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Applications of the Jordan's Lemma to physical problems

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The purpose of this note is to demonstrate how the Jordan's Lemma can be applied in order to find analytical solutions to integrals for which only numerical solutions are usually considered.

The Jordan's lemma can be applied in evaluating integrals along the real axis from $-\infty$ to $+\infty$. In order to do this, it is necessary to modify the integration path by including integrations along an imaginary axis.

In physics often we meet integrals of the type

$$I(t, t_0, m) = \exp(i\omega_0 t_0) \int_{-\infty}^{\infty} g^m(\omega)\, \rho(\omega)\, \exp[-i\omega(t + t_0)]\, d\omega \tag{1}$$

where A, a, and β are parameters which do not depend on ω; $g(\omega)$ and $\rho(\omega)$ are complex functions of the type

$$g(\omega) = \frac{i}{\omega - \omega_0}$$

$$\rho(\omega) = i\, \frac{A}{e^{a\omega + i\beta} - e^{-a\omega - i\beta}}\,. \tag{2}$$

Let us see how the Jordan's lemma can be utilized in order to solve the integral (1).

The Jorsan's lemma can be written as: [1]

- If $f(z) \to 0$ uniformly with regard to arg$z$ as $|z| \to \infty$ when $0 \leq \arg z \leq \pi$, and if $f(z)$ is analytic when both $|z| > c$ (constant) and $0 \leq \arg z \leq \pi$, then

$$\lim_{\rho \to \infty} \int_{\Gamma} \exp(miz) f(z) dz = 0$$

where Γ is a semicircle of radius ρ above the real axis with center at the origin. If the function $f(z)$ has poles within the closed contour, the values of the integral is different from zero and is equal to the sum of the residues.

In Eq. (1) we have to distinguish two cases:

1 - the case in which $t + t_0 < 0$

and

2 - the case in which $t + t_0 > 0$.

<u>First case</u>: for $t + t_0 < 0$, the integration path $-\infty$, $+\infty$ may be closed with a line at infinity in the Im$\omega > 0$ half-plane. Hence, $I(t, t_0, m)$ may be expressed with the sum of residues R of the integrand function at the poles in the upper half-plane, plus (one half of) a possible residue due to the function $g(\omega - \omega_0)$ on the real axis at $\omega = \omega_0$.

<u>Second case</u>: for $t + t_0 > 0$, $I(t, t_0, m)$ may be expressed with the sum of residues (with the sign changed) of the integrand function at the poles in the lower half-plane, minus one half of a possible residue of $g(\omega)$ on the real axis at $\omega = \omega_0$.

The value of the parameter m determines the existence of a residue at $\omega = \omega_0$. For $m = 0$, there is no pole and no residue at $\omega = \omega_0$; therefore,

$$R(\omega_0, 0) = 0\,. \tag{3}$$

For $m = 1$, there is a first-order pole and the corresponding residue $R(\omega_0, 1)$ is given by

$$R(\omega_0, 1) = -2\pi\,\rho(\omega_0)\exp(-i\omega_0 t)\,. \tag{4}$$

For $m > 1$, $I(t, t_0, m)$ diverges too rapidly at $\omega = \omega_0$. However, the difference between two such integrals, corresponding to different values of the parameter t_0, may compensate for the m-th order divergence and present a first order pole.

The poles in the complex ω-plane are located at

$$\omega_j = i\Omega_j\,, \tag{5}$$

where

$$\Omega_j = \frac{1}{a}(j\pi - 2\phi) \tag{6}$$

hence, they are in the upper half-plane for $j > 0$ and in the lower half-plane for $j \leq 0$.

The values of the residues, at these poles, are given by

$$R(\Omega_j) = F(m, j, t_0)\,(-1)^j \exp[\Omega_j(t + t_0)]\,, \quad m = 0, 1 \tag{7}$$

where

$$\begin{aligned} F(m, j, t_0) &= A' M_0^m \exp[i(m\psi + \omega_0 t_0)] \\ A' &= -\frac{\pi A}{a}\,, \quad M_0 = \frac{1}{\sqrt{\Omega_j^2 + \omega_0^2}} \\ \cos\psi &= \frac{\Omega_j}{M_0}\,, \quad \sin\psi = -\frac{\omega_0}{M_0}\,. \end{aligned} \tag{8}$$

We conclude that for $m = 0, 1$ and $t + t_0 < 0$,

$$I(t, t_0, m) = I^-(t, t_0, m) = \frac{1}{2}R(\omega_0, m) + \sum_{j>0} F(m, j, t_0)(-1)^j \exp[\Omega_j(t + t_0)] \tag{9}$$

whereas, for $t + t_0 > 0$,

$$I(t, t_0, m) = I^+(t, t_0, m) = -\frac{1}{2}R(\omega_0, m) - \sum_{j\leq 0} F(m, j, t_0)(-1)^j \exp[\Omega_j(t + t_0)]\,. \tag{10}$$

Equations (9) and (10) are the exact solutions of the integral (1).

• Let us now apply the technique explained above to two particular cases in the framework of the electromagnetic propagation.

Let us consider, for example, the electric field E_t transmitted through an air slab in the case of frustrated total reflection, that is for an incidence angle larger than the limit angle.

If the impinging wave is represented by a temporal pulse, rather than by a monochromatic wave, it is possible to demonstrate that E_t is given by [2] (apart from some unessential constants)

$$E_t \propto \int_{-\infty}^{\infty} g(\omega)\rho(\omega)\exp\left[i\omega\,\frac{n}{c}[\alpha x + \gamma(z - d)]\right]\exp(i\omega t)\,d\omega\,, \tag{11}$$

where he parameters A, a and β depends on the slab width d and on the refractive index n of the medium surrounding the slab; the function $g(\omega)$ is the incident spectrum, and $\rho(\omega)$ represents the transmission coefficient, which can be written in the same form as in Eq.(2).

Equation (11) refers to a two-dimensional Cartesian system **i, k** (coordinates x, z) where α and γ are the components of the incident vector of propagation. Thus by putting $t_0 = n[\alpha x + \gamma(z-d)]/c$, it is easy to verify that the transmitted field is given by an integral like the one in Eq. (1).

Let us see how to evaluate analitically the integral of Eq. (11) by using Jordan's Lemma.

Let us consider an incident pulse like a step function, the spectrum of which is

$$g(\omega) = g_0 \left[\pi\delta(\omega - \omega_0) + \frac{i}{\omega - \omega_0} \right] , \tag{12}$$

where g_0 is the amplitude of the pulse (located at $t = 0$) and ω_0 is the carrier frequency. By putting Eq. (12) into Eq. (11), the field E_t transmitted after the slab can be written as

$$E_t = \frac{1}{2} g_0 \rho(\omega_0) \exp(-i\omega_0 t) + \frac{1}{2\pi}\, g_0 \int_{-\infty}^{\infty} \frac{i}{\omega - \omega_0} \rho(\omega) \exp(-i\omega t)\, d\omega\,. \tag{13}$$

The integral

$$J = \int_{-\infty}^{\infty} \frac{i}{\omega - \omega_0} \rho(\omega) \exp(-i\omega t)\, d\omega$$

is of the same type as in Eq. (1), with $t_0 = 0$ and $m = 1$. Therefore, by taking into account Eqs. (4), (6), (9) and (10), we can write,

$$J = \frac{1}{2} R(\omega_0, 1) + \sum_{j>0} F(1, j, 0)(-1)^j \exp(\Omega_j t)\,, \qquad \text{for} \quad t < 0 \tag{14}$$

$$J = -\frac{1}{2} R(\omega_0, 1) - \sum_{j \leq 0} F(1, j, 0)(-1)^j \exp(\Omega_j t)\,, \qquad \text{for} \quad t > 0 \quad . \tag{15}$$

By introducing Eqs. (14) and (15) into Eq. (13), and considering that $F(1, j, 0) = A' M_0 \exp(i\psi)$, we can conclude that

$$\begin{aligned} E_t &= \frac{g_0}{2\pi} A' \sum_{j>0} (-1)^j M_0 \exp(i\psi) \exp(\Omega_j t)\,, \quad \text{for } t < 0 \\ E_t &= g_0 \left[\rho(\omega_0) \exp(-i\omega_0 t) - \frac{A'}{2\pi} \sum_{j \leq 0} (-1)^j M_0 \exp(i\psi) \exp(\Omega_j t) \right] , \quad \text{for } t > 0. \end{aligned} \tag{16}$$

This technique can be applied to more complicated functions.

• As another example, let us consider a rectangular pulse carried by a frequency ω_0.
For a rectangular pulse of height g_0 and duration from $-T$ to T, the spectrum $g(\omega)$ may be written as

$$g(\omega) = -i\, \frac{g_0}{\omega - \omega_0} \Big[\, \exp[i(\omega - \omega_0)T] - \exp[-i(\omega - \omega_0)T] \,\Big] \,. \tag{17}$$

By putting Eq. (17) into Eq. (11), the transmitted field is

$$E_t = -\frac{1}{2\pi} g_0 [J_1 - J_2]\,, \tag{18}$$

where J_1 and J_2 are of the same type as Eq. (1), that is,

$$J_1 = I(t, -T, 1)\,, \qquad J_2 = I(t, T, 1)\,.$$

Therefore, by applying Eqs. (9) and (10) we can write

$$\begin{aligned}
J_1 &= I^-(t, -T, 1)\,, \quad \text{for } t - T < 0,\ t < T \\
J_1 &= I^+(t, -T, 1)\,, \quad \text{for } t - T > 0,\ t > T \\
J_2 &= I^-(t, T, 1)\,, \quad \text{for } t + T < 0,\ t < -T \\
J_2 &= I^+(t, T, 1)\,, \quad \text{for } t + T > 0,\ t > -T.
\end{aligned} \tag{19}$$

By substituting into Eq. (18), we obtain

$$\begin{aligned}
E_t &= -\frac{g_0}{2\pi}\left[I^-(t, -T, 1) - I^-(t, T, 1)\right]\,, \quad \text{for } t < -T \\
E_t &= -\frac{g_0}{2\pi}\left[I^-(t, -T, 1) - I^+(t, T, 1)\right]\,, \quad \text{for } -T < t < T \\
E_t &= -\frac{g_0}{2\pi}\left[I^+(t, -T, 1) - I^+(t, T, 1)\right]\,, \quad \text{for } t > T\,.
\end{aligned} \tag{20}$$

It is interesting to note that the two terms $R(\omega_0, m)$ in the I-integrals cancel one another for $t < -T$ and $t > T$, whereas they sum in the interval $-T < t < T$.

The two examples considered above demonstrate how Jordan's Lemma can be a useful instrument in evaluating analytical solutions of complex integrals.

The procedure can also be applied to more complicated integrals, provided that they are of the type in Eq. (1): the total solution can always be expressed as the sum of functions like Eqs. (15) and (14), each of them working in a different temporal range.

1 J. A. Stratton, *Electromagnetic theory*, McGraw-Hill, New York (1941), Sec. 5.12.

2 D. Mugnai, A. Ranfagni, L. Ronchi, Atti della Fondazione Giorgio Ronchi **1**, 777 (1998).