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PRICING ARITHMETIC AVERAGE OPTIONS AND BASKET OPTIONS USING MONTE CARLO AND QUASI-MONTE CARLO METHODS

Maria Giuseppina Bruno e Antonio Grande

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Università degli Studi di Roma "La Sapienza"
Via del Castro Laurenziano 9, 00161 Roma (RM)
T (+39) 06 49766433 F (+39) 06 4957606
www.memotef.uniroma1.it

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PRICING ARITHMETIC AVERAGE OPTIONS AND BASKET OPTIONS USING MONTE CARLO AND QUASI-MONTE CARLO METHODS

Maria Giuseppina Bruno[1]
Antonio Grande [2]

ABSTRACT

In the present paper, we address the evaluation problem of multidimensional financial options. We apply in particular the Monte Carlo and Sobol Quasi-Monte Carlo numerical integration for pricing asian arithmetic average options and basket options and we show some numerical exemplifications in 4 and 12 dimensions. The paper is the occasion to furtherly test the algorithm for computing the quantile function of the standard gaussian distribution proposed by the authors in a previous publication.

Classification JEL: *C020, C630, C650, G130*.
Keywords: *Monte Carlo and Quasi-Monte Carlo numerical integration, Multidimensional financial options, Sobol low discrepancy sequences, Quantile function*.

1. INTRODUCTION

Arithmetic average options are options whose value depends on the arithmetic average of the prices of an underlying asset at pre-specified dates (called "reset points") over the time to maturity.

Basket options are instead options whose value depends on the price of a portfolio of correlated underlying assets at maturity.

In both cases, only numerical solutions exist for pricing them.

In this paper, we approach the problem as proposed by Bruno (2001) for the geometric average options in order to apply Monte Carlo and Quasi-Monte Carlo methods for solving it.

[1]Affiliation: Department of Methods and Models for Economics, Territory and Finance - Sapienza University of Rome - Via del Castro Laurenziano 9, 00161 Rome
(email: giuseppina.bruno@uniroma1.it)

[2]Affiliation: Department of Methods and Models for Economics, Territory and Finance - Sapienza University of Rome - Via del Castro Laurenziano 9, 00161 Rome
(email: antonio.grande@uniroma1.it)

We show the possibility of using the Control Variate as variance reduction technique for the crude Monte Carlo when pricing arithmetic average options. We also introduce the Cholesky matrix in order to manage correlation when pricing basket options. In the end, we show some numerical exemplifications in 4 and 12 dimentions.

We make the assumption of a geometric Brownian motion for the underlying assets dynamics and we apply the algorithm proposed by Bruno, Grande (2014) for inverting the standard gaussian cumulative distribution function. Notice however that the formalization and the calculation methodology can be easily generalized to other assumptions and it is intention of the authors to address these aspects in future developments of the paper.

2. ARITHMETIC AVERAGE OPTIONS

2.1. Assumptions and pricing formula

Let d be the number of reset points and $T_1, T_2, ..., T_d$ be the reset points. According to the "risk neutral" approach, the current value at time t (with $t = T_0 < T_1 < ... < T_d = T$=option maturity) of an arithmetic average call option is given by:

$$C_t^{(a)} = e^{-\delta(T-t)} E_Q \left[\max \left(0; \frac{1}{d} \sum_{k=1}^{d} S_{T_k} - X \right) \right] \tag{1}$$

where:

- δ is the instantaneous risk-less rate per unit of time;
- $T-t$ is the option time to maturity;
- E_Q is the mean value operator with respect to the equivalent martingale measure Q;
- S_{T_k} is the option underlying asset price at time $T_k (k = 1, 2, .., d)$;
- X is the option strike price.

By assuming that the underlying asset price follows a geometric Brownian motion with volatility per unit of time σ, equation (1) becomes:

$$C_t^{(a)} = e^{-\delta(T-t)} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} \max \left[0; \frac{1}{d} \sum_{k=1}^{d} \exp \left(\sum_{h=1}^{k} s_h y_h + m_k \right) - X \right] \varphi(y_1)\varphi(y_2)...\varphi(y_d) dy_1 dy_2 ... dy_d \tag{2}$$

where, for $k = 1, 2, ..., d, y_k$ are the numerical realizations of independent standard gaussian random variables and $\varphi(y_k)$ are the corresponding density functions. Besides:

$$
\begin{aligned}
m_k &= lnS_t + \left(\delta - \frac{\sigma^2}{2}\right)(T_k - t) && (3)\\
s_k &= \sigma\sqrt{T_k - T_{k-1}} && (4)
\end{aligned}
$$

2.2. Approximated solution using Crude Monte Carlo and Quasi-Monte Carlo

Using Crude Monte Carlo, we have:

$$
C_t^{(a)} \cong C_{t(M)}^{(a)} =
$$

$$
= e^{-\delta(T-t)}\frac{1}{N}\sum_{i=1}^{N}\max\left[0; \frac{1}{d}\sum_{k=1}^{d}\exp\left(\sum_{h=1}^{k}s_h\Phi^{-1}(x_h^{(i)}) + m_k\right) - X\right] \tag{5}
$$

where $\Phi^{-1}(\cdot)$ is the standard gaussian cumulative distribution inverse function, $x_k^{(i)}$ $(k = 1, \ldots, d$ and $i = 1, ..., N)$ is the k-th component of the i-th vector $x^{(i)} = (0 < x_k^{(i)} < 1; k = 1, ..., d)$ of d pseudo-random numbers and N the number of generated vectors.

Using Quasi-Monte Carlo methods, the solution is quite similar to equation (5) with the only difference that $x_k^{(i)}$ $(k = 1, \ldots, d$ and $i = 1, ..., N)$ is the component of a vector of quasi-random numbers resulting from a given d-dimensional low discrepancy sequence of points.

2.3. Approximated solution using Monte Carlo with Control Variate

We can apply the Control Variate as variance reduction technique for the solution (5) and we can use the value of a corresponding geometric average call option as control variable.

As known, the exact value of such an option is given by:

$$
C_t^{(g)} = e^{-\delta(T-t)}\left[e^{m+\frac{s^2}{2}}\Phi(d_1) - X\Phi(d_2)\right] \tag{6}
$$

where $\Phi(\cdot)$ is the standard gaussian cumulative distribution function and:

$$\begin{aligned} m &= \frac{1}{d}\sum_{k=1}^{d} m_k && (7)\\ s &= \frac{1}{d}\sqrt{\sum_{k=1}^{d}(d-k+1)^2 s_k^2} && (8)\\ d_1 &= \frac{m - logX + s^2}{s} && (9)\\ d_2 &= d_1 - s && (10)\end{aligned}$$

Besides, according to Bruno (2001), equation (6) can be approximated using Crude Monte Carlo as follows:

$$C_t^{(g)} \cong C_{t(M)}^{(g)} =$$
$$= e^{-\delta(T-t)}\frac{1}{N}\sum_{i=1}^{N}\max\left[0;\exp\left(\frac{1}{d}\sum_{k=1}^{d}(d-k+1)s_k\Phi^{-1}(x_k^{(i)}) + \frac{1}{d}\sum_{k=1}^{d}m_k\right) - X\right] \quad (11)$$

Then, by applying the Control Variate, we obtain the following approximated solution for the arithmetic average call option value:

$$C_t^{(a)} \cong C_{t(CV)}^{(a)} = C_{t(M)}^{(a)} + \left(C_t^{(g)} - C_{t(M)}^{(g)}\right) \quad (12)$$

3. BASKET OPTIONS

3.1. Assumptions, Cholesky decomposition and pricing formula

Let us take a portfolio of d correlated assets, each one with proportion $w_k \geq 0$ (with $\sum_{k=1}^{d} w_k = 1$).

According to the "risk neutral" approach, the current value at time t (with $t \leq T$) of a call option on this portfolio (called basket option) is given by:

$$C_t^{(b)} = e^{-\delta(T-t)}E_Q\left[\max\left(0;\sum_{k=1}^{d} w_k S_T^k - X\right)\right] \quad (13)$$

where S_T^k is the price of the k-th portfolio asset ($k = 1, 2, ..., d$) at the option maturity T and the rest of notations have the same meaning of Section 2.1.

Now, let us assume that each one of the portfolio assets follows a geometric Brownian motion. In other words, for $k = 1, 2, ..., d$, the following stochastic differential equations are satiesfied:

$$\frac{dS_t^k}{S_t^k} = \delta dt + \sigma_k dW_t^k \tag{14}$$

where S_t^k is the current price at time t of each portfolio asset, σ_k is the annual volatility of each asset and W_t^k is the standard Wiener process.

Let us also assume that the Wiener processes W_t^k, W_t^h $(k, h = 1, 2, ..., d)$ are correlated with correlation coefficient ρ_{kh}.

Under the above-mentioned assumptions, the price at maturity of each underlying asset can be written in the following way:

$$S_T^k = \exp\left(s'_k \sum_{h=1}^{k} a_{kh} N_h(0,1) + m'_k\right) \tag{15}$$

where, for $k = 1, 2, ..., d$, $N_k(0, 1)$ are independent standard gaussian random variables and:

$$m'_k = \ln S_t^k + \left(\delta - \frac{\sigma_k^2}{2}\right)(T - t) \tag{16}$$

$$s'_k = \sigma_k \sqrt{T - t} \tag{17}$$

where, for $k = 1, 2, ..., d$ and $h = 1, 2, ..., k$, we have:

$$a_{kk} = \begin{cases} \sqrt{\rho_{kk} - \sum_{h=1}^{k-1} a_{kh}^2} & \text{for} \quad h = k \text{ and } k \neq 1 \\ \sqrt{\rho_{kk}} & \text{for} \quad h = k \text{ and } k = 1 \end{cases} \tag{18}$$

$$a_{kh} = \begin{cases} \frac{1}{a_{hh}}\left(\rho_{kh} - \sum_{z=1}^{h-1} a_{kz} a_{hz}\right) & \text{for} \quad h < k \text{ and } h \neq 1 \\ \frac{1}{a_{hh}}\rho_{kh} & \text{for} \quad h < k \text{ and } h = 1 \end{cases} \tag{19}$$

Equations (18) and (19) are obtained by applying the Cholesky decomposition of the correlation matrix. This is a transformation converting correlated Wiener processes into independent ones. Remember however that this decomposition holds for positive definite matrixes only.

All above stated, equation (13) becomes:

$$C_t^{(b)} = e^{-\delta(T-t)} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} \max\left[0; \sum_{k=1}^{d} w_k \exp\left(s'_k \sum_{h=1}^{k} a_{kh} y_h + m'_k\right) - X\right] \varphi(y_1)\varphi(y_2)....\varphi(y_d) dy_1 dy_2 ....dy_d \tag{20}$$

where, for $k=1,2,...,d$, notations y_k and $\varphi(y_k)$ have the same meaning of Section 2.1.

3.2. Approximated solution using Crude Monte Carlo and Quasi-Monte Carlo

Using both Crude Monte Carlo and Quasi-Monte Carlo methods, we can approximate equation (20) as follows:

$$C_t^{(b)} \cong e^{-\delta(T-t)} \frac{1}{N} \sum_{i=1}^{N} \max \left[0; \sum_{k=1}^{d} w_k \exp\left(s'_k \sum_{h=1}^{k} a_{kh} \Phi^{-1}(x_h^{(i)}) + m'_k \right) - X \right] \quad (21)$$

where notations $\Phi^{-1}(\cdot)$ and $x_k^{(i)}$ ($k=1,\ldots,d$ and $i=1,...,N$) have the same meaning of Section 2.2. In particular, $x_k^{(i)}$ is the component of a vector of pseudo-random number when using Crude Monte Carlo while it is the component of a vector of quasi-random numbers when using Quasi-Monte Carlo methods.

4. APPLICATIONS

For both arithmetic average options and basket options, we show some numerical exemplifications in 4 and 12 dimensions using the crude Monte Carlo and the Sobol Quasi-Monte Carlo methods. Notice that in the previous sections the dimensions are denoted by d and they correspond to the number of reset points in the case of arithmetic average options while they are the number of assets in portfolio in the case of basket options.

As for the results illustrated in this paper, we put $(T-t)=3$ years, $\delta=0.09$ years^{-1} and $X=100$. Besides, for inverting the standard gaussian cumulative distribution function, we use the algorithm proposed in Bruno, Grande (2014).

4.1. Results for arithmetic average options

For arithmetic average options, we assume $S_t=100$ and $\sigma=0.20$ years^{-1}.

In Table 1, we show the results obtained with $d=4$ for different values of N, while in Table 2 we show the results with $d=12$.

Table 1. Value of the arithmetic average option in 4 dimensions

Replications	Crude Monte Carlo	Sobol
1 million	17.05705126	17.07117596
10 millions	17.07209584	17.07120674
100 millions	17.06807062	17.07121361

Table 2. Value of the aritmetic average option in 12 dimensions

Replications	Crude Monte Carlo	Sobol
1 million	14.86602066	14.86054748
10 millions	14.85816036	14.86080443
100 millions	14.86021608	14.86085704

4.2. Results for basket options

For basket options, we assume $w_k = \frac{1}{d}$ ($k = 1, ..., d$) and we use the following data:

a. in the case of $d = 4$ dimensions:

- current price of the 4 assets in the basket $(S_t^k, k = 1, 2, 3, 4) =$
$\begin{pmatrix} 110 & 120 & 97 & 133 \end{pmatrix}$
- volatility per unit of time of the 4 assets in the basket $(\sigma_k, k = 1, 2, 3, 4) =$
$\begin{pmatrix} 0.20 & 0.30 & 0.25 & 0.32 \end{pmatrix}$
- correlation matrix of the 4 assets in the basket $(\rho_{kh}, k, h = 1, 2, 3, 4) =$

$$\begin{pmatrix} 1 & 0.15 & 0.10 & 0.20 \\ 0.15 & 1 & -0.05 & 0.18 \\ 0.10 & -0.05 & 1 & 0.13 \\ 0.20 & 0.18 & 0.13 & 1 \end{pmatrix}$$

b. in the case of $d = 12$ dimensions:

- current price of the 12 assets in the basket $(S_t^k, k = 1, 2, ..., 12) =$
$\begin{pmatrix} 110 & 120 & 97 & 133 & 98 & 105 & 142 & 117 & 87 & 95 & 103 & 114 \end{pmatrix}$
- volatility per unit of time of the 12 assets in the basket $(\sigma_k, k = 1, 2, ..., 12) =$
$\begin{pmatrix} 0.20 & 0.30 & 0.25 & 0.32 & 0.13 & 0.12 & 0.55 & 0.42 & 0.10 & 0.09 & 0.03 & 0.41 \end{pmatrix}$
- correlation matrix of the 12 assets in the basket $(\rho_{kh}, k, h = 1, 2, ..., 12) =$

$$\begin{pmatrix} 1 & 0.15 & 0.10 & 0.20 & 0.21 & 0.19 & 0.28 & 0.33 & 0.21 & 0.25 & 0.41 & 0.45 \\ 0.15 & 1 & -0.05 & 0.18 & 0.03 & 0.20 & 0.25 & 0.22 & 0.10 & 0.28 & 0.40 & 0.35 \\ 0.10 & -0.05 & 1 & 0.13 & 0.22 & 0.17 & 0.21 & 0.14 & 0.11 & 0.14 & 0.36 & 0.28 \\ 0.20 & 0.18 & 0.13 & 1 & 0.54 & 0.71 & 0.11 & 0.18 & 0.02 & 0.18 & 0.31 & 0.27 \\ 0.21 & 0.03 & 0.22 & 0.54 & 1 & 0.90 & 0.02 & 0.21 & 0.14 & 0.22 & 0.29 & 0.24 \\ 0.19 & 0.20 & 0.17 & 0.71 & 0.90 & 1 & -0.07 & 0.10 & 0.05 & 0.20 & 0.27 & 0.29 \\ 0.28 & 0.25 & 0.21 & 0.11 & 0.02 & -0.07 & 1 & 0.02 & 0.21 & 0.13 & 0.16 & 0.18 \\ 0.33 & 0.22 & 0.14 & 0.18 & 0.21 & 0.10 & 0.02 & 1 & 0.12 & 0.05 & 0.14 & 0.25 \\ 0.21 & 0.10 & 0.11 & 0.02 & 0.14 & 0.05 & 0.21 & 0.12 & 1 & -0.04 & 0.11 & 0.14 \\ 0.25 & 0.28 & 0.14 & 0.18 & 0.22 & 0.20 & 0.13 & 0.05 & -0.04 & 1 & 0.08 & 0.11 \\ 0.41 & 0.40 & 0.36 & 0.31 & 0.29 & 0.27 & 0.16 & 0.14 & 0.11 & 0.08 & 1 & 0.13 \\ 0.45 & 0.35 & 0.28 & 0.27 & 0.24 & 0.29 & 0.18 & 0.25 & 0.14 & 0.11 & 0.13 & 1 \end{pmatrix}$$

In Table 3 and Table 4, we show the obtained results for different N.

Table 3. Value of the basket option in 4 dimensions

Replications	Crude Monte Carlo	Sobol
1 million	39.5005	39.50282
10 millions	39.509	39.50316
100 millions	39.4986	39.50319

Table 4. Value of the basket option in 12 dimensions

Replications	Crude Monte Carlo	Sobol
1 million	34.2212	34.20546
10 millions	34.2016	34.20580
100 millions	34.2041	34.20587

5. CONCLUSIONS

In the present paper, we address the evaluation problem of arithmetic average options and basket options using the Monte Carlo and Sobol Quasi-Monte Carlo numerical integration. We show the possibility of using the Control Variate as variance reduction technique for the crude Monte Carlo when pricing arithmetic average options. We also introduce the Cholesky matrix in order to manage correlation when pricing basket options. In the end, we show some numerical exemplifications in 4 and 12 dimensions. However, the primary purpose of the paper is to furtherly test the algorithm for computing the quantile function of the standard gaussian distribution proposed by the authors in a previous publication.

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