequence of maximising strategies chosen by players which take count of the strategic interdependence of their actions. Stable periodic solutions can emerge as consequence of rational behaviour also in this kind of economic environment

In the context of our Nash game, it is conceivable that the union, which represents a significant percentage of workers, should face an intertemporal problem of strategic nature: if the union increases wages for a certain period, the sum of the workers' utilities will also increase; in the subsequent period, the union will however experience a drop in employment and in membership which has a negative impact on the global level of its utility. Firms face a similar problem. At the beginning, they can expand production and profits by setting a high level of employment, but this encourages the trade union to incerease the wage. It is then convenient for the firms to reduce employment so as to decrease union membership and put a check on wage growth.

This scheme may be interpreted as an argument, different from those already present in literature (Lancaster, 1973; Hoel, 1978; Pohjola, 1984a and 1984b) in support of the dynamic inefficiency of a capitalist economy emerging from the distributive conflict between different groups of agents. When the Nash differential game presented in section 4 is interpreted along this lines, the resulting cyclical path, which represents the process through which the distributive conflict develops, may indeed be conceived as the consequence of rationally chosen strategies, and hence of the optimal behaviour that should be espoused by individual agents.

APPENDIX

We start by assuming that the conditions of Part I of the Hopf theorem are satisfied for the system in the generic form $\dot{x} = F(x,\xi)$, and that this system can be written as:

$$\dot{X} = A(\xi)X + f(X,\xi), \quad X = x - x^*, \quad A(\xi) = \partial F(x^*,\xi)/\partial x \qquad \text{(A.1)}$$

where $f(\,)$ is a polynomial in x and where the terms have order greater than or equal to 2. The aim of the proceeding is to facilitate the study of system (A.1) by lowering its dimension.

This can be done by applying the centre manifold theory according to which a centre manifold is a subset (of smaller dimension) of the phase space of (A.1) spanned by the eigenvalues of A with zero real part. Since the other eigenvalues of A are negative, instead of the whole system (A.1), we can study only the dynamics taking place in the centre manifold. To do so, the "reduced" system must be expressed in a form known as the Poincarè normal form. This requires to approximate the function defining the centre manifold.

Define the variables:

$$z(t) = \langle q^*, x(t) \rangle$$

and $\quad v(t) = x(t) - z(t)q - \underline{z}(t)\underline{q} = x(t) - 2\,\mathrm{Re}(zq)$

where q and q^* are the eigenvectors of A corresponding to the two complex eigenvalues $r(\xi)$; $x(t)$ is a generic solution of (A.1); $\langle .,. \rangle$ denotes the Hermitian product and the underlined variables are the conjugate of the original (complex) variables. Using the variables z and v, (A.1) can be expressed as:

$$\begin{aligned} \dot{z} &= r(\xi)z + G(z,\underline{z},v,\xi) \\ \dot{v} &= A(\xi)v + H(z,\underline{z},v,\xi) \end{aligned} \tag{A.2}$$

where:

$$G(\ ) = \langle q^*, f(v + 2\,\mathrm{Re}(zq), \xi) \rangle,\ H(\ ) = f(v + 2\,\mathrm{Re}(zq), \xi) - 2\,\mathrm{Re}(qG)$$

System (A.2) has n+2 variables, but two of them are a linear combination of the others.

A function $v = v(z, \underline{z}, \xi)$ describing the centre manifold can be derived from the second equation of (A.2). Substituting this function into the first equation of (A.2), we get:

$$\dot{z} = r(\xi)z + g(z, \underline{z}, \xi) \qquad \text{(A.3)}$$

where $g(z, \underline{z}, \xi) = G(z, \underline{z}, v(z, \underline{z}), \xi)$. We now have to calculate the coefficients of g and v (approximating v with a second order Taylor series expansion). It can be shown (Hassard et al., 1981, pp. 63-67) that coefficients g_{ij} of:

$$g(z, \underline{z}, \xi) = \sum_{i+j=2} \frac{g_{ij}}{i!\,j!} z^i \underline{z}^j + o(L+2)$$

are given by:

$$g_{ij} = \frac{\partial^2 G(0,0,0,\xi)}{\partial z^i \partial \underline{z}^j} \qquad i+j=2$$

while coefficients v_{ij} of the approximation of v are given by the systems:

$$v_{ij} = \left[(ri + \underline{r}j)I - A\right]^{-1} \frac{\partial^2 H(0,0,0,\xi)}{\partial z^i \partial \underline{z}^j} \quad i+j=2 \qquad \text{(A.3 b)}$$

Equation (A.3) is in Poincarè normal form.

Now consider the following equation, equivalent to (A.3):

$$\dot{z} = r(\xi)z + z\sum_1^M c_j(\xi)(z,\underline{z})^j \qquad \text{(A.4)}$$

where z is a complex variable, and $c_j(\xi)$ are coefficients. Calculating $d(z\underline{z})/dt$, from (A.4) we obtain:

$$d(z\underline{z})/dt = 2z\underline{z}\left[\operatorname{Re} r(\xi) + \sum_1^M \operatorname{Re} c_j(\xi)(z\underline{z})^j\right] \qquad \text{(A.4 b)}$$

which is equal to zero if z=0 or $\operatorname{Re} r(\xi) + \sum_1^M \operatorname{Re} c_j(\xi)(z\underline{z})^j = 0$. In the last case, (A.4 b) implies that $z\underline{z} = \varepsilon^2 \geq 0$. If we consider $\xi = \xi(\varepsilon)$, that is, we express it as a function of ε which we approximate with the polynomial $\xi = \sum_1^M \xi_j \varepsilon^j + o(\varepsilon^{M+1})$, and we substitute it in the last condition, this becomes:

$$\operatorname{Re} r(\xi(\varepsilon)) + \sum_1^M \operatorname{Re} c_j(\xi(\varepsilon))\varepsilon^{2j} = 0 \qquad \text{(A.5)}$$

Now expand the right hand side of (A.5) in powers of ε. Equating the result to zero and considering first $o(\varepsilon)$ and then $o(\varepsilon^2)$, we have:

$$\frac{d\operatorname{Re} r(\bar{\xi})}{d\xi}\xi_1 = 0 \Rightarrow \xi_1 = 0$$

and:

$$\frac{d\,\mathrm{Re}\,r(\bar{\xi})}{d\xi}\xi_2 + \mathrm{Re}\,c_1(\bar{\xi}) = 0 \Rightarrow \xi_2 = -\frac{\mathrm{Re}\,c_1(\bar{\xi})}{d\,\mathrm{Re}\,r(\bar{\xi})/d\xi}.$$

It can then be shown (Hassard et. al., 1981, pp. 45-47) that $c_1(\bar{\xi})$ is equal to:

$$i(2\,\mathrm{Im}\,r(\bar{\xi}))^{-1}\left[g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right] + \frac{g_{21}}{2}$$

The Poincarè-Bendixon theorem (Hassard et. al., 1981, pag. 36-38) applied to (A.4), allows us to state that, given $d\,\mathrm{Re}\,r(\bar{\xi})/d\xi > 0$, the stability of the limit cycle of (A.4) depends upon $\mathrm{Re}\,c_1(\bar{\xi})$: if it is greater than zero, the cycle is stable; if it is less than zero, the cycle is unstable.

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