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On some properties of Lagrangian dispersion models with non-Gaussian noise(*)(**)

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Summary. — The properties of a stochastic model with non-Gaussian random noise describing turbulent dispersion have been investigated, with reference to its mathematical structure and to its behaviour simulating the inertial subrange. The process is Markovian, mean-square continuous and with δ-correlated increments. The model is influenced by the turbulence inhomogeneities also at the smallest scales, that is, it does not correctly simulate the existence of a well-developed inertial subrange. Some numerical computations have been performed confirming the theoretical results.

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1. – Introduction

The Lagrangian description of turbulent dispersion by means of stochastic processes was initiated by Taylor (1921) [1] with reference to homogeneous turbulence. Obukhov (1959) [2] made explicit use of a stochastic differential (Langevin) equation to deal with the same problem, whereas more recently Thomson [3, 4] generalized the Langevin equation in the case of inhomogeneous turbulence.

Thomson [4] outlined some general conditions that have to be satisfied by a stochastic model for describing the turbulent dispersion of marked fluid particles, that is:

1. At small time scale, the accelerations of the fluid particles are independent of each other [5], so that the stochastic process describing the velocity is a Markov process.

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2. The small time behaviour of the statistics of the particle velocity has to be consistent with the Kolmogorov hypothesis referring to the inertial subrange, that is, the correlation of the velocity increments has to be independent of the initial conditions and grow linearly with time [5, 6].

3. A true fluid particle is characterized by a continuous trajectory, so that it seems reasonable to require the stochastic process to be continuous (this is the assumption leading to the model choice made by Thomson [4] and other authors).

4. If the particles of tracer are initially well mixed, they will remain so. This criterion is referred to as *well-mixed condition* and applies to general, unsteady situations. A weaker condition was used in [3], that is, for times larger than the time scale of the inertial subrange the unconditioned statistics of the particle velocities has to be the same as the Eulerian statistics of the fluid velocity.

A detailed analysis of the previous points has been performed by [4] and by [6] referring to stochastic models with noise described by a Wiener process. In particular, [4] proposed a model based on the previous assumptions, which has been used to investigate dispersion in complex flows (see, for istance, Luhar and Britter [7] and Näslund *et al.* [8]). The main shortcomings of this type of model have been also discussed in details, for istance by Sawford [9]. Namely, in the 3D case, the equation leading to define the drift term in the Langevin equation does not have in general a unique solution.

On the other hand, the model described by Thomson [3], characterized by a non-Gaussian noise, appears to be of easy application, although there are some problems with respect to the conditions outlined above. In particular, the trajectories of the stochastic process do not display sample-path continuity (a strong continuity condition, as will be discussed after), thus lacking to satisfy condition 3). In spite of this, implementations of this model turn out to describe quite well dispersion experiments in perturbed boundary layers (see Thomson [10], Tinarelli *et al.* [11], Brusasca *et al.* [12], Tampieri *et al.* [13]).

In this paper we shall discuss the structure of the stochastic process with non-Gaussian noise, as derived by Thomson [3], in order to discuss the applicability condition of the model.

2. – An outline of the model

According to the model proposed by Thomson [3], and referring to a monodimensional case, the motion of each particle in the phase space (x, v) is described by a Langevin equation, as follows:

$$\begin{cases} \mathrm{d}v = -\dfrac{v}{T}\mathrm{d}t + \mathrm{d}\xi(t)\,, \\ \mathrm{d}x = v\mathrm{d}t\,, \end{cases} \tag{2.1}$$

where T has the meaning of a decorrelation time (in homogeneous turbulence T is the Lagrangian correlation time) and ξ is a random function with

$$\langle \xi(t_i)\xi(t_j)\rangle = \delta(t_i - t_j)\,, \tag{2.2}$$

where $\langle\cdot\rangle$ indicates the ensemble average.

Information about the structure of turbulence is put into the explicit form of T and of the probability distribution (PDF) of ξ, so that in general they will be function of position x.

In the following we shall examine the properties of the discretized form of the model (2.1), that is:

$$\begin{cases} v_{i+1} = \left(1 - \dfrac{\Delta t}{T_{i+1}}\right) v_i + \xi_{i+1} \,, \\ x_{i+1} = v_i \Delta t + x_i \,, \end{cases} \tag{2.3}$$

where now ξ_k is a discrete random process whose PDF is determined, according to Thomson [3], imposing that the unconditioned steady-state distribution of the particles in the (x, v) space is the same as the Eulerian distribution of the fluid.

The expressions for the moments ξ^i are reported in [3], (eq. 10), at first order in ΔT, and in appendix A at the second order in Δt following [14] (eq. 3.7).

To generate the realizations of the process ξ_k we shall use random numbers sampled from a distribution with the first four moments $\xi^i, i = 1, \dots, 4$ given by (A.1). We note here that only using the moments of the random forcing expressed at order $(\Delta t)^2$ the Gaussian case can be recovered exactly (because ξ^4 turns to be equal to $3(\xi^2)^2$). Moreover the Schwartz inequality $\xi^2\xi^4 > (\xi^3)^2$ is satisfied, at least as far as the order of magnitudes are concerned, for all but the case of Gaussian inhomogeneous turbulence. In this respect, we note that this is a quite unphysical case, and does not describe the atmospheric boundary layer [15–17].

Condition (2.2) ensures that the process described by (2.1) is Markovian [18].

A few comments on the local structure of the increments of the process are in order. For small times $i\Delta t \ll T$, being T the scale of the decorrelation times T_i, there results

$$v_{i+1} - v_0 = \sum_{k=1}^{i+1} \xi_k - v_0 \Delta t \sum_{k=1}^{i+1} \frac{1}{T_k} + \mathrm{O}(\Delta t^2) \,. \tag{2.4}$$

The correlation of the velocity increments results, for $i \gg j + 1$

$$\langle (v_{i+1} - v_i)(v_{j+1} - v_j) \rangle = \frac{\Delta t^2}{T_{i+1} Tj + 1} \langle v_i v_j \rangle + \langle \xi_{i+1} \xi_{j+1} \rangle - \frac{\Delta t}{T_{j+1}} \delta_{j,i+1} \langle \xi_{i+1} \xi_j \rangle, \tag{2.5}$$

but $\langle \xi_{i+1}\xi_{j+1} \rangle = 0$ by (2.2) and $\langle v_i v_j \rangle = 0$ by Markovianity. Thus, the process is δ-correlated.

3. – Continuity of the trajectories

The continuity of the trajectories of the particles in the phase space is a physical requirement, if real trajectories have to be simulated. Because the stochastic model is used to describe the average properties over an ensemble of trajectories, we are not interested in the behaviour of the single trajectory, but only in the average behaviour of the ensemble, thus we argue that the relevant continuity condition for the process is a condition requiring the continuity in probability.

An appropriate continuity condition is a criterion [19] which reads

$$\lim_{\Delta t \to 0} \langle (v_{i+1} - v_i)^2 \rangle = 0 \,. \tag{3.1}$$

In fact there results

$$(v_{i+1} - v_i)^2 = \frac{\Delta t^2}{T_{i+1}^2} v_i^2 + \xi_{i+1}^2 - 2 \frac{\Delta t}{T_{i+1}} v_i \xi_{i+1} \tag{3.2}$$

and thus

$$(3.3) \qquad \langle (v_{i+1} - v_i)^2 \rangle = \Delta t^2 \left\langle \frac{v_i^2}{T_{i+1}^2} \right\rangle + \langle \xi_{i+1}^2 \rangle \,,$$

so

$$(3.4) \qquad \lim_{\Delta t \to 0} \langle (v_{i+1} - v_i)^2 \rangle = 0.$$

Thus, the present process is mean-square continuos, a weaker condition with respect to the sample path continuity, which could be expressed following a condition given by Kolmogorov [20]. This criterion implies the existence of positive constants p, r, L such that

$$(3.5) \qquad \langle (v_{i+1} - v_i)^p \rangle \le \frac{L|\Delta t|}{|lg_2|\Delta t||^{1+r}} \,, \quad p < r \,,$$

where v_{i+1}, v_i are the velocities at times $(i+1)\Delta t$ and $i\Delta t$. This stronger continuity condition is verified in the Gaussian case, and in general by the Thomson [4] model, because the noise is a Wiener process and the stochastic process has locally the same form as a process with independent increments.

4. – The small time behaviour

The existence of the inertial subrange has some implications on the behaviour of the trajectories of the stochastic model (as outlined by [6]). The Kolmogorov hypothesis states that in the inertial subrange the structure of the turbulence is locally homogeneous and isotropic. Because the accelerations are shown to be almost independent of each other [6], any process describing the trajectories has to be a Markovian process with δ-correlated increments.

In the inertial subrange the correlation of the velocity increments is given by

$$(4.1) \qquad \langle (v_i - v_0)^2 ; v_0 \rangle = c_0 \bar{\epsilon} i \Delta t \,,$$

where $\bar{\epsilon}$ is the dissipation rate, and the constant c_0 takes a value between 2 and 7, according to the literature.

In general, we have

$$(4.2) \qquad \langle (v_i - v_0)^q ; v_0 \rangle = (c_0 \bar{\epsilon} i \Delta t)^{\frac{q}{2}} \,,$$

but the odd moments have to be zero to the leading order due to isotropy.

In particular, we must note that the moments are independent of the value of the initial velocity v_0.

The present model is characterized by a non-Gaussian random noise, and then its PDF does not satisfy a Fokker-Planck equation (at variance with models with Gaussian noise). The relevant equation is a Kramers-Moyal equation, which does not give an expression for the PDF that can be used for the present purposes. More simply, it is possible to compute the first moments of the velocity increments directly from (2.3). The velocity increment may be written as

$$(4.3) \qquad v_{i+1} - v_0 = -v_0 \Delta t \left(\sum_{j=1}^{i+1} \frac{1}{T_j} \right) + \sum_{j=1}^{i+1} \xi_j + \mathrm{O}(\Delta t^2) \,.$$

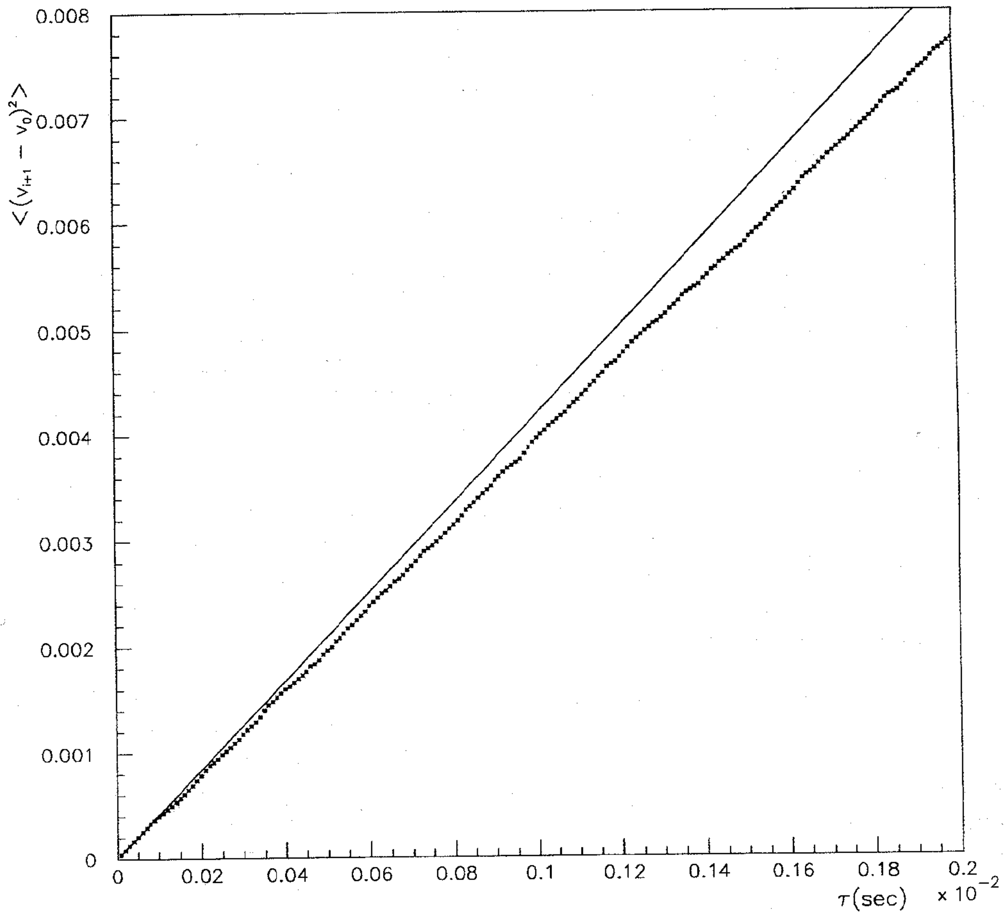

Fig. 1. – $\langle (v_{i+1} - v_0)^2 \rangle$ as a function of time τ, for $\tau \ll T$, in the case of homogeneous turbulence. Continuous line: first term of eq. (4.6); stars: numerical simulation average over 10^4 trajectories.

In general, the value of the decorrelation time T and the PDF of the noise vary with the position along the trajectory. However, for small times ($k\Delta t \ll T$) we assume that on average the total distance covered by each particle is small with respect to the spatial scale of variation of the turbulence. Consequently, in the following computations we will use the values of the time scale T and of the moments of the noise computed in the initial points of the trajectories. Obviously this is not a restriction in homogeneous turbulence. Averaging over the ensemble of trajectories we have

$$\langle (v_{i+1} - v_0)^2 \rangle = G_1 \tau - G_2 \Delta t \tau + \left(\frac{v_0^2}{T^2} - \frac{2 v_0 A_1}{T} \right) \tau^2 + \mathrm{O}(\Delta t)^2) \tag{4.4}$$

and

$$\begin{aligned} \langle (v_{i+1} - v_0)^3 \rangle = {} & H_1 \tau - H_2 \Delta t \tau - 3 v_0 \frac{G_1}{T} \tau^2 + 3 v_0 \frac{G_2}{T} \Delta t \tau^2 + \\ & + \left(3 v_0^2 \frac{A_1}{T^2} + 3 v_0 \frac{G_1}{T^2} \right) \tau^3 + \left(-3 v_0^2 \frac{A_2}{T^2} - 3 v_0 \frac{G_2}{T^2} \right) \Delta t \tau^3 + \mathrm{O}(\Delta t^2) \end{aligned} \tag{4.5}$$

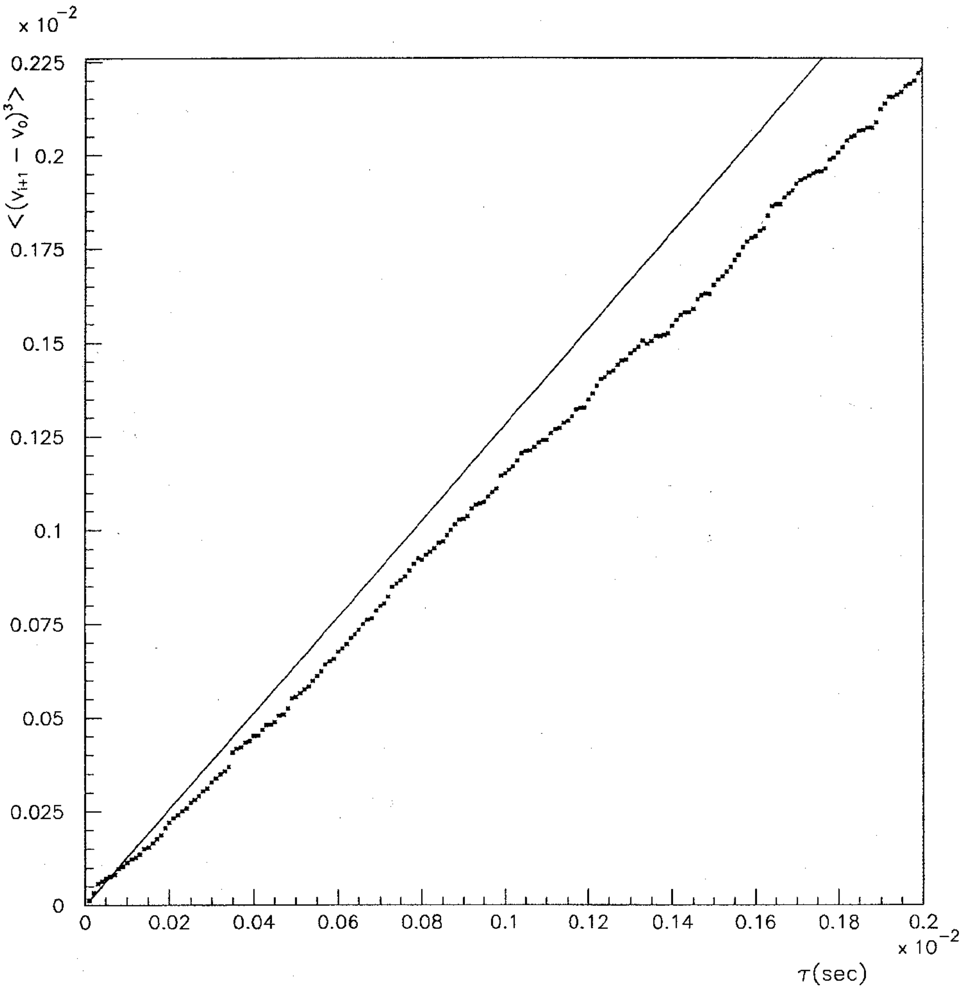

Fig. 2. – As in fig. 1, but for the third moment $\langle (v_{i+1} - v_0)^3 \rangle$, eq. (4.7).

where $\tau = (i+1)\Delta t$.

The expressions for the coefficients are reported in appendix A.

In homogeneous, non-Gaussian turbulence the previous expression reduces to

$$\langle (v_{i+1} - v_0)^2 \rangle = 2\frac{k_2}{T}\tau - \frac{k_2}{T^2}\Delta t\tau + \frac{v_0^2}{T^2}\tau^2 + \mathrm{O}(\Delta t^2) \tag{4.6}$$

and

$$\langle (v_{i+1} - v_0)^3 \rangle = 3\frac{k_3}{T}\tau - 3\frac{k_3}{T^2}\Delta t\tau - 6v_0\frac{k_2}{T^2}\tau^2 + 3v_0\frac{k^2}{T^3}\Delta t\tau^2 + 6v_0\frac{k_2}{T^3}\tau^3 - -3v_0\frac{k_2}{T^4}\Delta t\tau^3 + \mathrm{O}(\Delta t^2)\,. \tag{4.7}$$

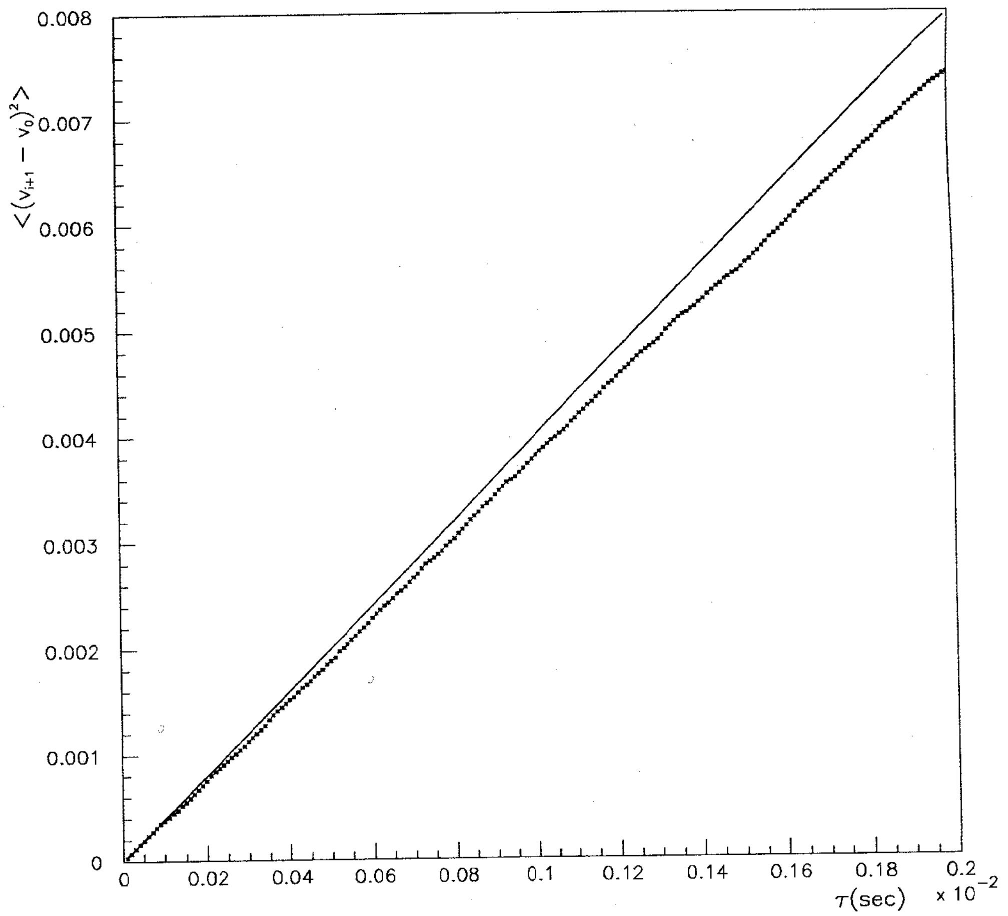

Fig. 3. – As in fig. 1, but in the case of inhomogeneous turbulence, eq. (4.4). The initial position of the trajectories is such that turbulence in the mixed region is strongly inhomogeneous and non-Gaussian.

At order τ the second moment is consistent with (4.2), holding in the inertial subrange, that is, it does not depend on the initial velocity v_0 and the coefficient of τ is exactly $c_0\bar{\epsilon}$. The dependence on the initial velocity appears at second order in τ. The independence from v_0 at first order has a relation with the mean-square continuity discussed above. In fact Borgas and Sawford [6] showed that in a discontinuos process (with jump) this dependence appears at first order in τ; the weak continuity condition of the present process is then consistent with a good behaviour of the velocity correlation.

The third moment is different from zero at leading order, contradicting the above considerations on the odd moments, and showing that in this model the physical information derived from the large scale structure of the turbulence has an effect on the small time behaviour, being present in the structure of the noise. Consistently, in inhomogeneous turbulence, the moments of the velocity increments depend at first order in τ on the derivatives of the moments of the random forcing.

We notice that (4.5) and (4.7) highlight a contradiction with the inertial subrange be-

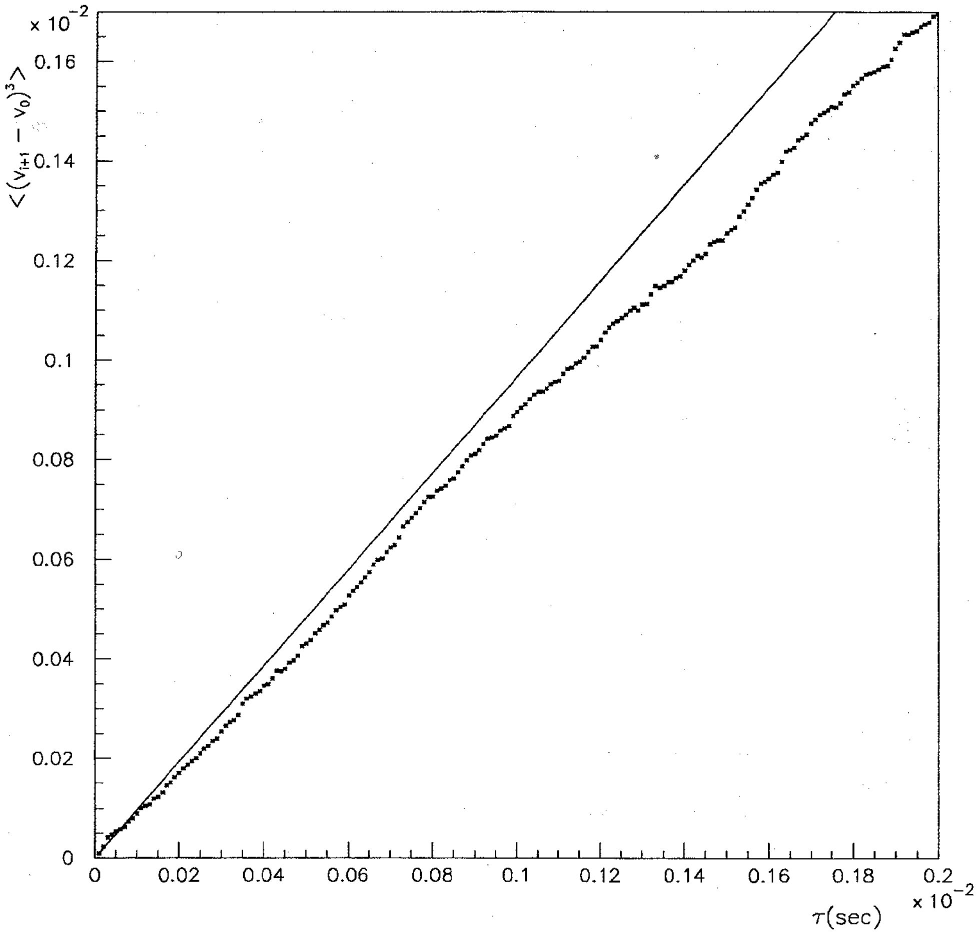

Fig. 4. – As in fig. 3, but for the third moment $\langle (v_{i+1} - v_0)^3 \rangle$, eq. (4.5).

haviour condition, namely (4.2) above. However, if the model is applied to describe dispersion over scales larger than the integral turbulence scale (or over times larger than T), we can neglect the small time behaviour of the dispersion. Thus, the model should be applied properly under these conditions.

The results obtained for the behaviour of the model at small times have been tested numerically using data on turbulence in a mixing region. The turbulent mixing region that develops when a high-velocity flow mixes with a low-velocity one presents a case of highly inhomogeneous turbulence. Detailed measurements of high-order moments and of many other quantities are available from laboratory experiments using high-pressure air inflated trough a nozzle in a chamber with still air [21].

For these experiments the PDF of the random forcing has been assumed to be given by the sum of two Gaussian distributions. Given only the first four moments the problem is not closed; we have adopted a closure presented by [14] wich minimizes the difference between the amplitude of the two Gaussian distributions. Because the time step was chosen to be quite small, the PDF of the random noise turns out to be quite skewned. This fact leads to random numbers chosen more frequently from one Gaussian curve than from the other one. On the other hand, small time steps are necessary to obtain stable numerical results in the integration of the Langevin equation. Comparing the statistics of the particle velocity with those prescribed for the fluid, a good agreement results, showing that the ensemble average is made on a sample large enough to ensure stability of the computed moments [14].

Equations (4.4) and (4.5) have been verified using different values of the initial velocity v_0. The numerical tests have shown the independence of velocity correlation on v_0.

In figs. 1-4 the behaviours of $\langle (v_{i+1}-v_0)^2\rangle$ and $\langle (v_{i+1}-v_0)^3\rangle$ are reported for $v_0 = 0$ as a function of time: there results that the theoretical formula at order Δt, *i.e.* the first term of eqs. (4.4)-(4.7) (continuous line), agrees with the numerical estimate (stars) within 10 %.

5. – Conclusions

Some properties of a stochastic model for turbulent dispersion based on the Langevin equation with non-Gaussian noise have been examined.

The model turns out to be Markovian, with δ-correlated increments. Moreover, a continuity condition for the trajectories has been verified.

The main difference with models characterized by a Gaussian noise lies in the behaviour in the inertial subrange. The non-Gaussian model shows odd moments of the velocity increments which account for the inhomogeneous structure of turbulence. Thus this model does not describe in a proper way the existence of the inertial subrange, but can be applied to dispersion problems with time scale larger than that of the inertial subrange. This was, in fact, done in the practical implementations of the model, already cited in the introduction. Moreover, the possibility to use such models in conditions when the inertial subrange is not developed, or is not explicitly described by the velocity field, has to be investigated.

APPENDIX A.

In this appendix we consider the explicit expression of the first four moments of the PDF of the random forcing used in (2.3). According to [14] (par. 3), we have

$$\text{(A.1)} \qquad \begin{cases} \xi^1 = \Delta t A_1 - (\Delta t)^2 A_2 \,, \\ \xi^2 = \Delta t G_1 - (\Delta t)^2 G_2 \,, \\ \xi^3 = \Delta t H_1 - (\Delta t)^2 H_2 \,, \\ \xi^4 = \Delta t L_1 - (\Delta t)^2 L_2 \,, \end{cases}$$

with

$$
(\mathrm{A}.2)\begin{cases}
A_1 = \dfrac{\mathrm{d}k_2}{\mathrm{d}x}, \\
A_2 = \left(\dfrac{\mathrm{d}k_2}{\mathrm{d}x}\dfrac{1}{T} + \dfrac{1}{2}\dfrac{\mathrm{d}^2k_3}{\mathrm{d}x^2}\right), \\
G_1 = \left(2\dfrac{k_2}{T} + \dfrac{\mathrm{d}k_3}{\mathrm{d}x}\right), \\
G_2 = \left[\dfrac{k_2}{T^2} + \dfrac{2}{T}\dfrac{\mathrm{d}k_3}{\mathrm{d}x} + \dfrac{5}{2}k_2\dfrac{\mathrm{d}^2k_2}{\mathrm{d}x^2} + \left(\dfrac{\mathrm{d}k_2}{\mathrm{d}x^2}\right)^2 + \dfrac{1}{2}\dfrac{\mathrm{d}^2k_4}{\mathrm{d}x^2}\right], \\
H_1 = \left(3\dfrac{k_3}{T} + \dfrac{\mathrm{d}k_4}{\mathrm{d}x} + 3k_2\dfrac{\mathrm{d}k_2}{\mathrm{d}x}\right), \\
H_2 = \left(\dfrac{3k_3}{T^2} + \dfrac{3}{T}\dfrac{\mathrm{d}k_4}{\mathrm{d}x} + \dfrac{3}{T}k_2\dfrac{\mathrm{d}k_2}{\mathrm{d}x} + \dfrac{7}{2}k_2\dfrac{\mathrm{d}^2k_3}{\mathrm{d}x^2} + \dfrac{9}{2}k_3\dfrac{\mathrm{d}^2k_2}{\mathrm{d}x^2} + 4\dfrac{\mathrm{d}k_2}{\mathrm{d}x}\dfrac{\mathrm{d}k_3}{\mathrm{d}x} + \dfrac{1}{2}\dfrac{\mathrm{d}^2k_5}{\mathrm{d}x^2}\right), \\
L_1 = \left(\dfrac{4}{T}k_4 + 6k_3\dfrac{\mathrm{d}k_2}{\mathrm{d}x} + \dfrac{\mathrm{d}k_5}{\mathrm{d}x} + 4k_2\dfrac{\mathrm{d}k_3}{\mathrm{d}x}\right), \\
L_2 = \left[-\dfrac{12}{T^2}k_2^2 + \dfrac{6k_4}{T^2} + \dfrac{4}{T}\left(\dfrac{\mathrm{d}k_5}{\mathrm{d}x} + k_2\dfrac{\mathrm{d}k_3}{\mathrm{d}x} + 3k_3\dfrac{\mathrm{d}k_2}{\mathrm{d}x}\right) + 7k_4\dfrac{\mathrm{d}^2k_2}{\mathrm{d}x^2} + 8k_3\dfrac{\mathrm{d}^2k_3}{\mathrm{d}x^2} + \right. \\
\left. +\dfrac{9}{2}k_2\dfrac{\mathrm{d}^2k_4}{\mathrm{d}x^2} + 6k_2^2\dfrac{\mathrm{d}^2k_2}{\mathrm{d}x^2} + 3k_2\left(\dfrac{\mathrm{d}k_2}{\mathrm{d}x}\right)^2 + 4\left(\dfrac{\mathrm{d}k_3}{\mathrm{d}x}\right)^2 + 7\dfrac{\mathrm{d}k_2}{\mathrm{d}x}\dfrac{\mathrm{d}k_4}{\mathrm{d}x}\right],
\end{cases}
$$

where $k_i, i = 1, \ldots, 5$ are the cumulants of the flow velocity, and we choose $k_1 = 0$ according to the data [10].

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