Antisymmetric wind stress fields and parity transformation of the correction streamfunctions in the western boundary layer of oceanic basins

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Summary. — Most models of the single-gyre, wind-driven oceanic circulation resort to wind stress fields that are strictly antisymmetric under a parity transformation with respect to a mid basin latitude. We show that, within the ambit of the weakly nonlinear quasi-geostrophic dynamics with a given frictional parametrization, this property of the wind stress implies a corresponding definite behaviour, under the same transform, of all the correction streamfunctions that constitute the perturbative expansion of the solution in the western boundary layer. The result generalizes what can be observed in some truncated expansions found in the literature and in the presence of an explicitly defined wind stress field.

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1. – Introduction

The basic picture of the wind-driven oceanic circulation in a basin bounded by meridional walls presupposes the matching between the southward interior flow governed, at least in the ambit of weak nonlinearity, by the Sverdrup balance and the western boundary layer (WBL), where a strongly intensified northward returned flow takes place. Denoting the wind stress field as

$$\boldsymbol{\tau} = \tau^x \mathbf{i} + \tau^y \mathbf{j}\,, \tag{1}$$

(the β-plane reference is understood) the nondimensional meridional interior velocity ν_{I} given by the Sverdrup balance is

$$\nu_{\mathrm{I}} = \mathbf{k} \cdot \mathrm{curl}\, \boldsymbol{\tau}\,, \tag{2}$$

k being the upward unit vector.

In the single-gyre models, the functional dependence of the components τ^x and τ^y on the spatial coordinates is demanded to take into account, at least in a rudimentary manner, the easterlies-westerlies structure of the wind field above the gyre, in accordance with a well-known phenomenology (see, for instance, Gill, 1982, sect. 2.3).

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Usually, both in analytical and numerical models, the convenience of dealing with a mathematically simple form of $\tau^x(x, y)$ and $\tau^y(x, y)$ introduces, at the same time, a definite symmetry property of the functions τ^x and τ^y themselves under a parity transformation of the kind

$$(x, y) \to (x, 2\theta - y), \tag{3}$$

that affects not only the structure of the interior flow via eq. (2) but also that of the WBL. Transform (3) represents a mirror reflection of the β-plane with respect to a proper latitude θ that can be easily singled out once the explicit form of τ^x and τ^y is known. We are referring to the following symmetry property under transform (3):

$$\tau^x(x, 2\theta - y) = -\tau^x(x, y),$$

$$\tau^y(x, 2\theta - y) = \tau^y(x, y).$$

As a consequence, the wind stress curl $\partial\tau^x/\partial y - \partial\tau^y/\partial x$ is left unchanged and the same trivially holds also for $v_{\rm I}$ given in eq. (2). For instance, Bryan, 1963, considers

$$\tau^x(y) = -\cos\left(\frac{\pi y}{2L}\right), \qquad \tau^y \equiv 0, \qquad 0 \leqslant y \leqslant 2L;$$

analogously, Moore, 1963, puts

$$\tau^x(y) = -W_0 \cos\left(\frac{\pi y}{L}\right), \qquad \tau^y \equiv 0, \qquad 0 \leqslant y \leqslant L$$

and Blandford, 1971, chooses

$$\tau^x(x, y) = -\frac{1}{2}\sin(\pi x)\cos(\pi y), \qquad \tau^y(x, y) = \frac{1}{2}\cos(\pi x)\sin(\pi y), \qquad 0 \leqslant y \leqslant 1.$$

It is trivial to check that all these examples verify the above-introduced symmetry property. What about the behaviour of the flow pattern in the WBL in the presence of transform (3)? Perhaps, the best known examples are given in the papers of Munk *et al.*, 1950, and Veronis, 1966. In the first case $\tau^x = -\Gamma\cos(y)$, $\tau^y \equiv 0$, $0 \leqslant y \leqslant \pi$ so that $2\theta = \pi$. The first three terms of the truncated perturbative expansion of the stream-function in the WBL are, respectively, proportional to $\sin(y)$, $\sin(2y)$ and to a linear combination of $\sin(y)$ and $\sin(3y)$. In the second case

$$\tau^x = -\frac{W}{2}\sin\frac{x}{L}\cos\frac{y}{L}, \qquad \tau^y = -\frac{W}{2}\cos\frac{x}{L}\sin\frac{y}{L}, \qquad 0 \leqslant y \leqslant \pi L$$

so that $2\theta = \pi L$. Also in this case the first two terms of the truncated WBL expansion are proportional to $\sin(y)$ and $\sin(2y)$, respectively.

We see that the zeroth-order correction is invariant under transform (3) while that of the first order is antisymmetric under the same transform. Moreover, the second-order term of Munk's model is again invariant. Thus, although the circulation pattern does not seem to have a definite symmetry property under the mirror reflection (3) *on the whole*, the memory of the definite symmetry property of the wind stress field (if it exists) is separately retained in each term of the truncated expansions of the WBL solution. This fact leads us to conjecture the existence of a sort of regularity in the behaviour of the *single* WBL *corrections* under transform (3) for the whole series. The main aim of the present note is to deduce such behaviour assuming the above symmetry property of the wind stress field with respect to a given latitude θ but releasing any explicit form of the functions $\tau^x(x, y)$ and $\tau^y(x, y)$. The result will be obtained by resorting to the properties of the WBL correction equations and to the mathematical induction principle.

2. – The WBL correction equations

We consider the well-known quasi-geostrophic equation (Pedlosky, 1987)

$$\left(\frac{\delta_{\mathrm{I}}}{L}\right)^2 J(\psi, \nabla^2\psi) + \frac{\partial\psi}{\partial x} = \mathbf{k}\cdot\mathrm{curl}\,\boldsymbol{\tau} - \frac{\delta_{\mathrm{S}}}{L}\nabla^2\psi + \left(\frac{\delta_{\mathrm{M}}}{L}\right)^3\nabla^4\psi \tag{4}$$

in the domain $(x_{\mathrm{W}} \leqslant x \leqslant x_{\mathrm{E}}) \times (-\infty < y < +\infty)$, with special regard to the solution for x close to x_{W}, and boundary conditions

$$\psi(x_{\mathrm{W}}, y) = 0\,, \tag{5}$$

$$\frac{\partial\psi}{\partial x} = 0 \qquad \text{in } x = x_{\mathrm{W}}. \tag{6}$$

The unbounded latitudinal domain simply means that we are interested only in the WBL solution, so we do not take into account possible boundary layers associated with zonal boundaries.

Hereafter we assume

$$\delta_{\mathrm{S}} = \delta_{\mathrm{M}} \equiv \delta_F\,, \tag{7}$$

$$\left(\frac{\delta_{\mathrm{I}}}{\delta_F}\right)^2 < 1\,, \tag{8}$$

$$\left(\frac{\delta_F}{L}\right)^2 \ll 1\,, \tag{9}$$

and define the boundary layer coordinate ξ of the western boundary through the equation

$$\delta_F\,\xi = L(x - x_{\mathrm{W}})\,, \tag{10}$$

so that

$$\frac{\partial}{\partial x} = \frac{L}{\delta_F}\frac{\partial}{\partial\xi} \tag{11}$$

and so on. In the WBL, transform (3) becomes $(\xi, y) \to (\xi, 2\theta - y)$.

For x close to x_{W}, the total solution ψ can be written as

$$\psi = \psi_{\mathrm{I}}(x, y) + \phi_{\mathrm{W}}(\xi, y)\,, \tag{12}$$

where $\phi_W(x, y)$ is the WBL correction while ψ_I satisfies the Sverdrup balance (2), ψ_I being the interior streamfunction. Note that $\psi_I(x, y) = -\int_x^{x_E} \nu_I(\eta, y)\, d\eta$ is symmetric in the presence of an antisymmetric wind stress. Substitution of position (12) into eq. (4) gives, to the zeroth-order in $(\delta_F/L)^2$ and recalling hypothesis (7),

$$\left(\frac{\delta_I}{\delta_F}\right)^2 \left[J_{\xi y}\left(\phi_W, \frac{\partial^2}{\partial \xi^2}\phi_W\right) - \frac{\partial \psi_I}{\partial y}\frac{\partial^3 \phi_W}{\partial \xi^3}\right] + \frac{\partial \phi_W}{\partial \xi} = -\frac{\partial^2 \phi_W}{\partial \xi^2} + \frac{\partial^4 \phi_W}{\partial \xi^4}\,. \tag{13}$$

The Jacobian determinant $J_{\xi y}$ operates on the variables ξ and y.

Because of inequality (8), ϕ_W can be expanded as

$$\phi_W = \phi^{(0)} + \left(\frac{\delta_I}{\delta_F}\right)^2 \phi^{(1)} + \dots\,. \tag{14}$$

By substituting expansion (14) into eq. (13) and equating like order terms in $(\delta_I/\delta_F)^{2n}$, the equations for $\phi^{(0)}, \phi^{(1)} \dots$ can be found. Boundary conditions are the following:

$$\psi_I(x_W, y) + \phi^{(0)}(0, y) = 0\,, \tag{15}$$

or

$$\phi^{(n)}(0, y) = 0\,, \qquad \text{if } n \geqslant 1\,; \tag{16}$$

$$\lim_{\xi \to +\infty} \phi^{(n)}(\xi, y) = 0\,, \qquad n \geqslant 0\,; \tag{17}$$

$$\frac{\partial}{\partial \xi}\phi^{(n)} = 0 \quad \text{in } \xi = 0\,, \qquad n \geqslant 0\,. \tag{18}$$

Boundary conditions (15) and (16) easily follow from eqs. (5), (12) and expansion (14). Boundary condition (17) expresses the matching condition between the WBL solution and the interior solution far from the western coast (recall position (12)) and finally boundary condition (18) comes from the corresponding boundary condition (6), eq. (11) and inequality (9).

Putting

$$L_\xi = -\frac{\partial}{\partial \xi} - \frac{\partial^2}{\partial \xi^2} + \frac{\partial^4}{\partial \xi^4}\,,$$

the problem for $\phi^{(0)}$ is given by the equation

$$L_\xi \phi^{(0)} = 0 \tag{19}$$

plus boundary conditions (15) and (17), (18) with $n = 0$.

In the same way the problem for $\phi^{(1)}$ is given by the equation

$$J_{\xi y}\left(\phi^{(0)}, \frac{\partial^2}{\partial \xi^2}\phi^{(0)}\right) - \frac{\partial \psi_I}{\partial y}\frac{\partial^3 \phi^{(0)}}{\partial \xi^3} = L_\xi \phi^{(1)} \tag{20}$$

plus boundary conditions (16) and (17), (18) with $n = 1$.

Using the identiy

$$J_{xy}\left(\sum_{n\geqslant 0}\alpha^n f^{(n)}, \sum_{k\geqslant 0}\alpha^k g^{(k)}\right) = \sum_{l\geqslant 0}\alpha^l\left[\sum_{n+k=l} J(f^{(n)}, g^{(k)})\right],$$

where α is a real parameter and $f^{(n)}$, $g^{(k)}$ are functions of x and y; it follows that, in general, the problem for $\phi^{(l+1)}$ is given by the equation

$$\sum_{n+k=l} J_{\xi y}\left(\phi^{(n)}, \frac{\partial^2}{\partial\xi^2}\phi^{(k)}\right) - \frac{\partial\psi_{\mathrm{I}}}{\partial y}\frac{\partial^3\phi^{(l)}}{\partial\xi^3} = L_\xi\phi^{(l+1)}, \qquad l\geqslant 0\,, \tag{21}$$

plus boundary conditions (16), (17) and (18). In this way we have deduced for every $l\geqslant 0$ the problem for the individual WBL corrections.

3. – The parity-transformed corrections

In this section we prove the main result, that is the relation

$$\phi^{(m)}(\xi, 2\theta - y) = (-)^m\phi^{(m)}(\xi, y)\,, \qquad m\geqslant 0\,. \tag{22}$$

First, we prove that relation (22) holds for $m=0$ and $m=1$. Then, assuming its validity for $m\leqslant l$, we show that it is true also for $m=l+1$.

Relation (22) for $m=0$ can be directly checked by resorting to the explicit solution of problem (15), (17), (18) and (19). The general integral of eq. (19) is

$$\phi^{(0)} = C_1(y) + C_2(y)\,e^{(S+T)\xi} + C_3(y)\,e^{-(S+T)\frac{\xi}{2}}\sin\left(\frac{\sqrt{3}}{2}(S-T)\,\xi\right) + \\ + C_4(y)\,e^{-(S+T)\frac{\xi}{2}}\cos\left(\frac{\sqrt{3}}{2}(S-T)\,\xi\right), \tag{23}$$

where

$$S = \left(\frac{1}{2} + \left(\frac{23}{108}\right)^{1/2}\right)^{1/3} \qquad \text{and} \qquad T = \left(\frac{1}{2} - \left(\frac{23}{108}\right)^{1/2}\right)^{1/3}.$$

Boundary condition (17) demands $C_1(y) = C_2(y) = 0$, while boundary condition (18) implies

$$C_3(y) = \frac{S+T}{\sqrt{3}(S-T)}\,C_4(y)\,. \tag{24}$$

Finally boundary condition (15) gives

$$\psi_{\mathrm{I}}(x_{\mathrm{W}}, y) + C_4(y) = 0 \tag{25}$$

so both functions $C_3(y)$ and $C_4(y)$ can be expressed via $\psi_{\mathrm{I}}(x_{\mathrm{W}}, y)$ and we have

$$\phi^{(0)}(\xi, y) = \tag{26}$$

$$= -\psi_{\mathrm{I}}(x_{\mathrm{W}}, y)\, e^{-(S+T)\frac{\xi}{2}} \left[\frac{S+T}{\sqrt{3}(S-T)} \sin\left(\frac{\sqrt{3}}{2}(S-T)\,\xi \right) + \cos\left(\frac{\sqrt{3}}{2}(S-T)\,\xi \right) \right].$$

At this point, from the invariance of ψ_{I} under transform (3), the same invariance immediately follows for $\phi^{(0)}$:

$$\phi^{(0)}(\xi, 2\theta - y) = \phi^{(0)}(\xi, y)\,. \tag{27}$$

About relation (22) for $m = 1$, we observe that transform (3) applied to eq. (20) gives

$$-\left[J_{\xi y}\left(\phi^{(0)}, \frac{\partial^2}{\partial \xi^2}\phi^{(0)} \right) - \frac{\partial \psi_{\mathrm{I}}}{\partial y} \frac{\partial^3 \phi^{(0)}}{\partial \xi^3} \right] = L_\xi \phi^{(1)}(\xi, 2\theta - y)\,, \tag{28}$$

where use has been made of the invariance of $\phi^{(0)}$ and ψ_{I} and the antisymmetry of the y-derivative. Therefore, addition of eq. (20) with eq. (27) gives

$$L_\xi[\phi^{(1)}(\xi, y) + \phi^{(1)}(\xi, 2\theta - y)] = 0\,. \tag{29}$$

Putting

$$s(\xi, y) = \phi^{(1)}(\xi, y) + \phi^{(1)}(\xi, 2\theta - y)\,, \tag{30}$$

the boundary conditions for s easily follow from those of $\phi^{(1)}$, that is to say

$$s(0, y) = 0\,, \tag{31}$$

$$\lim_{\xi \to +\infty} s(\xi, y) = 0\,, \tag{32}$$

$$\frac{\partial s}{\partial \xi} = 0 \quad \text{in } \xi = 0\,. \tag{33}$$

In general, the integral of eq. (29) coincides with the r.h.s. of eq. (23). Again, boundary condition (32) demands $C_1(y) = C_2(y) = 0$ and boundary condition (33) implies $C_3(y) = \big((S+T)/\sqrt{3}(S-T)\big)C_4(y)$. The basic difference with respect to the problem for $\phi^{(0)}$ is that, now, boundary condition (31) is satisfied by $C_4(y) = 0$, so problem (29), (31), (32), (33) has only the null solution $s(\xi, y) = 0$, that is, recalling position (30),

$$\phi^{(1)}(\xi, 2\theta - y) = -\phi^{(1)}(\xi, y)\,. \tag{34}$$

The antisymmetry of $\phi^{(1)}$ is thus proved.

To prove relation (22) for $m \geqslant 2$ consider transform (3) applied to eq. (21):

$$-\sum_{n+k=l} J_{\xi y}\left(\phi^{(n)}(\xi, 2\theta - y), \frac{\partial^2}{\partial \xi^2}\phi^{(k)}(\xi, 2\theta - y)\right) + \frac{\partial \psi_{\mathrm{I}}}{\partial y}\frac{\partial^3}{\partial \xi^3}\phi^{(l)}(\xi, 2\theta - y) = L_\xi \phi^{(l+1)}(\xi, 2\theta - y). \tag{35}$$

Now assume l even and relation (22) for $m \leqslant l$. In this case eq. (35) takes the form

$$-\sum_{n+k=l} J_{\xi y}\left(\phi^{(n)}(\xi, y), \frac{\partial^2}{\partial \xi^2}\phi^{(k)}(\xi, y)\right) + \frac{\partial \psi_{\mathrm{I}}}{\partial y}\frac{\partial^3}{\partial \xi^3}\phi^{(l)}(\xi, y) = L_\xi \phi^{(l+1)}(\xi, 2\theta - y). \tag{36}$$

Addition of eq. (21) to eq. (36) gives

$$L_\xi[\phi^{(l+1)}(\xi, y) + \phi^{(l+1)}(\xi, 2\theta - y)] = 0\,. \tag{37}$$

Following the same way as for $\phi^{(1)}$, we can easily check that eq. (37) with boundary conditions (16), (17), (18) has only the null solution, that is to say

$$\phi^{(l+1)}(\xi, 2\theta - y) = -\,\phi^{(l+1)}(\xi, y)\,, \qquad \text{for } l+1 \text{ odd}\,. \tag{38}$$

Finally, assume l odd and relation (22) for $m \leqslant l$. Now eq. (35) takes the form

$$\sum_{n+k=l} J_{\xi y}\left(\phi^{(n)}(\xi, y), \frac{\partial^2}{\partial \xi^2}\phi^{(k)}(\xi, y)\right) - \frac{\partial \psi_{\mathrm{I}}}{\partial y}\frac{\partial^3}{\partial \xi^3}\phi^{(l)}(\xi, y) = L_\xi \phi^{(l+1)}(\xi, 2\theta - y). \tag{39}$$

Subtraction of eq. (39) from eq. (21) gives

$$L_\xi[\phi^{(l+1)}(\xi, y) - \phi^{(l+1)}(\xi, 2\theta - y)] = 0\,. \tag{40}$$

Equation (40) with boundary conditions (16), (17), (18) has only the null solution, that is to say

$$\phi^{(l+1)}(\xi, 2\theta - y) = \phi^{(l+1)}(\xi, y) \qquad \text{for } l+1 \text{ even}\,. \tag{41}$$

According to the mathematical induction principle, we conclude that relation (22) holds for every integer m.

4. – Discussion and concluding remarks

Equation (22) is obtained with assumption (7) that corresponds to a well-defined, particular frictional parametrization, that is $-(\delta_F/L)\,\nabla^2\psi + (\delta_F/L)^3\,\nabla^4\psi$. However, if bottom dissipation only is taken into account, *i.e.* $\delta_{\mathrm{S}} = \delta_F$ and $\delta_{\mathrm{M}} = 0$ or if lateral diffusion of relative vorticity only is considered, *i.e.* $\delta_{\mathrm{S}} = 0$ and $\delta_{\mathrm{M}} = \delta_F$, then equation (22) still holds. In fact, if $\delta_{\mathrm{M}} = 0$ we have only to redefine the operator L_ξ as $L_\xi = -\partial/\partial\xi - \partial^2/\partial\xi^2$, while if $\delta_{\mathrm{S}} = 0$, then we put $L_\xi = -\,\partial/\partial\xi + \partial^4/\partial\xi^4$. Obviously, if $\delta_{\mathrm{M}} = 0$, boundary conditions (6), (18), (33) must be released but, in any case, the method to prove eq. (22) is left unchanged. So, what was observed in sect. **1** about the approximated solutions of Munk *et al.*, 1950, and Veronis, 1966, in the WBL can be immediately fit into our result (22).

The actual streamlines coming from expansions (12) and (14) can be imagined as the superposition of virtual paths, each produced by the single terms $\phi^{(k)}$ of the expansion. Unlike the linear solution $\psi = \psi_{\mathrm{I}} + \phi^{(0)}$, the so-obtained overall solution is able to take into account both the northward splitting of the gyre due to inertial effects and the dissipation of the surplus of relative vorticity acquired in the WBL, by means of the formation of meanders in the North-western region of the basin. Relation (22) gives some features of the current field associated with the different streamfunctions $\phi^{(k)}$. For m odd and $y = \theta$, relation (22) implies $\phi^{(m)}(\xi, \theta) = 0$. This is equivalent to say that $y = \theta$ is a stagnation line for the antisymmetrical streamfunctions. Note that the point $(0, \theta)$ belongs both to the coastline $\xi = 0$ where the current is identically vanishing because of boundary conditions (5), (6) and to the line $y = \theta$; the horizontally nondivergent nature of the flow implies that no current flows along the line $y = \theta$. Therefore, the northward transport in the WBL at the latitude θ is entirely due to the terms of the expansions (12) and (14) which are invariant under transform (3).

From relation (22) the following relation for the zonal current $u^{(m)} = -\partial\phi^{(m)}/\partial y$ can be easily deduced

$$u^{(m)}(\xi, \theta - y) = (-)^{m+1} u^{(m)}(\xi, \theta + y). \tag{42}$$

For m even, eq. (42) shows that the flux incoming into the WBL below the latitude θ is outgoing above the same latitude, with a symmetrical latitudinal distribution.

For m odd, eq. (42) exhibits the presence of two identical zonal currents, both incoming into the WBL or outgoing from it, arranged in a symmetrical way with respect to the latitude θ. Since, for m odd, $y = \theta$ is a stagnation line, these zonal currents belong to two different virtual current systems, each being confined in one of the half planes $y > \theta$, $y < \theta$.

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