From the Maxwell's equations to the String Theory: new possible mathematical connections

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Abstract

*In this paper in the **Section 1**, we describe the possible mathematics concerning the unification between the Maxwell's equations and the gravitational equations. In this Section we have described also some equations concerning the gravitomagnetic and gravitoelectric fields. In the **Section 2**, we have described the mathematics concerning the Maxwell's equations in higher dimension (thence Kaluza-Klein compactification and relative connections with string theory and Palumbo-Nardelli model). In the **Section 3**, we have described some equations concerning the noncommutativity in String Theory, principally the Dirac-Born-Infeld action, noncommutative open string actions, Chern-Simons couplings on the brane, D-brane actions and the connections with the Maxwell electrodynamics, Maxwell's equations, B-field and gauge fields. In the **Section 4**, we have described some equations concerning the noncommutative quantum mechanics regarding the particle in a constant field and the noncommutative classical dynamics related to quadratic Lagrangians (Hamiltonians) connected with some equations concerning the Section 3. In conclusion, in the **Section 5**, we have described the possible mathematical connections between various equations concerning the arguments above mentioned, some links with some aspects of Number Theory (Ramanujan modular equations connected with the phisycal vibrations of the superstrings, various relationships and links concerning π, ϕ thence the Aurea ratio), the zeta strings and the Palumbo-Nardelli model that link bosonic and fermionic strings.*

Introduction

James Clerk Maxwell

Maxwell's Equations are an important connection between mathematics and physics, an instrument without which we would not have reached the present state of knowledge such as relativity, quantum mechanics and string theory.

These equations are of extraordinary importance, especially from a physical point of view, as well as mathematics.
The authors, in this paper, show how to treat them from a mathematical point of view with techniques of vector calculus.

In the second part the authors examine, with the method of the conceptual experiments and the "umbral calculus", a "symmetrization" of these equations and we have the question of the correctness of the approach and the existence of phenomena that should be linked then this fact. All this is done without introducing the Minkowski space and the equations of Einstein-Maxwell, but keeping for simplicity only on Maxwell's equations.

Another basic question that accompanies the whole paper is the possibility or not to create anti-gravity and propulsion systems, in this regard we have mentioned the EHT theory, a theory to unify relativity and quantum mechanics, that is multi-dimensional (8 dimensions) and that can be related with M-theory and superstring theory.

1. On the possibile mathematics concerning the unification between the Maxwell’s equations and the gravitational equations.

Maxwell's equations

Maxwell's equations are:

Law	Equation
Gauss's law of electric field	$\nabla \bullet \vec{E} = \frac{\rho}{\varepsilon_0}$ (1)
Gauss's law of magnetic field	$\nabla \bullet \vec{B} = 0$ (2)
Faraday’s law on the electric field	$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ (3)
Ampere’s law on the magnetic field	$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$ (4)

where:

∇ it’s is the mathematical symbol "nabla" (gradient)
$\nabla \bullet$ it’s the mathematical symbol for the divergence
$\nabla \times$ it’s is the mathematical symbol of the rotor

Then:

$\vec{E}$ the electric field
$\vec{B}$ the magnetic induction
$\vec{H}$ the magnetic field
$\vec{J}$ the current density
ε_0 the dielectric constant of vacuum 8,854188 x 10^{-12} F/m
μ_0 the magnetic permeability of vacuum 4π x 10^{-7} H/m
c_0 the speed of light 3 x 10^{8} m/s

Some useful rule for the follow:

- The divergence of a rotor is always zero or $\nabla \bullet \left(\nabla \times \vec{C}\right) = 0$
- The rotor of a gradient is always zero or $\nabla \times (\nabla C) = 0$
- The divergence of a gradient is the Laplacian or $\nabla \bullet (\nabla C) = \nabla^2 C$
- The rotor of a rotor instead is $\nabla \times \left(\nabla \times \vec{C}\right) = \nabla\left(\nabla \bullet \vec{C}\right) - \nabla^2 \vec{C}$

Maxwell's equations in vacuum

We start from equation (4). We divide both sides by μ_0 and bearing in mind that:

$$\frac{1}{c_0^2} = \mu_0 \varepsilon_0$$

where c_0 is the speed of light, we obtain:

$$c_0^2 \nabla \times \vec{B} = \frac{\vec{J}}{\varepsilon_0} + \frac{\partial \vec{E}}{\partial t} \qquad (4')$$

Recall that the Lorentz's force is given by:

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) \qquad (5)$$

The equations, including (4 ') and (5), are the equations in a local form and permit the calculation of electromagnetic fields in vacuum from known values of charge density ρ and current density J.

Maxwell's equations in materials

If we are inside of the materials then the electromagnetic waves must take account of other physical phenomena due to the electric induction D, the polarization P, the magnetic induction M such that:

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$
$$\vec{B} = \mu_0\left(\vec{H} + \vec{M}\right)$$

P and M are the mean value of electric and magnetic dipole per unit volume.

If we consider the medium as linear and isotropic material, the equations are simplified in:

$$\begin{aligned} \vec{D} &= \varepsilon_0 \varepsilon_r \vec{E} \\ \vec{B} &= \mu_0 \mu_r \vec{H} \qquad (6) \\ \mu &= \mu_0 \mu_r \\ \varepsilon &= \varepsilon_0 \varepsilon_r \end{aligned}$$

Thence, we obtain:

Law	Equation
Gauss's law of electric field	$\nabla \bullet \vec{D} = \rho$ (1')
Gauss's law of magnetic field	$\nabla \bullet \vec{B} = 0$ (2)
Faraday's law on the electric field	$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ (3)
Ampere's law on the magnetic field	$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$ (4'')

where

μ_r is the relative magnetic permeability

The eq. (3) can lead to something more interesting through (6) and the precedent rules :

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\nabla\times\left(\nabla\times\vec{E}\right) = -\mu_0\mu_r\frac{\partial}{\partial t}\left(\nabla\times\vec{H}\right) = -\mu\frac{\partial}{\partial t}\left(\vec{J}+\frac{\partial\vec{D}}{\partial t}\right) = -\mu\frac{\partial}{\partial t}\left(\vec{J}+\varepsilon\frac{\partial\vec{E}}{\partial t}\right) = -\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}$$

thence, we obtain:

$$\nabla^2\vec{E} = -\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} \qquad (7)$$

In the vacuum ρ=0, J=0 and $\mu\varepsilon = \frac{1}{v^2}$ with v that is the speed of light , then we obtain that:

$$\nabla^2\vec{E} = \nabla\left(\nabla\bullet\vec{E}\right) - \nabla\times\left(\nabla\times\vec{E}\right) = \frac{1}{v^2}\frac{\partial^2\vec{E}}{\partial t^2} \qquad (8)$$

which is the *D'Alembert wave equation* in vacuum (v = c_0). For the vector B we can use a similar procedure.

Vector potential and scalar potential

If in (2) the divergence is zero using the fact that the divergence of a rotor always gives a null result, then there exists a *vector potential* A, such that:

$$\vec{B} = \nabla\times\vec{A}$$

$$\nabla\bullet\vec{B} = 0 \qquad (9)$$

From (3) thence, we have that:

$$\nabla\times\vec{E} = -\frac{\partial}{\partial t}\nabla\times\vec{A}$$

$$\nabla\times\vec{E} + \frac{\partial}{\partial t}\nabla\times\vec{A} = 0$$

$$\nabla\times\left(\vec{E}+\frac{\partial}{\partial t}\vec{A}\right) = 0$$

With the last expression, using the rule that the rotor of a gradient is always zero, then, introducing a *scalar potential* ϕ is:

$$\vec{E}+\frac{\partial}{\partial t}\vec{A} = -\nabla\phi$$

thence, we have that:

$$\vec{E} = -\left(\nabla\phi+\frac{\partial}{\partial t}\vec{A}\right) \qquad (10)$$

Utilizing the eqs. (9),(10) we can rewrite the equations (1),(4) as follows:

$$\nabla \bullet \vec{E} = -\nabla \bullet \left(\nabla \phi + \frac{\partial}{\partial t} \vec{A} \right) = -\left(\nabla^2 \phi + \frac{\partial}{\partial t} \vec{A} \right) = \frac{\rho}{\varepsilon_0}$$

It’s equivalent to:

$$\nabla^2 \phi + \frac{\partial}{\partial t} \vec{A} = -\frac{\rho}{\varepsilon_0} \qquad (11)$$

From (4), instead, is:

$$\nabla \times \nabla \times \vec{A} = \mu_0 \vec{J} - \frac{1}{c_0^2} \frac{\partial}{\partial t} \left(\nabla \phi + \frac{\partial}{\partial t} \vec{A} \right)$$

$$c_0^2 \left(\nabla \times \nabla \times \vec{A} \right) = \frac{\vec{J}}{\varepsilon_0} - \frac{\partial}{\partial t} \left(\nabla \phi + \frac{\partial}{\partial t} \vec{A} \right)$$

$$-c_0^2 \nabla^2 \vec{A} + c_0^2 \nabla \left(\nabla \bullet \vec{A} \right) + \frac{\partial}{\partial t} \nabla \phi + \frac{\partial^2 \vec{A}}{\partial t^2} = \frac{\vec{J}}{\varepsilon_0} \qquad (12)$$

Gauge transformations

Let's see what happens if on the vector potential A and scalar potential ϕ are made changes like:

$$\vec{A} \rightarrow \vec{A} + \nabla \psi$$

$$\phi \rightarrow \phi - \frac{\partial \psi}{\partial t}$$

Considering that the divergence of a rotor is null and that the rotor of a gradient is zero, then the equations of vectors B and E do not change (see (9) and (10)). So we speak of *gauge invariance*.

We can use gauge invariance and choose the vector A as appropriate, for example:

$$\nabla \bullet \vec{A} = -\frac{1}{c_0^2} \frac{\partial \phi}{\partial t}$$

Now with (1) and (10) we obtain that:

$$\vec{E} = -\left(\nabla \phi + \frac{\partial}{\partial t} \vec{A} \right)$$

$$\nabla \bullet \vec{E} = -\nabla \bullet \left(\nabla \phi + \frac{\partial}{\partial t} \vec{A} \right) = -\nabla^2 \phi - \frac{\partial}{\partial t} \nabla \vec{A} = -\nabla^2 \phi + \frac{1}{c_0^2} \frac{\partial^2 \phi}{\partial t^2} = \frac{\rho}{\varepsilon}$$

so

$$\nabla^2 \phi - \frac{1}{c_0^2} \frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\varepsilon} \qquad (13)$$

If we substitute in (12) we obtain:

$$\nabla^2 \vec{A} - \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J} \qquad (14)$$

Now (13) and (14) constitute a system of 4 equations or a *four-vector* that describes the waves advancing in space-time with speed c_0. In fact in (14) the vectors A and J can be decomposed into the components in the direction x, y, z.

In fact, you solve the Maxwell equations by introducing a vector potential and a scalar potential, then gauge invariance is exploited by reducing everything to a system of differential equations in four scalar functions as follows:

$$\nabla^2\phi - \frac{1}{c_0^2}\frac{\partial^2\phi}{\partial t^2} = -\frac{\rho}{\varepsilon}$$

$$\nabla^2 A_x - \frac{\partial^2 A_x}{\partial t^2} = -\mu_0 J_x$$

$$\nabla^2 A_y - \frac{\partial^2 A_y}{\partial t^2} = -\mu_0 J_y$$

$$\nabla^2 A_z - \frac{\partial^2 A_z}{\partial t^2} = -\mu_0 J_z$$

These equations put in evidence that the behavior of electromagnetic waves and of light were aspects of the same phenomenon. So far the classical theory.

With regard the problem of symmetrization of the equations.

Following Heim and Hauser (see [1]), we start our simplistic flight of fancy, a little as Maxwell, who discovered from a mathematical point of view that missing something to the Ampere's law and introduced the temporal variation of electric induction D.

The aim is to see if we can obtain the symmetrization of equations, adding the terms used and that in practice are negligible, and introduce a *formulation of linearized gravity*, without necessarily introducing the *Minkowski space* and the *equations of Einstein-Maxwell*, but starting simply by classical equations of Maxwell.

If we look at (1 ') (2) (3) (4''), described below for convenience, we note that, in "a sense", these are two by two similar, but between (1') and (2) there is an obvious "symmetry breaking", this at least from a formal mathematical point of view:

$$\nabla \bullet \vec{D} = \rho \qquad (1')$$

$$\nabla \bullet \vec{B} = 0 \qquad (2)$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad (3)$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \qquad (4'')$$

In (2), for example, there isn't a "density of magnetic charge" ρ_m as in (1 '), while in (3) compared to (4") there isn't a corresponding "density of magnetic current" J_m.

Furthermore between (3) and (4") are opposite also the signs of the partial derivatives.

In the reality should be borne in mind that a magnetic charge is non-existent, but in a fantasy of symmetrization, especially for a conceptual experiment and "umbral calculus", leads us to say that the equations (2) and (3), which are true compared with experiments, are such because the boundary conditions involving the negligible or null terms as "density of magnetic charge" and "density of magnetic current".

We start writing a first form of equations of symmetrization:

$$\nabla \bullet \vec{D} = \rho \qquad (1')$$

$$\nabla \bullet \vec{B} = \rho_m \qquad (2a)$$

$$\nabla \times \vec{E} = \overrightarrow{J_m} - \frac{\partial \vec{B}}{\partial t} \qquad (3a)$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \qquad (4'')$$

where in the (2a) (3a) we have introduced the density missing.

Can we add the gravity strength to the Maxwell's equations?

We begin another flight of fancy: it is possible to envisage the introduction of gravity in these equations? Or at least an effect of fields that interacts with it?

We know that there exist an analogy between the *Coulomb's law* and the *Gravity's law*; the Coulomb's law expresses the interaction between two electric charges q_1 and q_2:

$$F = \frac{1}{4\pi\varepsilon_0} \frac{q_1 \cdot q_2}{r^2}$$

The *strength of gravity* expresses the interaction between two masses m_1 and m_2:

$$F = G \frac{m_1 \cdot m_2}{r^2}$$

where G = 6,670 x 10^{-11} Nm^2Kg^{-2}

In correspondence of the Coulomb's strength we have an electric field expressible by:

$$E = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2}$$

The gravitational field can be expressed with:

$$X_g = G \frac{m}{r^2} \qquad (15)$$

Can we create an electric field equivalent to a gravitational field? If we equate the two fields we see that we must introduce a "charge equivalent to the mass":

$$\frac{1}{4\pi\varepsilon_0} q = G \cdot m \rightarrow q_{eq} = 4\pi\varepsilon_0 G \cdot m$$

so:

$$\begin{aligned} q_{eq} &= k_g m \\ k_g &= 4\pi\varepsilon_0 G \end{aligned} \qquad (16)$$

g= 6.67422 x 10^{-11} m^3/s^2 kg

We would also include a charge density equivalent to the mass ρ_{em} through a mass density per unit volume ρ_{mv}:

$$\rho_{em} = k_g \rho_{mv} \qquad (17)$$

Furthermore, if we want to write, with regard the gravity, something of equivalent to (1 ') we must to consider a parameter $\eta_0 = \varepsilon_0$ such that:

$$\eta_0 = \frac{k_g}{4\pi G} \qquad (18)$$

The (15), then, could be written as follows:

$$X_g = \frac{1}{4\pi\eta_0} \frac{q_{eq}}{r^2} \qquad (19)$$

At this point for the gravity you need to locate an equivalent of $\vec{D} = \varepsilon_0 \vec{E}$ (for linear and isotopic materials), that is $\vec{R} = \eta_0 \overrightarrow{X_g}$; thence the equivalent of (1') for the gravity would be:

$$\nabla \bullet \vec{R} = k_g \rho_{mv} = \rho_{em} \qquad (20)$$

while:

$$\nabla \times \vec{X} = 0 \qquad (21)$$

The (21) is a result of the fact that the gravitational field should be irrotational or conservative, but we will see that we have to work again on this up and perhaps discover news unexpected.

Physical correctness of the idea.

Is it correct from the physical point of view, what we have obtained from a formal point of view? In practice, because of the similarity of the fields, it was said that you could get an electric field equivalent to gravity (at least in terms of strength).

What are the differences? In the Coulomb strength we can have both positive and negative electric charges and strengths of attraction if the charges are of a different sign, repulsive if the charges are of equal sign, while in the gravitational strength we have only positive mass and the gravitational strength is only attractive.

However, isn't the direction of the force, which reduces or invalidates the idea, the direction must be different, but it is certainly true that "a flow of charges is also a flow of masses", so there are electric currents and current of masses, the current of masses are in this case both sources of gravity field and of electric and magnetic field. So the only valid physical sense here is that we are introducing fields equivalent to gravitational fields, or *gravito-electric and gravito-magnetic fields*.

We try to unify the equations [1b]

Now suppose that we could put together all the equations are those of Maxwell, or to obtain an unification of electromagnetic and gravitational equations ensuring the symmetrization and the signs. How to proceed with a hypothetical unification of equations?

It involves:

- to introduce the terms of connection:
  - ✓ the "current density of mass displacement", represented by the partial derivative in time of R;
  - ✓ the "current density mass" J_k;
- to introduce a constant γ_0 for the dimensional consistency of the physical dimensions
- maintain a symmetry of the equations and signs (X should have a formal appearance as similar to E ¸ E and H must have, with opposite signs, contributions of link because the behaviour in the classical Maxwell's equations)

We get a unified system of six equations:

$$\nabla \bullet \vec{D} = \rho \qquad (1')$$

$$\nabla \bullet \vec{B} = \rho_m \qquad (2a)$$

$$\nabla \bullet \vec{R} = k_g \rho_{mv} = \rho_{em} \qquad (2b)$$

$$\nabla \times \vec{X} = k_g (\vec{J_m} + \frac{\partial \vec{B}}{\partial t}) - k_g \gamma_0 (\vec{J} + \frac{\partial \vec{D}}{\partial t}) \qquad (2c)$$

$$\nabla \times \vec{E} = \gamma_0 (\vec{J_k} + \frac{\partial \vec{R}}{\partial t}) - \vec{J_m} - \frac{\partial \vec{B}}{\partial t} \qquad (3a)$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} - \vec{J_k} - \frac{\partial \vec{R}}{\partial t} \qquad (4''')$$

$$\gamma_0 = \frac{1}{4\pi\varepsilon_0 c_0} = \frac{\mu_0}{4\pi c_0}$$

The verification of laws [1b]

After the formal-mathematical symmetrization and unification we must verify that the equations do not violate the laws of conservation of charge and irrotational fields. The previous equations, in the absence of currents of mass and time-varying gravity field, will be such that J_k=0 and $\frac{\partial \vec{R}}{\partial t}$ = 0; the variations are due to movement of mass. An electric current is either a flow of charge but also of mass.

Conservation of total charge

Take the (4'' ') and calculate the divergence of a rotor (that is null). We obtain that:

$$\nabla \bullet \left(\nabla \times \vec{H}\right) = \nabla \bullet \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} - \vec{J}_k - \frac{\partial \vec{R}}{\partial t} \right) = \nabla \bullet \vec{J} + \frac{\partial}{\partial t} \nabla \bullet \vec{D} - \nabla \bullet \vec{J}_k - \frac{\partial}{\partial t} \nabla \bullet \vec{R} = 0$$

From (1') (2b), we have that:

$$\nabla \bullet \vec{J} + \frac{\partial \rho}{\partial t} - \nabla \bullet \vec{J}_k - \frac{\partial \rho_{em}}{\partial t} = 0$$

If we pass to the integral of surface, obtain the law of conservation of charge:

$$I - I_k - = \frac{\partial Q_{em}}{\partial t} - \frac{\partial Q}{\partial t} \quad (22)$$

It 's clear that in cases where J_k=0 and $\frac{\partial \vec{R}}{\partial t}$ = 0 we get the actual law of conservation of charge $I = -\frac{\partial Q}{\partial t}$ that we obtain from (4''), and that represents an equation of continuity.

In the event that we wanted to consider all the terms of (22) we observe that the flow of electrons is made up of electronic mass m = 9,1091 x 10^{-31} Kg, while the constant $4\pi\varepsilon_0 G$ is 1,8544 x 10^{-21} Kg^{-3} $m^{-1}s^2C^{2N}$ for which the mass flow of electrons is of the order of 10^{-51}. If we assume the presence only of the magnetostatic field, therefore only no time-varying magnetic field and in the absence of electric charges, then only J_k is different from zero and $I_k = -\frac{\partial Q_{em}}{\partial t}$; but also here it is negligible, because taking $4\pi\varepsilon_0 G$, a mass of 1 kg per second causes a current 6.18 x 10^{-21} A.

Apparent irrotational electric field?

In particular, the (3a) would seem to violate the irrotational electric field, saying that one has an electric field perpendicular to the mass movement and arranged in a circle around the mass. In fact the irrotational electric field in a magnetostatic field is valid, this is because that the mass flow is significant is need to achieve enormous amounts like 10^{12} tons.

New phenomenon?

The (2c) says that in the presence of non static electric and magnetic fields, then the gravitational field is not irrotational, taking into account that there is no a magnetic current density (Jm = 0). In fact from what has so far suggest to us the formulas and calculations, the mass flows very little influence on electromagnetic fields, but it is certainly possible otherwise. An idea, therefore, is to exploit the (2c) to create the opposite strengths to the gravitation, or to obtain an electromagnetic levitation (not just the antigravity! But we are very close like effect).

1.1 On some equations concerning the Gravitoelectric and Gravitomagnetic fields [1]

Considering the Einstein-Maxwell formulation of linearized gravity, a remarkable similarity to the mathematical form of the electromagnetic Maxwell equations can be found. In analogy to electromagnetism there exist a gravitational scalar and vector potential, denoted by Φ_g and A_g, respectively. Introducing the gravitoelectric and gravitomagnetic fields

$$e = -\nabla\Phi_g \quad \text{and} \quad b = \nabla\times A_g \quad (23)$$

the gravitational Maxwell equations can be written in the following form:

$$\nabla\cdot e = -4\pi G\rho, \quad \nabla\cdot b = 0, \quad \nabla\times e = 0, \quad \nabla\times b = -\frac{16\pi G}{c^2} j \quad (24)$$

where $j = \rho v$ is the mass flux and G is the gravitational constant. The field e describes the gravitational field form a stationary mass distribution, whereas b describes an extra gravitational field produced by moving masses.
At critical temperature T_c some materials become superconductors that is, their resistance goes to 0. Superconductors have an energy gap of some $E_g \approx 3.5kT_c$. This energy gap separates superconducting electrons below from normal electrons above the gap. At temperatures below T_c, electrons (that are **fermions**) are coupled in pairs, called Cooper pairs, which are **bosons**.
<u>We note that, in this case, there is the application of the relationship of Palumbo-Nardelli model, i.e.</u>

$$-\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]=$$
$$=\int_0^\infty\frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{\kappa_{10}^2}{g_{10}^2}Tr_\nu\left(\left|F_2\right|^2\right)\right], \quad (25)$$

<u>a general relationship that links bosonic and fermionic strings acting in all natural systems</u>.
A rotating superconductor generates a magnetic induction field, the so called London moment

$$B = -\frac{2m_e}{e}\omega \quad (26)$$

where ω is the angular velocity of the rotating ring and e denotes elementary charge. It should be noted that the magnetic field in Tajmar's experiment [1] is produced by the rotation of the ring, and not by a current of Cooper pairs that are moving within the ring.
Tajmar and his colleagues simply postulate an equivalence between the generated B field, eq. (26) with a gravitational field by proposing a so called *gravitomagnetic London effect*.

Let R denote the radius of the rotating ring, then eq. (26) puts a limit on the maximal allowable magnetic induction, $B_{\max}$, which is given by

$$B_{\max}^2 = 14 \frac{m_e}{e^2} \frac{k_B T_C}{R^2}. \quad (27)$$

If the magnetic induction exceeds this value, the kinetic energy of the Cooper pairs exceeds the maximum energy gap, and the Cooper pairs are destroyed. The rotating ring is no longer a superconductor. Moreover, the magnetic induction must not exceed the critical value $B_c(T)$, which is the maximal magnetic induction that can be sustained at temperature T, and is dependent on the material. In this possible scenario, the magnetic induction field B is equivalent to a gravitophoton (gravitational) field b_{gp}. Therefore, the following relation holds, provided that B is smaller than $B_{\max}$

$$b_{gp} \propto \frac{B}{B_{\max}} B. \quad (28)$$

As soon as B exceeds $B_{\max}$ the gravitophoton field vanishes. Can be derived the following general relationship between a magnetic and the neutral gravitophoton field, b_{gp}:

$$b_{gp} = \left[\frac{1}{(1-k)(1-ka)} - 1 \right] \frac{e}{m_e} \frac{B}{B_{\max}} B \quad (29)$$

where $k = 1/24$ and $a = 1/8$. The dimension of b_{gp} is s^{-1}. Inserting eq. (26) into eq. (29), using eq. (28), and differentiating with respect to time, results in

$$\frac{\partial b_{gp}}{\partial t} = \left[\frac{1}{(1-k)(1-ka)} - 1 \right] \frac{2e}{m_e} \frac{B}{B_{\max}} \frac{\partial B}{\partial t}. \quad (30)$$

Integrating over an arbitrary area A yields

$$\int \frac{\partial b_{gp}}{\partial t} \cdot dA = \oint g_{gp} \cdot ds \quad (31)$$

where it was assumed that the gravitophoton field, since it is a gravitational field, can be separated according to eqs. (23), (24). Combining eqs. (30) and (31) gives the following relationship

$$\oint g_{gp} \cdot ds = \left[\frac{1}{\left(1-\frac{1}{24}\right)\left(1-\frac{1}{24 \cdot 8}\right)} - 1 \right] \frac{2e}{m_e} \int \frac{B}{B_{\max}} \frac{\partial B}{\partial t} \cdot dA. \quad (32)$$

From eq. (26) one obtains

$$\frac{\partial B}{\partial t} = -\frac{2m_e}{e}\dot{\omega} . \quad (33)$$

Next, we apply eqs. (32) and (33) calculating the gravitophoton acceleration for the in-ring accelerometer. It is assumed that the accelerometer is located at distance r from the origin of the coordinate system. From eq. (26) it can be directly seen that the magnetic induction has a z-component only. From eq. (32) it is obvious that the gravitophoton acceleration is in the r-θ plane. Because of symmetry reasons the gravitophoton acceleration is independent on the azimuthal angle θ, and thus only has a component in the circumferential (tangential) direction, denoted by $\hat{e}_\theta$. Since the gravitophoton acceleration is constant along a circle with radius r, integration is over the area $A = \pi r^2 \hat{e}_z$. Inserting eq. (33) into eq. (32), and carrying out the integration the following expression for the gravitophoton acceleration is obtained

$$g_{gp} = -\frac{1}{10}\frac{B}{B_{\max}}\dot{\omega} r \quad (34)$$

where the minus sign indicates an acceleration opposite to the original one and it was assumed that the B fields is homogeneous over the integration area.
The ratio of the magnetic fields was calculated from the following formula, obtained by dividing eq. (26) by the square root of eq. (27)

$$\frac{B}{B_{\max}} = \frac{1}{7}\sqrt{\left(\frac{m_e}{k_B T_C}\right)}\omega R . \quad (35)$$

Inserting an estimated average value of $\omega = 175$ *rad/s*, $m_e = 9\times10^{-31}$ *kg*, $k_B = 1.38\times10^{-23}$ *J/K*, $T_C = 9.4$ *K*, and $R = 7.2\times10^{-2}$ *m*, this ratio is calculated as 3.97×10^{-4}.

2. On the mathematics concerning the Maxwell's equations in higher dimensions [2]

In (3+1)-D, the field strength tensor **F** can be represented as an anti-symmetric 4×4 matrix:

$$\mathbf{F} = \begin{pmatrix} 0 & F^{01} & F^{02} & F^{03} \\ -F^{01} & 0 & F^{12} & F^{13} \\ -F^{02} & -F^{12} & 0 & F^{23} \\ -F^{03} & -F^{13} & -F^{23} & 0 \end{pmatrix} . \quad (36)$$

We define the controvariant spacetime position vector as $\{x^\mu\} \equiv \{x^0 \equiv ct, x^1, x^2, x^3\}$, where the lowercase Greek letters represent spacetime indices $\{\mu,\nu\} = \{0,1,2,3\}$. We choose the metric tensor **g** = diag $\{-1,1,1,1\}$ to have positive spatial components such that the raising and lowering of indices only changes the sign of the temporal components. The components of the field strength tensor $\{F^{\mu\nu}\}$ are related to the components of the spacetime potential $\{A^{\mu}\}=\{A^0 \equiv V/c, A^1, A^2, A^3\}$ via

$$F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu} . \quad (37)$$

The field strength tensor **F** is naturally divided into temporal and spatial components. The three independent temporal components are associated with the vector electric field

$$E^i/c \equiv F^{0i} = \partial^0 A^i - \partial^i A^0 \quad (38)$$

while the three spatial components naturally form a 3×3 second rank anti-symmetric tensor magnetic field

$$B^{ij} \equiv F^{ij} = \partial^i A^j - \partial^j A^i . \quad (39)$$

Maxwell's inhomogeneous equations, in vacuum, are expressed concisely in terms of the field strength tensor:

$$\partial_{\nu}F^{\mu\nu} = \mu_0 J^{\mu} \quad (40)$$

where the spacetime current density has components $J^{\mu} = \{J^0 \equiv c\rho, J^1, J^2, J^3\}$.
In (3+1)-D, ρ is the volume charge density and $\vec{J} = J_1\hat{x}_1 + J_2\hat{x}_2 + J_3\hat{x}_3$ represents the distribution of the current over a cross-sectional area. Maxwell's homogeneous equations follow from the relationship between the fields and the spacetime potential (eqs. 38-39). In terms of the vector electric field **E** and tensor magnetic field **B**, Maxwell's equations in differential form are:

$$\partial_i E_i = \frac{\rho}{\varepsilon_0}, \qquad \partial_i E_j - \partial_j E_i = -c\partial_0 B_{ij} ,$$
$$\varepsilon_{ijk}\partial_i B_{jk} = 0 , \qquad -\partial_i B_{ij} = \frac{\partial_0 E_j}{c} + \mu_0 J_j . \qquad (41)$$

We do not distinguish between covariant and controvariant spatial indices $\{i, j, k\} \in \{1,2,3\}$ since only raising or lowering of temporal components involves a sign change. With the line element ds expressed as an anti-symmetric second-rank tensor and the differential area $d\vec{A}$ expressed as a vector via the following equation

$$C_{ij} = \varepsilon_{ijk}C_k , \qquad C_k = \frac{1}{2!}\varepsilon_{ijk}C_{ij} ,$$

the integral form of Maxwell's equations are:

$$\oint_S E_i dA_i = \frac{1}{\varepsilon_0}\int_{V_{enc}} \rho dV , \qquad \varepsilon_{ijk}\oint_C E_i ds_{jk} = -c\varepsilon_{ijk}\partial_0 \int_{S_{enc}} B_{ij} dA_k ,$$
$$\varepsilon_{ijk}\oint_S B_{ij} dA_k = 0 , \qquad \oint_C B_{ij} ds_{ij} = 2!\int_{S_{enc}} \left(\mu_0 J_i + \frac{\partial_0 E_i}{c}\right) dA_i . \qquad (42)$$

When generalizing eq. (42) to (N+1)-D, the higher-dimensional analog of the differential line element ds will have $N-2$ spatial dimensions and the higher-dimensional analog of the differential area $d\vec{A}$ will have $N-1$ spatial dimensions. It is also useful to work with an anti-symmetric second-rank tensor **G** that is dual to **F**:

$$G^{\mu\nu} = \frac{1}{2!}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} . \quad (43)$$

Maxwell's homogeneous equations are obtained from the dual tensor **G** via

$$\partial_\nu G^{\mu\nu} = 0 . \quad (44)$$

Alternatively, Maxwell's homogeneous equations may be extracted from

$$\varepsilon^{\lambda\mu\nu\rho}T_{\lambda\mu\nu} = 0 , \quad (45)$$

where the anti-symmetric third-rank tensor **T** is defined as

$$T_{\lambda\mu\nu} \equiv \partial_\lambda F_{\mu\nu} . \quad (46)$$

The most straightforward generalization of Maxwell's equations to (4+1)-D begins by extending eq. (38) to

$$\partial_\Gamma F^{\Phi\Gamma} = \mu_{4+1}J^\Phi \quad (47)$$

where the uppercase Greek letters $\{\Phi,\Gamma\} \in \{0,1,2,3,4\}$ represent (4+1)-D spacetime indices and the (4+1)-D permittivity and permeability of free space ε_{4+1} and μ_{4+1}, respectively, have different dimensions than their (3+1)-D counterparts ε_0 and μ_0.

The 5×5 (4+1)-D field strength tensor **F** has 10 independent components, which naturally divides into a 4-component vector electric field

$$E^I / c = F^{0I} = \partial^0 A^I - \partial^I A^0 \quad (48)$$

and a 4×4 anti-symmetric second rank tensor magnetic field

$$B^{IJ} = F^{IJ} = \partial^I A^J - \partial^J A^I \quad (49)$$

with 6 independent components. In terms of the vector electric field $\vec{E}$ and tensor magnetic field **B**, Maxwell's equations in (4+1)-D in differential form become

$$\partial_I E_I = \frac{\rho}{\varepsilon_{4+1}} , \qquad \partial_I E_J - \partial_J E_I = -c\partial_0 B_{IJ} ,$$
$$\varepsilon_{IJKL}\partial_I B_{JK} = 0 , \quad -\partial_I B_{IJ} = \frac{\partial_0 E_J}{c} + \mu_{4+1}J_J . \qquad (50)$$

Gauss's law in magnetism, which states that the net magnetic flux is always zero as a consequence that magnetic monopoles have not been observed, can alternatively be expressed in differential form as:

$$\partial_I B_{JK} + \partial_J B_{KI} + \partial_K B_{IJ} = 0 . \quad (51)$$

Alternatively, the differential volume may also be expressed as a vector $d\vec{V}$ via the following equation

$$C_{IJK} = \varepsilon_{IJKL} C_L , \qquad C_L = \frac{1}{3!} \varepsilon_{IJKL} C_{IJK} .$$

With the fields and differential elements expressed in terms of appropriate vectors and tensors, (4+1)-D Maxwell's equations can be expressed in integral form as:

$$\oint_V E_I dV_I = \frac{1}{\varepsilon_{4+1}} \int_{W_{enc}} \rho dW , \qquad \varepsilon_{IJKL} \oint_S E_I dA_{JK} = -c \varepsilon_{IJKL} \partial_0 \int_{V_{enc}} B_{IJ} dV_K ,$$
$$\varepsilon_{IJKL} \oint_V B_{IJ} dV_K = 0 , \qquad \oint_S B_{IJ} dA_{IJ} = 2! \int_{V_{enc}} \left(\mu_{4+1} J_I + \frac{\partial_0 E_I}{c} \right) dV_I . \quad (52)$$

While electric flux remains a scalar, magnetic flux has four components in (4+1)-D:

$$\Phi^e = \int_V E_I dV_I , \qquad \Phi_L^m = \varepsilon_{IJKL} \int_V B_{IJ} dV_K . \quad (53)$$

In (4+1)-D, the continuity equation reads

$$\partial_I J_I = -c \partial_0 \rho . \quad (54)$$

In (4+1)-D, the dual tensor **G** is an anti-symmetric third-rank tensor:

$$G^{\Phi\Gamma\Lambda} \equiv \frac{1}{3!} \varepsilon^{\Phi\Gamma\Lambda\Sigma\Omega} F_{\Sigma\Omega} . \quad (55)$$

Maxwell's homogeneous equations may be expressed in terms of the dual tensor **G** via

$$\partial_\Lambda G^{\Phi\Gamma\Lambda} = 0 \quad (56)$$

or, equivalently, in terms of an anti-symmetric third-rank tensor **T** via

$$\varepsilon^{\Phi\Gamma\Lambda\Sigma\Omega} T_{\Lambda\Sigma\Omega} = 0 \quad (57)$$

where **T** is defined as

$$T_{\Phi\Gamma\Lambda} \equiv \partial_\Phi F_{\Gamma\Lambda} . \quad (58)$$

In (5+1)-D, the differential form of Maxwell's inhomogeneous equations are expressed concisely in terms of the 6×6 field strength tensor **F** as

$$\partial_{\Gamma_{5+1}} F^{\Phi_{5+1}\Gamma_{5+1}} = \mu_{5+1} J^{\Phi_{5+1}} \quad (59)$$

where $\{\Phi_{5+1}, \Gamma_{5+1}\} \in \{0,1,2,3,4,5\}$ represent (5+1)-D spacetime indices. Maxwell's homogeneous equations can be expressed in terms of the dual tensor **G** or, equivalently, its counterpart **T**, as

$$\partial_{\Xi_{5+1}} G^{\Phi_{5+1}\Gamma_{5+1}\Lambda_{5+1}\Xi_{5+1}} = 0, \qquad \varepsilon^{\Phi_{5+1}\Gamma_{5+1}\Lambda_{5+1}\Sigma_{5+1}\Omega_{5+1}\Xi_{5+1}} T_{\Lambda_{5+1}\Sigma_{5+1}\Omega_{5+1}} = 0, \quad (60)$$

where

$$G^{\Phi_{5+1}\Gamma_{5+1}\Lambda_{5+1}\Sigma_{5+1}} \equiv \frac{1}{4!} \varepsilon^{\Phi_{5+1}\Gamma_{5+1}\Lambda_{5+1}\Sigma_{5+1}\Omega_{5+1}\Xi_{5+1}} F_{\Omega_{5+1}\Xi_{5+1}}, \qquad T_{\Phi_{5+1}\Gamma_{5+1}\Lambda_{5+1}} \equiv \partial_{\Phi_{5+1}} F_{\Gamma_{5+1}\Lambda_{5+1}}. \quad (61)$$

In terms of the 5-component vector electric field **E** and the 5×5 tensor magnetic field **B**, the differential form of Maxwell's equations in (5+1)-D are:

$$\partial_{I_5} E_{I_5} = \frac{\rho}{\varepsilon_{5+1}}, \qquad \partial_{I_5} E_{J_5} - \partial_{J_5} E_{I_5} = -c\partial_0 B_{I_5 J_5},$$
$$\varepsilon_{I_5 J_5 K_5 L_5 M_5} \partial_{I_5} B_{J_5 K_5} = 0, \qquad -\partial_{I_5} B_{I_5 J_5} = \frac{\partial_0 E_{J_5}}{c} + \mu_{5+1} J_{J_5}, \quad (62)$$

where $\{I_5, J_5\} = \{1,2,3,4,5\}$. In integral form, Maxwell's equations in (5+1)-D are:

$$\oint_W E_{I_5} dW_{I_5} = \frac{1}{\varepsilon_{5+1}} \int_{X_{enc}} \rho dX, \qquad \varepsilon_{I_5 J_5 K_5 L_5 M_5} \oint_V E_{I_5} dV_{J_5 K_5} = -c\varepsilon_{I_5 J_5 K_5 L_5 M_5} \partial_0 \int_{W_{enc}} B_{I_5 J_5} dW_{K_5},$$
$$\varepsilon_{I_5 J_5 K_5 L_5 M_5} \oint_W B_{I_5 J_5} dW_{K_5} = 0, \qquad \oint_V B_{I_5 J_5} dV_{I_5 J_5} = 2! \int_{W_{enc}} \left(\mu_{5+1} J_{I_5} + \frac{\partial_0 E_{I_5}}{c} \right) dW_{I_5}, \quad (63)$$

where dX is the differential five-dimensional volume element.
For an isolated point charge in (4+1)-D, the electric field lines radiate outward in four-dimensional space. For a Gaussian hypersphere of radius r, the electric field $\vec{E}$ is parallel to the differential volume element $d\vec{V}$ everywhere on the three-dimensional volume $x_1^2 + x_2^2 + x_3^2 + x_4^2 = r^2$ bounding the (four-dimensional) hypersphere defined by $x_1^2 + x_2^2 + x_3^2 + x_4^2 \le r^2$. In (4+1)-D, the electric field $\vec{E}$ due to a point charge is thus found to be

$$\oint_V E_I dV_I = E 2\pi^2 r^3 = \frac{1}{\varepsilon_{4+1}} \int_{W_{enc}} \rho dW = \frac{q}{\varepsilon_{4+1}}, \qquad \vec{E} = \frac{q}{2\pi^2 \varepsilon_{4+1} r^3} \hat{r}, \quad (64)$$

thence, we obtain

$$\oint_V E_I dV_I = E2\pi^2 r^3 = \frac{1}{\varepsilon_{4+1}} \int_{W_{enc}} \rho dW = \frac{\vec{E}2\pi^2 r^3}{\hat{r}}. \quad (64b)$$

We have utilized the result that the (N–1)-dimensional volume $\sum_{i=1}^{N} x_i^2 = r^2$ bounding a solid sphere in N-dimensional space $\sum_{i=1}^{N} x_i^2 \le r^2$ is

$$V_{N-1}^{sphere} = \frac{2\pi^{N/2} r^{N-1}}{\Gamma(N/2)} \quad (65)$$

where $\Gamma(z)$ is the gamma function.
Generalizing to (N+1)-D, the electric field $\vec{E}$ due to a point charge is

$$\vec{E} = \frac{\Gamma(N/2)}{2\pi^{N/2}} \frac{q}{\varepsilon_{N+1} r^{N-1}} \hat{r} \quad (66)$$

where the (N+1)-D permittivity of free space ε_{N+1} has different dimensionality than its (3+1)-D counterpart ε_0:

$$[\varepsilon_{N+1}] = \frac{[\varepsilon_0]}{[L^{N-3}]}. \quad (67)$$

Thus, Coulomb's law for the force exerted by one point charge q_1 on another point charge q_2 displaced by the relative position vector $\vec{R}_{21}$ away from the first is a $1/r^{N-1}$ force law:

$$\vec{F}_{q_2,q_1} = \frac{\Gamma(N/2)}{2\pi^{N/2}} \frac{q_1 q_2}{\varepsilon_{N+1} R_{21}^{N-1}} \hat{R}_{21}. \quad (68)$$

In (N+1)-D, the electric field $\vec{E}(\vec{r})$ at a field point located at $\vec{r}$ may alternatively be derived through direct integration over the differential source element dq':

$$\vec{E}(\vec{r}) = \frac{\Gamma(N/2)}{2\pi^{N/2} \varepsilon_{N+1}} \int_0^{q'} \frac{1}{R^{N-1}} \hat{R} dq' \quad (69)$$

where the relative position vector $\vec{R} = \vec{r} - \vec{r}\,'$ extends from the differential source element dq' located at $\vec{r}\,'$ to the field point located at $\vec{r}$.
The (N+1)-D electric field is related to the (N+1)-D scalar potential via

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}. \quad (70)$$

In electrostatics, the (N+1)-D scalar potential is derived from the following integral:

$$V(r) = \frac{\Gamma(N/2)}{2\pi^{N/2}(N-2)\varepsilon_{N+1}} \int_0^{q'} \frac{1}{R^{N-2}} dq' . \quad (71)$$

The traditional Ampèrian loop is promoted to a two-dimensional Ampèrian surface in (4+1)-D. A suitable choice for a steady filamentary current I_0 running along the x_1-axis in the two-dimensional surface $x_2^2 + x_3^2 + x_4^2 = r^2$ of a solid sphere $x_2^2 + x_3^2 + x_4^2 \le r^2$ in the three-dimensional subspace $x_2 x_3 x_4$. In this case, the only non-vanishing components of the magnetic field tensor **B** are $\{B_{1I}\}$ and $\{B_{I1}\}$. By symmetry, the magnetic field scalar, defined by the tensor contraction $B \equiv \sqrt{B_{IJ} B_{IJ} / 2!}$, is constant everywhere on the surface $x_2^2 + x_3^2 + x_4^2 = r^2$ and the components of the magnetic field tensor are related by $B_{1I} = B_{1J} = -B_{I1} = -B_{J1}$, where $I \neq J$, such that $B = \sqrt{B_{12}^2 + B_{13}^2 + B_{14}^2} = B_{12}\sqrt{3}$. However, it is simpler to work in spherical coordinates, where $B_{IJ} dA_{IJ} = 2Br^2 d\Omega$. Thus, application of Ampère's law yields

$$\oint_S B_{IJ} dA_{IJ} = 2\int_S Br^2 d\Omega = B8\pi r^2 = 2\mu_{4+1} I_{enc} , \qquad B = \frac{\mu_{4+1} I_0}{4\pi r^2} . \quad (72)$$

In (3+1)-D, the magnetic flux Φ^m is a scalar and the magnetic field lines for this infinite steady filamentary current I_0 are traditionally drawn as a concentric circles $x_2^2 + x_3^2 = r^2$ in the $x_2 x_3$ plane. However, in (4+1)-D the magnetic flux is a four-component vector $\{\Phi_I^m\}$ according to eq. (53). This corresponds to the fact that circles $x_2^2 + x_3^2 = r^2$, $x_2^2 + x_4^2 = r^2$, and $x_3^2 + x_4^2 = r^2$ lying in the $x_2 x_3$, $x_2 x_4$, and $x_3 x_4$ planes, respectively, are all orthogonal to the current running along the x_1-axis. In (N+1)-D, the magnetic flux becomes an anti-symmetric (N-3)-rank tensor.
Generalizing to (N+1)-D, the magnetic field due to a steady filamentary current I_0 is

$$B(r) = \frac{1}{2\pi^{(N-1)/2}} \Gamma\left(\frac{N-1}{2}\right) \frac{\mu_{4+1} I_0}{r^{N-2}} . \quad (73)$$

In (N+1)-D, the force per unit length ℓ that one steady filamentary current I_1 exerts on a parallel steady filamentary current I_2 separated by the relative position vector $\vec{R}_{21}$ is:

$$\frac{\vec{F}_{I_2,I_1}}{\ell} = \frac{1}{2\pi^{(N-1)/2}} \Gamma\left(\frac{N-1}{2}\right) \frac{\mu_{4+1} I_1 I_2}{R_{21}^{N-2}} \hat{R}_{21} . \quad (74)$$

In (4+1)-D, the tensor magnetic field $B(\vec{r})$ at a field point located at $\vec{r}$ may alternatively be derived through direct integration. For a steady filamentary current,

$$B_{IJ}(\vec{r}) = \frac{1}{4\pi^{(N-1)/2}} \Gamma\left(\frac{N-1}{2}\right) \mu_{N+1} I_0 \int \frac{1}{R^4} R_J ds_I - R_I ds_J \quad (75)$$

where the relative position vector $\vec{R} = \vec{r} - \vec{r}\,'$ extends from the differential source element $d\vec{s}\,'$ located at $\vec{r}\,'$ to the field point located at $\vec{r}$. If the current is instead distributed over a one-,two, or three-dimensional cross section, the tensor magnetic field $B(\vec{r})$ is

$$B_{IJ}(\vec{r}) = \frac{1}{4\pi^{(N-1)/2}}\Gamma\left(\frac{N-1}{2}\right)\mu_{N+1}\int \frac{1}{R^3} J_1 dA_{IJ} ds\,,\quad B_{IJ}(\vec{r}) = \frac{1}{4\pi^{(N-1)/2}}\Gamma\left(\frac{N-1}{2}\right)\mu_{N+1}\int \frac{1}{R^3} J_2 dA_{IJ} ds\,,$$

$$B_{IJ}(\vec{r}) = \frac{1}{4\pi^{(N-1)/2}}\Gamma\left(\frac{N-1}{2}\right)\mu_{N+1}\int \frac{1}{R^4} J_3 (R_J dV_I - R_I dV_J) ds\,, \quad (76)$$

where J_1, J_2, and J_3 are the corresponding to the current densities.

With a single non-compact extra dimension x_4, two (4+1)-D electric charges q_1 and q_2 communicate via the exchange of a (4+1)-D photon. Classically, there is a single straight-line path with which the (4+1)-D photon can reach q_2 from another q_1. One result of this single non-compact extra dimension x_4 is that Coulomb's law becomes a $1/r^3$ force law. If instead the extra dimension x_4 is compactified on a torus with radius R, there exists an infinite discrete set of paths with which a classical (4+1)-D photon could travel from q_1 to q_2.

The net force exerted on q_2 is the superposition of the (4+1)-D $1/r^3$ Coulomb forces (eq. 68) associated with each path:

$$\vec{F} = \frac{1}{2\pi^2 \varepsilon_{4+1}} q_1 q_2 \sum_{n=-\infty}^{\infty} \frac{1}{R_n^3} \hat{R}_n \quad (77)$$

where $R_n = \sqrt{R_3^2 + (\Delta x_4 + 4\pi n R)^2}$ is the path length corresponding to a classical (4+1)-D photon winding around the torus n times and $\hat{R}_n = \dfrac{R_3 \hat{R}_3 + \Delta x_4 \hat{x}_4}{R_n}$ is a unit vector directed from q_1 to q_2 (and $\hat{R}_3$ is a unit vector along the axis of the torus). In terms of its three-dimensional and extra-dimensional components, the net force is

$$\vec{F} = \frac{1}{2\pi^2 \varepsilon_{4+1}} q_1 q_2 \sum_{n=-\infty}^{\infty} \frac{R_3 \hat{R}_3 + \Delta x_4 \hat{x}_4}{\left[R_3^2 + (\Delta x_4 + 4\pi n R)^2\right]^2}\,. \quad (78)$$

The net force can also be expressed in terms of the (4+1)-D scalar potential $V(R_4)$ via

$$\vec{F} = -q_2 \vec{\nabla} V\,. \quad (79)$$

The (4+1)-D scalar potential due to the source q_1 at the location of q_2 is

$$V(R_4) = \frac{1}{4\pi^2 \varepsilon_{4+1}} q_1 \sum_{n=-\infty}^{\infty} \frac{1}{R_3^2 + (\Delta x_4 + 4\pi n R)^2}\,. \quad (80)$$

Note that the (4+1)-D scalar potential is periodic in the extra dimension:

$$V(R_3, \Delta x_4) = V(R_3, \Delta x_4 \pm 2\pi nR). \quad (81)$$

In the limit that the charges are very close compared to the radius of the extra dimension, i.e. $R_3 << R$ and $\Delta x_4 << R$, the $n = 0$ term dominates and the net force is approximately the (4+1)-D $1/r^3$ Coulomb force:

$$\vec{F}_{q_1,q_2} \approx \frac{1}{2\pi^2 \varepsilon_{4+1} R_4^3} q_1 q_2 \hat{R}_4. \quad (82)$$

This is the limit where the underlying (4+1)-D theory of electrodynamics – i.e. Maxwell's equations with a non-compact extra dimension – governs the motion. In the opposite extreme, where the compact extra dimension is very small compared to the separation of the charges, i.e. $R_3 >> R$, an integral provides a good approximation for the sum:

$$V(R_4) \approx \frac{1}{8\pi^3 \varepsilon_{4+1} R} q_1 \int_{y=-\infty}^{\infty} \frac{1}{R_3^2 + y^2} dy = \frac{1}{8\pi^2 \varepsilon_{4+1} R R_3} q_1. \quad (83)$$

This is the limit where the effective (3+1)-D theory of electrodynamics – i.e. the usual (3+1)-D form of Maxwell's equations – governs the motion. In order for the effective (3+1)-D scalar potential in eq. (83) to be consistent with the usual (3+1)-D form of Coulomb's law, it is necessary that the (4+1)-D permittivity of free space ε_{4+1} be related to the usual (3+1)-D permittivity ε_0 via

$$\varepsilon_{4+1} = \frac{\varepsilon_0}{2\pi R}. \quad (84)$$

It follows that

$$\mu_{4+1} = 2\pi \mu_0 R \quad (85)$$

such that an electromagnetic wave propagates at the usual speed of light in vacuum

$$c = \frac{1}{\sqrt{\varepsilon_{4+1}\mu_{4+1}}} = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}. \quad (86)$$

In (N+1)-D, Coulomb's law is a $1/r^{N-1}$ force law if the extra dimensions are non-compact. If instead there is toroidal compactification and the extra dimensions are symmetric . i.e. they all have the same radius R - then, in the limit that $R_3 >> R$, Coulomb's law is approximately a $1/r^2$ force law in the effective (3+1)-D theory. In this case, the (N+1)-D permittivity of free space ε_{N+1} is related to the usual (3+1)-D permittivity ε_0 via

$$\varepsilon_{N+1} = \frac{\varepsilon_0}{\left(2R\sqrt{\pi}\right)^{N-3} \Gamma\left(\frac{N-3}{2}\right)} \quad (87)$$

where $N > 3$.
Thence, the eq. (71), for the eq. (87), can be written also as follow:

$$V(r) = \frac{\Gamma(N/2)}{2\pi^{N/2}(N-2)} \frac{\left(2R\sqrt{\pi}\right)^{N-3}\Gamma\left(\frac{N-3}{2}\right)}{\varepsilon_0} \int_0^{q'} \frac{1}{R^{N-2}} dq', \quad (87b)$$

where $\Gamma(z)$ is the gamma function.

The deviation in the usual $1/r^2$ form of Coulomb's law can be computed for a specified geometry of extra dimensions through the lowest-order corrections to the integration in eq. (83). The effects are similar to the deviation in the usual $1/r^2$ form of Newton's law of universal gravitation.

3. On some equations concerning the noncommutativity in String Theory: the Dirac-Born-Infeld action, connections with the Maxwell electrodynamics and the Maxwell's equations, noncommutative open string actions and D-brane actions [3] [4] [5] [6] [7]

3.1 *The Dirac-Born-Infeld action: connections with the Maxwell's equations*

To obtain the low-energy effective action for the gauge field, we need to expand the worldsheet theory about a background field $\overline{X}^i$, and compute the (divergent) one-loop counterterm:

$$-i\int d\tau \Gamma_i(A(X),\Lambda)\partial_\tau X^i \quad (88)$$

where Λ is an ultraviolet cutoff. Setting Γ_i to zero gives a condition on the gauge field $A_i(X)$, equivalent to worldsheet conformal invariance, or vanishing of the β-function.
That gives the spacetime equation of motion, from which the action can be reconstructed.
Performing the background field expansion

$$X^i = \overline{X}^i + \xi^i \quad (89)$$

where $\overline{X}^i$ is an arbitrary classical solution of the worldsheet equations of motion, we find:

$$S(X) = S(\overline{X}) + \int \frac{\delta S}{\delta X^i}_{|X=\overline{X}} \xi^i + \frac{1}{2}\int \frac{\delta^2 S}{\delta X^i \delta X^j}_{|X=\overline{X}} \xi^i \xi^j + \ldots \quad (90)$$

The linear term in ξ^i vanishes as $\overline{X}$ is a solution of the equation of motion. The quadratic term is easily evaluated:

$$\frac{1}{2}\int \frac{\delta^2 S}{\delta X^i \delta X^j}_{|X=\overline{X}} \xi^i \xi^j = \frac{1}{4\pi\alpha'}\left[\int d\sigma d\tau g_{ij}\partial_a \xi^i \partial_a \xi^j + i\int d\tau\left(\partial_i \mathcal{F}_{jk}\partial_\tau \overline{X}^j \xi^i \xi^k + \mathcal{F}_{ij}\xi^j \partial_\tau \xi^i\right)\right]. \quad (91)$$

The correction to the worldsheet action is:

$$\frac{i}{4\pi\alpha'}\int d\varpi \partial_i \mathcal{F}_{jk}\partial_\tau \overline{X}^j K^{ik}(\tau,\tau')\Big|_{\tau=\tau'}, \quad (92)$$

where we have ignored possible UV finite terms.
Recalling the formula for the boundary propagator, we have:

$$\lim_{\tau\to\tau'} K^{ij}(\tau;\tau') = -2\alpha' G^{ij}(\mathcal{F})\ln\Lambda + \text{(finite)} \quad (93)$$

and hence the equation of motion is:

$$\partial_i \mathcal{F}_{jk} G^{ik}(\mathcal{F}) \equiv \partial_i \mathcal{F}_{jk}\left(\frac{1}{g+\mathcal{F}} g \frac{1}{g-\mathcal{F}}\right)^{ik} = 0. \quad (94)$$

When we think of $G^{ij}(\mathcal{F})$ as the (inverse) open-string metric, then this looks just like the free Maxwell' equations. It turns out that the desired open-string effective action is:

$$S_{NS-NS}[A_i; g_{ij}, B_{ij}] = \frac{1}{g_s}\int d^{10}x\sqrt{\det(g+\mathcal{F})}. \quad (95)$$

It is possible to show that:

$$\partial_i\left[\frac{\delta}{\delta(\partial_i A_j)}\sqrt{\det(g+\mathcal{F})}\right] = -\sqrt{\det(g+\mathcal{F})}G^{jk}(\mathcal{F})\partial_i \mathcal{F}_{kl} G^{li}(\mathcal{F}). \quad (96)$$

Since the factor $\sqrt{\det(g+\mathcal{F})}$ is nonzero and the matrix $G^{jk}(\mathcal{F})$ is invertible, it follows that setting the above expression to zero is equivalent to:

$$\partial_i \mathcal{F}_{kl} G^{li}(\mathcal{F}) = 0 \quad (97)$$

which is the desired equation of motion. At lowest order in $\mathcal{F}$ this is equivalent to the Maxwell equations:

$$\partial_i \mathcal{F}_{kl} g^{li} = 0 \quad (98)$$

but in general, as we noted, it has nonlinear corrections.
The action:

$$S_{NS-NS}[A_i; g_{ij}, B_{ij}] = \frac{1}{g_s}\int d^{10}x\sqrt{\det(g+\mathcal{F})} = \frac{1}{g_s}\int d^{10}x\sqrt{\det(g+2\pi\alpha'(B+F))} \quad (99)$$

is called the Dirac-Born-Infeld (DBI) action. Expanding this action to quadratic order in F, it is easily seen that it is proportional to the usual action of free Maxwell electrodynamics:

$$S_{NS-NS}\left[A_i; g_{ij}, B_{ij}\right] \approx \int F_{ij}F^{ij} + \ldots \quad (100)$$

i.e. we can write the eq. (100) also as follow:

$$S_{NS-NS}\left[A_i; g_{ij}, B_{ij}\right] \approx \int F_{ij}F^{ij} + \ldots \approx \oint_C B \cdot dl = \mu_0 I_{enc} + \mu_0 \varepsilon_0 \frac{d}{dt}\int_S E \cdot \hat{n} dA, \quad (100b)$$

thence, we obtain the following interesting mathematical connection:

$$S_{NS-NS}\left[A_i; g_{ij}, B_{ij}\right] = \frac{1}{g_s}\int d^{10}x\sqrt{\det(g + \mathcal{F})} = \frac{1}{g_s}\int d^{10}x\sqrt{\det(g + 2\pi\alpha'(B + F))} \approx$$
$$\approx \oint_C B \cdot dl = \mu_0 I_{enc} + \mu_0 \varepsilon_0 \frac{d}{dt}\int_S E \cdot \hat{n} dA. \quad (100c)$$

This is consistent with the fact that the linearized equations of motion are just Maxwell's equations.
We will now recast the DBI action in a different form. For this, let us first define

$$G^{ij} \equiv G^{ij}(F = 0) = \left(\frac{1}{g + 2\pi\alpha' B} g \frac{1}{g - 2\pi\alpha' B}\right)^{ij} \quad (101)$$
$$\frac{\theta^{ij}}{2\pi\alpha'} \equiv \frac{\theta^{ij}(F = 0)}{2\pi\alpha'} = -\left(\frac{1}{g + 2\pi\alpha' B} 2\pi\alpha' B \frac{1}{g - 2\pi\alpha' B}\right)^{ij}. \quad (102)$$

We abbreviate G^{ij} by G^{-1}. We also define the matrix G_{ij}, abbreviated G, to be the matrix inverse of G^{ij}. Thus we have defined two new constant tensors G^{-1}, θ in terms of the original constant tensors g, B. In particular,

$$\left(G^{-1} + \frac{\theta}{2\pi\alpha'}\right)^{ij} = \left(\frac{1}{g + 2\pi\alpha' B}\right)^{ij}. \quad (103)$$

It is illuminating to rewrite the DBI Lagrangian in terms of the new tensors. This is achieved by writing:

$$\frac{1}{g_s}\sqrt{\det(g + 2\pi\alpha'(B + F))} = \frac{1}{g_s}\sqrt{\det\left(\frac{1}{G^{-1} + \frac{\theta}{2\pi\alpha'}} + 2\pi\alpha' F\right)} =$$
$$= \frac{1}{g_s}\frac{1}{\sqrt{\det\left(1 + \frac{G\theta}{2\pi\alpha'}\right)}}\sqrt{\det(G(1 + \theta F) + 2\pi\alpha' F)} = \frac{1}{g_s}\frac{\sqrt{\det(1 + \theta F)}}{\sqrt{\det\left(1 + \frac{G\theta}{2\pi\alpha'}\right)}}\sqrt{\det\left(G + 2\pi\alpha' F \frac{1}{1 + \theta F}\right)}.$$
$$(104)$$

Defining

$$\hat{F} = F\,\frac{1}{1+\theta F}\,,\quad G_s = g_s\sqrt{\det\left(1+\frac{G\theta}{2\pi\alpha'}\right)} \quad (105)$$

we end up with the relation

$$\frac{1}{g_s}\sqrt{\det\left(g+2\pi\alpha'(B+F)\right)} = \frac{1}{G_s}\sqrt{\det(1+\theta F)}\sqrt{\det\left(G+2\pi\alpha'\hat{F}\right)}\,. \quad (106)$$

The relation between F and $\hat{F}$ can be easily inverted, leading to:

$$F = \hat{F}\,\frac{1}{1-\theta\hat{F}} \quad (107)$$

from which it also follows that:

$$1+\theta F = \frac{1}{1-\theta\hat{F}}\,. \quad (108)$$

Hence we have:

$$\frac{1}{g_s}\sqrt{\det\left(g+2\pi\alpha'(B+F)\right)} = \frac{1}{G_s}\frac{1}{\sqrt{\det\left(1-\theta\hat{F}\right)}}\sqrt{\det\left(G+2\pi\alpha'\hat{F}\right)}\,. \quad (109)$$

In what follows, we must be careful to remember that the above equations were obtained in the strict DBI approximation of constant F .

Apart from the factor $\sqrt{\det\left(1-\theta\hat{F}\right)}$ in the denominator, the right hand-side looks like a DBI Lagrangian with a new string coupling G_s, metric G_{ij} and gauge field strength $\hat{F}$, and no B-field. Let us therefore tentatively define the action:

$$\hat{S}_{DBI} = \frac{1}{G_s}\int\sqrt{\det\left(G+2\pi\alpha'\hat{F}\right)}\,. \quad (110)$$

Let us start by expanding the relation through which we defined $\hat{F}$, to lowest order in θ:

$$\hat{F}_{ij} = F_{ij} - F_{ik}\theta^{kl}F_{lj} + \mathrm{O}\left(\theta^2\right). \quad (111)$$

Inserting the definition of F_{ij}, we get:

$$\hat{F}_{ij} = \partial_i A_j - \partial_j A_i + \theta^{kl}\left(\partial_i A_k \partial_j A_l - \partial_i A_k \partial_l A_j - \partial_k A_i \partial_j A_l + \partial_k A_i \partial_l A_j\right) + \mathrm{O}\left(\theta^2\right). \quad (112)$$

We can make a nonlinear redefinition of A_i to this order, which absorbs three of the four terms linear in θ:

$$\hat{A}_i = A_i - \theta^{kl}\left(A_k \partial_l A_i + \frac{1}{2} A_k \partial_i A_l \right) + O(\theta^2). \quad (113)$$

We can find that

$$\hat{F}_{ij} = \partial_i \hat{A}_j - \partial_j \hat{A}_i + \theta^{kl} \partial_k \hat{A}_i \partial_l \hat{A}_j . \quad (114)$$

It is clear that there is no further redefinition of $\hat{A}$ that will absorb the last term. However, we note that this term is:

$$\theta^{kl} \partial_k \hat{A}_i \partial_l \hat{A}_j = \{\hat{A}_i, \hat{A}_j\} \quad (115)$$

where $\{...,\}$ is the Poisson bracket with Poisson structure θ. Thus, to linear order in θ, we have found that:

$$\hat{F}_{ij} = \partial_i \hat{A}_j - \partial_j \hat{A}_i + \{\hat{A}_i, \hat{A}_j\}. \quad (116)$$

This looks like a non-Abelian gauge field strength, except that there is a Poisson bracket instead of a commutator.
We can say that the field strength $\hat{F}_{ij}$ is a noncommutative field strength related to its gauge potential $\hat{A}_i$ by:

$$\hat{F}_{ij} = \partial_i \hat{A}_j - \partial_j \hat{A}_i - i[\hat{A}_i, \hat{A}_j]_* . \quad (117)$$

The map

$$\hat{F}_{ij} = F_{ik}\left(\frac{1}{1+\theta F}\right)^k{}_j \quad (118)$$

$$\hat{A}_i = A_i - \theta^{kl}\left(A_k \partial_l A_i + \frac{1}{2} A_k \partial_i A_l \right) + O(\theta^2) \quad (119)$$

is known as the Seiberg-Witten map.

We remember that the **Seiberg-Witten gauge theory** is a set of calculations that determine the low-energy physics — namely the moduli space and the masses of electrically and magnetically charged supersymmetric particles as a function of the moduli space.

This is possible and nontrivial in gauge theory with $N = 2$ extended supersymmetry by combining the fact that various parameters of the Lagrangian are holomorphic functions (a consequence of supersymmetry) and the known behavior of the theory in the classical limit.

The moduli space in the full quantum theory has a slightly different structure from that in the classical theory

The noncommutative description is parametrized by the noncommutativity parameter θ, the open-string metric G_{ij}, the open-string coupling G_s, and a "description parameter" Φ, in terms of which the relationship between closed-string and open-string parameters is given by:

$$N^{ij} \equiv \left(\frac{1}{g+2\pi\alpha' B}\right)^{ij} = \frac{\theta}{2\pi\alpha'} + \frac{1}{G+2\pi\alpha'\Phi}\ , \quad \frac{\sqrt{\det(g+2\pi\alpha' B)}}{g_s} = \frac{\sqrt{\det(G+2\pi\alpha'\Phi)}}{G_s}\ . \quad (120)$$

Now we wish to compare the sum of the commutative DBI action S_{DBI} plus the derivative corrections to it ΔS_{DBI} with the noncommutative DBI action $\hat{S}_{DBI}$, after taking the Seiberg-Witten limit on both sides.
The dilaton couples to the entire Lagrangian density, so we need to consider the full DBI action. We will start by restricting to terms quadratic in F. To this order, we have:

$$S_{DBI} = \int \frac{\sqrt{\det(g+2\pi\alpha' B)}}{g_s}\left[1+\frac{2\pi\alpha'}{2}tr(NF) - \frac{(2\pi\alpha')^2}{4}tr(NFNF) + \frac{(2\pi\alpha')^2}{8}(trNF)^2 + \ldots\right]. \quad (121)$$

In the Seiberg-Witten limit we have $N^{ij} \to \frac{\theta^{ij}}{2\pi\alpha'}$ and therefore:

$$S_{DBI}\big|_{SW} = \int \frac{\sqrt{\det(g+2\pi\alpha' B)}}{g_s}\left[1+\frac{1}{2}tr(\theta F) - \frac{1}{4}tr(\theta F\theta F) + \frac{1}{8}(tr\theta F)^2 + \ldots\right]. \quad (122)$$

Let us now convert the commutative field strengths F appearing in this expression into noncommutative field strengths $\hat{F}$, using the Seiberg-Witten map. To the order that we need it, this map is:

$$F_{ab} = \hat{F}_{ab} + \theta^{kl}\left(\left\langle \hat{A}_k, \partial_l \hat{F}_{ab}\right\rangle_{*2} - \left\langle \hat{F}_{ak}, \hat{F}_{bl}\right\rangle_{*2}\right) \quad (123)$$

where

$$\hat{F}_{ab} = \partial_a \hat{A}_b - \partial_b \hat{A}_a + \theta^{kl}\left\langle \partial_k \hat{A}_a, \partial_l \hat{A}_b\right\rangle_{*2}. \quad (124)$$

Here we have used an identity relating the Moyal $*$ commutator and the $*_2$ product:

$$-i[f,g]_* = \theta^{ij}\langle \partial_i f, \partial_i g\rangle_{*2}. \quad (125)$$

Inserting the Seiberg-Witten map into eq. (122), we find

$$S_{DBI}\Big|_{SW} = \int \frac{\sqrt{\det(g+2\pi\alpha' B)}}{g_s}\Bigg[1+\theta^{ij}\partial_j \hat{A}_i + \frac{1}{2}\theta^{ba}\theta^{kl}\left\langle \partial_k \hat{A}_a, \partial_l \hat{A}_b\right\rangle_{*2} +$$
$$+\frac{1}{2}\theta^{ab}\theta^{kl}\left(\left\langle \hat{A}_k, \partial_l \hat{F}_{ab}\right\rangle_{*2} - \left\langle \hat{F}_{ak}, \hat{F}_{bl}\right\rangle_{*2}\right) - \frac{1}{4}\theta^{ij}\theta^{kl}\hat{F}_{jk}\hat{F}_{li} + \frac{1}{8}\left(\theta^{ij}\hat{F}_{ij}\right)^2\Bigg]. \quad (126)$$

Some manipulation of the last few terms permits us to rewrite this as:

$$S_{DBI}\Big|_{SW} = \int \frac{\sqrt{\det(g+2\pi\alpha' B)}}{g_s}\Bigg[1+\theta^{ij}\partial_j \hat{A}_i + \frac{1}{2}\theta^{ij}\theta^{kl}\left\langle \partial_j \hat{A}_k, \partial_l \hat{A}_i\right\rangle_{*2} + \frac{1}{2}\theta^{ba}\theta^{kl}\left\langle \hat{A}_k, \partial_l \hat{F}_{ab}\right\rangle_{*2} +$$

$$+\frac{1}{4}\theta^{ij}\theta^{kl}\left(\left\langle\hat{F}_{jk},\hat{F}_{li}\right\rangle_{*2}-\hat{F}_{jk}\hat{F}_{li}\right)+\frac{1}{8}\theta^{ij}\theta^{kl}\left\langle\hat{F}_{ji},\hat{F}_{lk}\right\rangle_{*2}-\frac{1}{8}\theta^{ij}\theta^{kl}\left(\left\langle\hat{F}_{ji},\hat{F}_{lk}\right\rangle_{*2}-\hat{F}_{ji}\hat{F}_{lk}\right)\Bigg] \quad (127)$$

which is the form in which it will be useful.
Let us now turn to the noncommutative side. Here, we only need to keep the terms arising from expansion of the Wilson line, since all other terms are suppressed by powers of α' in the Seiberg-Witten limit. The Wilson line give us:

$$\hat{S}_{DBI}\big|_{SW}=\int\frac{\sqrt{\det(G+2\pi\alpha'\Phi)}}{G_s}\left[1+\theta^{ij}\partial_j\hat{A}_i+\frac{1}{2}\theta^{ij}\theta^{kl}\partial_j\partial_l\left\langle\hat{A}_i,\hat{A}_k\right\rangle_{*2}\right]. \quad (128)$$

After some rearrangements of terms, this can be written:

$$\hat{S}_{DBI}\big|_{SW}=\int\frac{\sqrt{\det(G+2\pi\alpha'\Phi)}}{G_s}\Bigg[1+\theta^{ij}\partial_j\hat{A}_i+\frac{1}{2}\theta^{ij}\theta^{kl}\left\langle\partial_j\hat{A}_k,\partial_l\hat{A}_i\right\rangle_{*2}+$$
$$+\frac{1}{2}\theta^{ba}\theta^{kl}\left\langle\hat{A}_k,\partial_l\hat{F}_{ab}\right\rangle_{*2}+\frac{1}{8}\theta^{ij}\theta^{kl}\left\langle\hat{F}_{ji},\hat{F}_{lk}\right\rangle_{*2}\Bigg]. \quad (129)$$

Now we can take the difference of eqs. (129) and (127). The prefactor in front of each expression is the same, by virtue of eq. (120). Apart from this factor and the integral sign, the result is:

$$\hat{S}_{DBI}\big|_{SW}-S_{DBI}\big|_{SW}=\frac{1}{4}\theta^{ij}\theta^{kl}\left(\left\langle\hat{F}_{jk},\hat{F}_{li}\right\rangle_{*2}-\hat{F}_{jk}\hat{F}_{li}\right)-\frac{1}{8}\theta^{ij}\theta^{kl}\left(\left\langle\hat{F}_{ji},\hat{F}_{lk}\right\rangle_{*2}-\hat{F}_{ji}\hat{F}_{lk}\right). \quad (130)$$

To the order in which we are working, we may replace $\hat{F}$ by F everywhere in this expression. This, then, is our prediction for the correction ΔS_{DBI}, to order $(\alpha')^2$ and to quadratic order in the field strength F, after taking the Seiberg-Witten limit. We note that this is manifestly a higher-derivative correction: it vanishes for constant F, for which the $*_2$ product reduces to the ordinary product. Expanding the $*_2$ product to 4-derivative order, we find that

$$\Delta S_{DBI}\Big|_{SW}=-\frac{1}{96}\left[\theta^{ij}\theta^{kl}\theta^{mn}\theta^{rs}\partial_m\partial_r F_{jk}\partial_n\partial_s F_{li}-\frac{1}{2}\theta^{ij}\theta^{kl}\theta^{mn}\theta^{rs}\partial_m\partial_r F_{ji}\partial_n\partial_s F_{lk}\right] \quad (131)$$

which gives:

$$\Delta S_{DBI}\Big|_{SW}=-\frac{(2\pi\alpha')^4}{96}\left[h^{ij}h^{kl}h^{mn}h^{rs}\partial_m\partial_r F_{jk}\partial_n\partial_s F_{li}-\frac{1}{2}h^{ij}h^{kl}h^{mn}h^{rs}\partial_m\partial_r F_{ji}\partial_n\partial_s F_{lk}\right] \quad (132)$$

where the matrix h^{ij} is defined as:

$$h^{ij}\equiv\left(\frac{1}{g+2\pi\alpha'(B+F)}\right)^{ij}. \quad (133)$$

Taking the Seiberg-Witten limit, which amounts to the replacement $2\pi\alpha' h \to (1+\theta F)^{-1}\theta$, and further restricting to terms quadratic in F, we find exact agreement with eq. (131) above.
Now we will compare the coupling of the bulk graviton to the energy-momentum tensor on the commutative and noncommutative sides. On the commutative side, we start again with the expression in eq. (121), but this time we use the full form of N as defined in eq. (120):

$$N^{ij} \equiv \left(\frac{1}{g+2\pi\alpha'(B+F)}\right)^{ij} = \frac{\theta^{ij}}{2\pi\alpha'} + M^{ij} \quad (134)$$

where

$$M^{ij} \equiv \left(\frac{1}{G+2\pi\alpha'\Phi}\right)^{ij}. \quad (135)$$

Thence, we obtain:

$$N^{ij} \equiv \left(\frac{1}{g+2\pi\alpha'(B+F)}\right)^{ij} = \frac{\theta^{ij}}{2\pi\alpha'} + \left(\frac{1}{G+2\pi\alpha'\Phi}\right)^{ij}. \quad (135b)$$

As the linear coupling to the graviton starts at order $(\alpha')^2$, we now have to go beyond the leading term in the Seiberg-Witten limit. Hence we will keep terms up to order M^2. Expanding S_{DBI} around this limit and keeping terms to order $(\alpha')^2$, and using the Seiberg-Witten map, we find:

$$S_{DBI} = \int \frac{\sqrt{\det(g+2\pi\alpha' B)}}{g_s}\left[1+\frac{2\pi\alpha'}{2}\left\{M^{ji}\left(F_{ij}+\theta^{kl}\left\langle A_k,\partial_l F_{ij}\right\rangle_{*2}+\theta^{kl}\left\langle F_{jk},F_{li}\right\rangle_{*2}\right)\right\}+\frac{(2\pi\alpha')^2}{8}\cdot\right.$$
$$\left.\cdot\left\{\frac{2}{2\pi\alpha'}(trMF)(tr\theta F)+(trMF)^2\right\}-\frac{(2\pi\alpha')^2}{4}\left\{trMFMF+\frac{2}{2\pi\alpha'}trMF\theta F\right\}+\text{ terms not involving } M + \text{order } F^3\right]. \quad (136)$$

Turning now to the noncommutative action, the graviton coupling is obtained by expanding the DBI action around the Seiberg-Witten limit to order $(\alpha')^2$. We have, in momentum space:

$$\hat{S}_{DBI} = \frac{1}{G_s}\int L_*\left[\sqrt{\det\left(G+2\pi\alpha'\left(\hat{F}+\Phi\right)\right)}W(x,C)\right]*e^{ik.x} =$$
$$= \frac{1}{G_s}\int \sqrt{\det G}\,L_*\left[\left(1-\frac{1}{4}(2\pi\alpha')^2 trG^{-1}\left(\hat{F}+\Phi\right)G^{-1}\left(\hat{F}+\Phi\right)\right)W(x,C)\right]*e^{ik.x}+\ldots \quad (137)$$

The piece of the above expression that is order 1 in α' has already been computed earlier for the dilaton coupling. It contributes to the coupling of the trace of the graviton. The new non trivial coupling is given by the order $(\alpha')^2$ term.
To compare with the commutative side, it is convenient to expand the above action differently, in terms of M rather than G. We get:

$$\hat{S}_{DBI} = \frac{1}{G_s}\int\sqrt{\det(G+2\pi\alpha'\Phi)}\left[L_*(W(x,C)) + \frac{2\pi\alpha'}{2}\left\{trMF + M^{kl}\theta^{ij}\langle\partial_j F_{lk}, A_i\rangle_{*2} + \frac{1}{2}\langle trMF, tr\theta F\rangle_{*2}\right\} + \right.$$
$$\left. -\frac{(2\pi\alpha')^2}{4}tr\langle MF, MF\rangle_{*2} + \frac{(2\pi\alpha')^2}{8}\langle trMF, trMF\rangle_{*2} + \ldots\right]. \quad (138)$$

Now taking the difference of the noncommutative and commutative actions in eqs. (138) and (136), and expanding the result to 4-derivative order, we get the prediction:

$$\Delta S_{DBI}\big|_{SW} = -\frac{2\pi\alpha'}{48}\left\{M^{ij}\theta^{kl}\theta^{mn}\theta^{rs}\partial_m\partial_r F_{jk}\partial_n\partial_s F_{li} - \frac{1}{2}M^{ij}\theta^{kl}\theta^{mn}\theta^{rs}\partial_m\partial_r F_{ji}\partial_n\partial_s F_{lk}\right\} +$$
$$-\frac{(2\pi\alpha')^2}{96}\left\{M^{ij}M^{kl}\theta^{mn}\theta^{rs}\partial_m\partial_r F_{jk}\partial_n\partial_s F_{li} - \frac{1}{2}M^{ij}M^{kl}\theta^{mn}\theta^{rs}\partial_m\partial_r F_{ji}\partial_n\partial_s F_{lk}\right\}. \quad (139)$$

Note that contrary to appearances, both of the above terms are of order $(\alpha')^2$. This is because if one inserts G in place of M in the first line, the result vanishes.

3.2 *The noncommutative open string actions, Chern-Simons theory and D-brane actions*

Let us focus on a particular Chern-Simons coupling, the one involving the Ramond-Ramond 6-form $C^{(6)}$. In the commutative theory, this is just

$$\frac{1}{2}\int C^{(6)}\wedge F\wedge F. \quad (140)$$

The noncommutative version of this coupling is obtained from the commutative one by making the replacement:

$$F \to \hat{F}\frac{1}{1-\theta\hat{F}} \quad (141)$$

where $\hat{F}$ is the noncommutative gauge field strength, multiplying the action by a factor $\sqrt{\det(1-\theta\hat{F})}$, and using the Moyal $*$-product defined in the following equation:

$$f(x)*g(x) \equiv f(x)e^{\frac{i}{2}\overleftarrow{\partial}_p\theta^{pq}\overrightarrow{\partial}_q}g(x). \quad (142)$$

To make a coupling that is gauge-invariant even for nonconstant fields, this has to be combined with an open Wilson line. The resulting expression for the coupling to $C^{(6)}$ is more conveniently expressed in momentum space, where $\tilde{C}^{(6)}(k)$ is the Fourier transform of $C^{(6)}(x)$:

$$\frac{1}{2}\tilde{C}^{(6)}(-k)\wedge\int L_*\left[\sqrt{\det(1-\theta\hat{F})}\left(\hat{F}\frac{1}{1-\theta\hat{F}}\right)\wedge\left(\hat{F}\frac{1}{1-\theta\hat{F}}\right)W(x,C)\right]*e^{ik.x}. \quad (143)$$

Here $W(x,C)$ is an open Wilson line, and L_* is the prescription of smearing local operators along the Wilson line and path-ordering with respect to the Moyal product. Evaluation of the L_* prescription leads to $*_n$ products.
The above expression can easily be re-expressed in position space, and it turns into:

$$\frac{1}{2}\int C^{(6)} \wedge (F \wedge F)_{*2} \,. \quad (144)$$

Let us denote the sum of all derivative corrections to S_{CS} as ΔS_{CS}. Our starting point is the expression

$$S_{CS} + \Delta S_{CS} = \left\langle C \middle| e^{-\frac{i}{2\pi\alpha'}\int d\sigma d\theta D\phi^i A_i(\phi)} \middle| B \right\rangle_R \qquad (145)$$

where $|C\rangle$ represents the RR field, and $|B\rangle_R$ is the Ramond-sector boundary state for zero field strength. We are using superspace notation, for example $\phi^i = X^i + \theta\psi^i$ and D is the supercovariant derivative. It is possible write the following expression:

$$S_{CS} + \Delta S_{CS} = \left\langle C \middle| e^{\frac{i}{2\pi\alpha'}\int d\sigma d\theta \sum_{k=0}^{\infty} \frac{1}{(k+1)!}\frac{k+1}{k+2} D\tilde{\phi}^j \tilde{\phi}^i \tilde{\phi}^{a_1} \ldots \tilde{\phi}^{a_k} \partial_{a_1} \ldots \partial_{a_k} F_{ij}(x)} \times e^{\frac{i}{2\pi\alpha'}\int d\sigma \left[\tilde{\Psi}^i \psi_0^j + \psi_0^i \psi_0^j\right] \sum_{k=0}^{\infty} \frac{1}{k!} \tilde{X}^{a_1} .. \tilde{X}^{a_k} \partial_{a_1} .. \partial_{a_k} F_{ij}(x)} \middle| B \right\rangle_R$$

(146)

where nonzero modes have a tilde on them, while the zero modes are explicitly indicated.
We can drop the first exponential factor in eq. (146) above, as well as the first fermion bilinear $\tilde{\Psi}^i \psi_0^j$ in the second exponential. Then, expanding the exponential to second order, we get:

$$S_{CS} + \Delta S_{CS} = \frac{1}{2}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\left(\frac{i}{2\pi\alpha'}\right)^2 \int_0^{2\pi} d\sigma_1 \int_0^{2\pi} d\sigma_2 \Big\langle C \Big| \left(\frac{1}{2}\psi_0^i \psi_0^j\right)\left(\frac{1}{2}\psi_0^k \psi_0^l\right) \times$$
$$\times \frac{1}{n!}\tilde{X}^{a_1}(\sigma_1)\ldots\tilde{X}^{a_n}(\sigma_1)\frac{1}{p!}\tilde{X}^{b_1}(\sigma_2)\ldots\tilde{X}^{b_p}(\sigma_2) \times \partial_{a_1}\ldots\partial_{a_n} F_{ij}(x)\partial_{b_1}\ldots\partial_{b_p} F_{kl}(x)\Big| B\Big\rangle_R \,. \quad (147)$$

Now we need to evaluate the 2-point functions of the $\tilde{X}$. The relevant contributions have non-logarithmic finite parts and come from propagators for which there is no self-contraction. This requires that $n = p$. Then we get a combinatorial factor of $n!$ from the number of such contractions in $\left\langle \left(\tilde{X}(\sigma_1)\right)^n \left(\tilde{X}(\sigma_2)\right)^n \right\rangle$. The result is:

$$S_{CS} + \Delta S_{CS} = \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{i}{2\pi\alpha'}\right)^2 \int_0^{2\pi} d\sigma_1 \int_0^{2\pi} d\sigma_2 D^{a_1 b_1}(\sigma_1 - \sigma_2)\ldots D^{a_n b_n}(\sigma_1 - \sigma_2) \times$$
$$\times \partial_{a_1}\ldots\partial_{a_n} F_{ij}(x)\partial_{b_1}\ldots\partial_{b_n} F_{kl}(x)\Big\langle C \Big| \left(\frac{1}{2}\psi_0^i \psi_0^j\right)\left(\frac{1}{2}\psi_0^k \psi_0^l\right)\Big| B\Big\rangle_R \,. \quad (148)$$

The fermion zero mode expectation values are evaluated using the recipe:

$$\frac{1}{2}\psi_0^i\psi_0^j F_{ij} \to (-i\alpha')F \quad (149)$$

where the F on the right hand side is a differential 2-form. Thus we are led to:

$$S_{CS} + \Delta S_{CS} = T^{a_1 \ldots a_n; b_1 \ldots b_n} \partial_{a_1} \ldots \partial_{a_n} F \wedge \partial_{b_1} \ldots \partial_{b_n} F \quad (150)$$

where

$$T^{a_1 \ldots a_n; b_1 \ldots b_n} \equiv \frac{1}{2}\frac{1}{n!}\left(\frac{i}{2\pi\alpha'}\right)^2 (-i\alpha')^2 \int_0^{2\pi} d\sigma_1 \int_0^{2\pi} d\sigma_2 D^{a_1 b_1}(\sigma_1 - \sigma_2) \ldots D^{a_n b_n}(\sigma_1 - \sigma_2). \quad (151)$$

Thence, we obtain:

$$S_{CS} + \Delta S_{CS} = \partial_{a_1} \ldots \partial_{a_n} F \wedge \partial_{b_1} \ldots \partial_{b_n} F \times$$
$$\times \frac{1}{2}\frac{1}{n!}\left(\frac{i}{2\pi\alpha'}\right)^2 (-i\alpha')^2 \int_0^{2\pi} d\sigma_1 \int_0^{2\pi} d\sigma_2 D^{a_1 b_1}(\sigma_1 - \sigma_2) \ldots D^{a_n b_n}(\sigma_1 - \sigma_2). \quad (151b)$$

Next we insert the expression for the propagator:

$$D^{ab}(\sigma_1 - \sigma_2) = \alpha' \sum_{m=1}^{\infty} \frac{e^{-\varepsilon m}}{m}\left(h^{ab} e^{im(\sigma_2 - \sigma_1)} + h^{ba} e^{-im(\sigma_2 - \sigma_1)}\right) \quad (152)$$

where ε is a regulator, and

$$h^{ij} \equiv \frac{1}{g + 2\pi\alpha' B}. \quad (153)$$

As we have seen, this tensor when expanded about large B has the form:

$$h^{ij} \approx \frac{\theta^{ij}}{2\pi\alpha'} - \frac{(\theta g \theta)^{ij}}{2\pi\alpha'^2} + \ldots \quad (154)$$

where the terms in the expansion are alternatively antisymmetric and symmetric, and the two terms exhibited above are description-independent. Now we neglect all but the first term above. It follows that

$$T^{a_1 \ldots a_n; b_1 \ldots b_n} = \frac{1}{2}\frac{1}{n!}(\alpha')^n \int_0^{2\pi} \frac{d\sigma}{2\pi} \prod_{i=1}^{n}\left(h^{a_i b_i}\sum_{m=1}^{\infty}\frac{e^{-\varepsilon m + im\sigma}}{m} + h^{b_i a_i}\sum_{m=1}^{\infty}\frac{e^{-\varepsilon m - im\sigma}}{m}\right). \quad (155)$$

After evaluating the sum over m, the result, depending on the regulator ε, is

$$T^{a_1 \ldots a_n; b_1 \ldots b_n} = \frac{1}{2}\frac{1}{n!}(\alpha')^n \int_0^{2\pi} \frac{d\sigma}{2\pi} \prod_{i=1}^{n}\left(-h^{[a_i b_i]}\ln\frac{(1 - e^{-\varepsilon + i\sigma})}{(1 - e^{-\varepsilon - i\sigma})} - h^{(a_i b_i)}\ln\left|1 - e^{-\varepsilon + i\sigma}\right|^2\right). \quad (156)$$

Here, $h^{[ab]}$ and $h^{(ab)}$ are, respectively, the antisymmetric and symmetric parts of h^{ab}. The large-B or Seiberg-Witten limit consists of the replacements:

$$h^{[ab]} \to \frac{\theta^{ab}}{2\pi\alpha'}, \qquad h^{(ab)} \to 0. \quad (157)$$

By virtue of the fact that

$$\lim_{\varepsilon\to 0} \ln \frac{\left(1-e^{-\varepsilon+i\sigma}\right)}{\left(1-e^{-\varepsilon-i\sigma}\right)} = i(\sigma-\pi) \quad (158)$$

this limit leads to an elementary integral. Evaluating it, one finally obtains the result:

$$S_{CS} + \Delta S_{CS} = \frac{1}{2}\int C^{(6)} \wedge \sum_{j=0}^{\infty} (-1)^j \frac{1}{2^{2j}(2j+1)!} \theta^{a_1 b_1} \ldots \theta^{a_{2j} b_{2j}} \partial_{a_1} \ldots \partial_{a_{2j}} F \wedge \partial_{b_1} \ldots \partial_{b_{2j}} F =$$
$$= \frac{1}{2}\int C^{(6)} \wedge (F \wedge F)_{*2} \quad (159)$$

where the product $*_2$ is defined by the following equation:

$$\left\langle F_{ij}(x), F_{kl}(x)\right\rangle_{*2} \equiv F_{ij}(x) \frac{\sin\left(\frac{1}{2}\overleftarrow{\partial}_p \theta^{pq} \vec{\partial}_q\right)}{\frac{1}{2}\overleftarrow{\partial}_p \theta^{pq} \vec{\partial}_q} F_{kl}(x). \quad (160)$$

This agree perfectly with eq. (144), the prediction from noncommutativity.
Now we extend the calculation from the point of eq. (156). To go to first order beyond the Seiberg-Witten limit, we make the replacements:

$$h^{[ab]} \to \frac{\theta^{ab}}{2\pi\alpha'}, \qquad h^{(ab)} \to -\frac{\theta^{ac} g_{cd} \theta^{db}}{(2\pi\alpha')^2} \quad (161)$$

and keep all terms that are first order in $h^{(ab)}$. Denote by $T_{(1)}^{a_1 \ldots a_n ; b_1 \ldots b_n}$ the first correction to T (defined in eq. (151)) away from the Seiberg-Witten limit. Then, we see that

$$T_{(1)}^{a_1 \ldots a_n ; b_1 \ldots b_n} = \frac{(-i)^{n-1}}{2} \frac{1}{(n-1)!} \frac{1}{(2\pi)^n} \frac{(\theta g \theta)^{a_1 b_1}}{2\pi\alpha'} \theta^{a_2 b_2} \ldots \theta^{a_n b_n} \int_0^{2\pi} \frac{d\sigma}{2\pi} (\sigma-\pi)^{n-1} \ln\left|1-e^{i\sigma}\right|^2. \quad (162)$$

The integral in the above expression vanishes for even n. For odd $n = 2p+1$, we find that

$$T_{(1)}^{a_1 \ldots a_{2p+1} ; b_1 \ldots b_{2p+1}} = \frac{(-1)^p}{2} \frac{1}{(2p)!} \frac{1}{(2\pi)^{2p+1}} \frac{(\theta g \theta)^{a_1 b_1}}{2\pi\alpha'} \theta^{a_2 b_2} \ldots \theta^{a_{2p+1} b_{2p+1}} I_{2p+1} \quad (163)$$

where

$$I_{2p+1} \equiv \int_0^{2\pi} \frac{d\sigma}{2\pi} (\sigma - \pi)^{2p} \ln\left|1 - e^{i\sigma}\right|^2 = 2(-1)^p (2p)! \sum_{j=0}^{p-1} (-1)^j \frac{\pi^{2j}}{(2j+1)!} \zeta(2p - 2j + 1), \quad (164)$$

where ζ is the Riemann's zeta function.
Thence, we obtain:

$$T_{(1)}^{a_1 \ldots a_{2p+1}; b_1 \ldots b_{2p+1}} = \frac{(-1)^p}{2} \frac{1}{(2p)!} \frac{1}{(2\pi)^{2p+1}} \frac{(\theta g \theta)^{a_1 b_1}}{2\pi\alpha'} \theta^{a_2 b_2} \ldots \theta^{a_{2p+1} b_{2p+1}} \times$$

$$\times \int_0^{2\pi} \frac{d\sigma}{2\pi} (\sigma - \pi)^{2p} \ln\left|1 - e^{i\sigma}\right|^2 = 2(-1)^p (2p)! \sum_{j=0}^{p-1} (-1)^j \frac{\pi^{2j}}{(2j+1)!} \zeta(2p - 2j + 1). \quad (164b)$$

It is convenient to define a 4-form W_4 that encodes the derivative corrections for the coupling to $C^{(6)}$:

$$S_{CS} + \Delta S_{CS} = \frac{1}{2} \int C^{(6)} \wedge F \wedge F + \int C^{(6)} \wedge W_4. \quad (165)$$

The leading-order term in W_4 in the large-B limit is:

$$W_4^{(0)} = \langle F \wedge F \rangle_{*2} - F \wedge F. \quad (166)$$

The calculations leading to eq. (164) amount to computing W_4 to first order (in α') around the Seiberg-Witten limit:

$$W_4^{(1)} = \sum_{p=1}^{\infty} \frac{1}{(2\pi)^{2p+1}} \frac{(\theta g \theta)^{a_1 b_1}}{2\pi\alpha'} \theta^{a_2 b_2} \ldots \theta^{a_{2p+1} b_{2p+1}} \partial_{a_1} \ldots \partial_{a_{2p+1}} F \wedge \partial_{b_1} \ldots \partial_{b_{2p+1}} F \times \sum_{j=0}^{p-1} (-1)^j \frac{\pi^{2j}}{(2j+1)!} \zeta(2p - 2j + 1) \quad (167)$$

Interchanging the order of the two summations, we find that the sum over j can be performed and leads to the appearance of the familiar $*_2$ product. The result, after some relabeling of indices, is:

$$W_4^{(1)} = \sum_{p=0}^{\infty} \frac{1}{(2\pi)^{2p+3}} \frac{(\theta g \theta)^{cd}}{2\pi\alpha'} \zeta(2p+3) \theta^{a_1 b_1} \ldots \theta^{a_{2p+2} b_{2p+2}} \times \left\langle \partial_c \partial_{a_1} \ldots \partial_{a_{2p+2}} F \wedge \partial_d \partial_{b_1} \ldots \partial_{b_{2p+2}} F \right\rangle_{*2}. \quad (168)$$

Unlike the leading term eq. (159), which is a single infinite series in derivatives summarised by the $*_2$ product, here we see a double infinite series. After forming the $*_2$ product we still have an additional series whose coefficients are ζ-functions (Riemann's zeta functions) of odd argument. We will now use this to extract the term corresponding to our computation in eq. (168) and compare the two expressions. The computation is an evaluation of the amplitude:

$$A_2 \equiv \int_{-\infty}^{\infty} dy \left\langle V_{RR}^{-\frac{1}{2}, -\frac{3}{2}}(q; i) V_O^0(a_1, k_1; 0) V_O^0(a_2, k_2; y) \right\rangle \quad (169)$$

where $V_{RR}^{-\frac{1}{2},-\frac{3}{2}}$ is the vertex operator for an RR potential of momentum q, in the $\left(-\frac{1}{2},-\frac{3}{2}\right)$ picture, and V_O^0 are vertex operators for massless gauge fields of momentum k_i and polarizations a_i, $i=1,2$. We define:

$$t \equiv \alpha' k_1 \cdot k_2 = \alpha' k_{1i} G^{ij} k_{2j}; \qquad a \equiv \frac{1}{2\pi} k_1 \times k_2 = \frac{1}{2\pi} k_{1i}\theta^{ij} k_{2j} \quad (170)$$

and change integration variables via $y = -\cot \pi\tau$. Then it follows that the coefficient of $F \wedge F$ provided by this computation, to be compared with the coefficient of eq. (168), is:

$$2^{2t}\int_0^{\frac{1}{2}} d\tau (\cos \pi\tau)^{2t} \cos 2\pi a \tau = \frac{1}{2}\frac{\Gamma(1+2t)}{\Gamma(1+a+t)\Gamma(1-a+t)}. \quad (171)$$

The right hand side can be expanded in powers of t and, up to terms of $\mathrm{O}(t^2)$, one has:

$$\frac{\Gamma(1+2t)}{\Gamma(1+a+t)\Gamma(1-a+t)} = \frac{1}{\Gamma(1-a)\Gamma(1+a)}\left[1-(2\gamma+\psi(1-a)+\psi(1+a))t+\mathrm{O}(t^2)\right] \quad (172)$$

where γ is the Euler constant and $\psi(x)$ is the digamma function $\frac{d}{dx}\ln\Gamma(x)$.
The first term can be recognised as the kernel of the $*_2$-product, using the relation:

$$\frac{1}{\Gamma(1-a)\Gamma(1+a)} = \frac{\sin \pi a}{\pi a}. \quad (173)$$

Let us now examine the second term more carefully. We use the fact that:

$$\psi(1+x) = -\gamma + \sum_{k=2}^{\infty}(-1)^k \zeta(k) x^{k-1} \quad (174)$$

to write:

$$2\gamma + \psi(1-a) + \psi(1+a) = -\sum_{k=2}^{\infty}\left(1-(-1)^k\right)\zeta(k)a^{k-1} = -2\sum_{p=0}^{\infty}\zeta(2p+3)a^{2p+2}. \quad (175)$$

Putting everything together, we find that:

$$W_4^{(1)} = \frac{\sin\left(\frac{k_1\times k_2}{2}\right)}{\frac{k_1\times k_2}{2}}\sum_{p=0}^{\infty}\frac{1}{2\pi\alpha'(2\pi)^{2p+3}}\zeta(2p+3)(k_1\times k_2)^{2p+2}\left(k_{1i}(\theta g\theta)^{ij}k_{2j}\right)\tilde{F}(k_1)\wedge\tilde{F}(k_2). \quad (176)$$

On Fourier transforming, this is identical to eq. (168).
Thence, we obtain:

$$\frac{1}{2}\frac{1}{\Gamma(1-a)\Gamma(1+a)}\left[1-\left(-2\sum_{p=0}^{\infty}\zeta(2p+3)a^{2p+2}\right)t+O(t^2)\right]=2^{2t}\int_0^{\frac{1}{2}}d\tau(\cos\pi\tau)^{2t}\cos 2\pi a\tau, \quad (176b)$$

$$S_{CS}+\Delta S_{CS}=\frac{1}{2}\int C^{(6)}\wedge F\wedge F+\int C^{(6)}\wedge$$

$$\wedge\frac{\sin\left(\frac{k_1\times k_2}{2}\right)}{\frac{k_1\times k_2}{2}}\sum_{p=0}^{\infty}\frac{1}{2\pi\alpha'(2\pi)^{2p+3}}\zeta(2p+3)(k_1\times k_2)^{2p+2}\left(k_{1i}(\theta g\theta)^{ij}k_{2j}\right)\tilde{F}(k_1)\wedge\tilde{F}(k_2), \quad (176c)$$

where ζ is the Riemann's zeta function.

Now we consider Chern-Simons gauge theory with the gauge group G a complex Lie group such as $SL(2,C)$. Let A be a connection on a G-bundle E over a three-manifold M. Such a connection has a complex-valued Chern-Simons invariant

$$W(A)=\frac{1}{4\pi}\int_M Tr\left(A\wedge dA+\frac{2}{3}A\wedge A\wedge A\right), \quad (177)$$

which we have normalized to be gauge-invariant modulo 2π. Just as in the case of a compact group, $W(A)$ is gauge-invariant modulo 2π. The indeterminacy in $W(A)$ is real, even if the gauge group is complex, because a complex Lie group is contractible to its maximal compact subgroup. The quantum theory is based upon integrating the expression $\exp(iI)$, and for this function to be well-defined, I must be defined $\mathrm{mod}\,2\pi$. Because of the indeterminacy in $\mathrm{Re}W$, its coefficient must be an integer, while the coefficient of $\mathrm{Im}W$ may be an arbitrary complex number. The action therefore has the general form

$$I=-s\,\mathrm{Im}W+\ell\,\mathrm{Re}W.\quad s\in C,\ \ell\in Z. \quad (178)$$

Alternatively, we can write

$$I=\frac{tW}{2}+\frac{\tilde{t}\,\overline{W}}{2}=\frac{t}{8\pi}\int_M Tr\left(A\wedge dA+\frac{2}{3}A\wedge A\wedge A\right)+\frac{\tilde{t}}{8\pi}\int_M Tr\left(\overline{A}\wedge d\overline{A}+\frac{2}{3}\overline{A}\wedge\overline{A}\wedge\overline{A}\right), \quad (179)$$

with

$$t=\ell+is,\quad \tilde{t}=\ell-is. \quad (180)$$

Introduce a complex variable z and a complex-valued polynomial

$$g(z)=\sum_{j=0}^{n}a_j z^j. \quad (181)$$

Now consider the integral

$$Z_g = \int \left| d^2 z \right| \exp\left(g(z) - \overline{g}(\overline{z}) \right). \quad (182)$$

This again is a convergent oscillatory integral and a closer analogous of Chern-Simons theory, with $-ig(z)$ and $i\overline{g}(\overline{z})$ corresponding to the terms tW and $\tilde{t}\,\overline{W}$ in the action (179).
Define a new polynomial

$$\tilde{g}(z) = \sum_{j=0}^{n} \tilde{a}_j z^j , \quad (183)$$

and generalize the integral Z_g to

$$Z_{g,\tilde{g}} = \int \left| d^2 z \right| \exp\left(g(z) - \tilde{g}(\overline{z}) \right). \quad (184)$$

Then $Z_{g,\tilde{g}}$ coincides with the original Z_g if $\tilde{g} = \overline{g}$.
Denoting the independent complex variables as z and $\tilde{z}$, the analytically continued integral is

$$Z_{g,\tilde{g}} = \int_{\mathcal{C}} dz d\tilde{z} \exp\left(g(z) - \tilde{g}(\tilde{z}) \right). \quad (185)$$

The integral is over a two-dimensional real integration cycle $\mathcal{C}$, which is the real slice $\tilde{z} = \overline{z}$ if $\tilde{g} = \overline{g}$, and in general must be deformed as g and $\tilde{g}$ vary so that the integral remains convergent.
Just as we promoted $\overline{z}$ to a complex variable $\tilde{z}$ that is independent of z, we will have to promote $\overline{\mathrm{A}}$ to a new G-valued connection $\tilde{\mathrm{A}}$ that is independent of A. Thus we consider the classical theory with independent G-valued connections A and $\tilde{\mathrm{A}}$ (we recall that G is a complex Lie group such as $SL(2,C)$) and action

$$I\left(\mathrm{A}, \tilde{\mathrm{A}}\right) = \frac{t}{8\pi} \int_M Tr\left(\mathrm{A} \wedge d\mathrm{A} + \frac{2}{3} \mathrm{A} \wedge \mathrm{A} \wedge \mathrm{A} \right) + \frac{\tilde{t}}{8\pi} \int_M Tr\left(\tilde{\mathrm{A}} \wedge d\tilde{\mathrm{A}} + \frac{2}{3} \tilde{\mathrm{A}} \wedge \tilde{\mathrm{A}} \wedge \tilde{\mathrm{A}} \right). \quad (186)$$

Then we have to find an appropriate integration cycle $\mathcal{C}$ in the path integral

$$\int_{\mathcal{C}} D\mathrm{A} D\tilde{\mathrm{A}} \exp\left(iI\left(\mathrm{A}, \tilde{\mathrm{A}}\right) \right). \quad (187)$$

The cycle $\mathcal{C}$ must be equivalent to $\tilde{\mathrm{A}} = \overline{\mathrm{A}}$ if s is real, and in general must be obtained by deforming that one as s varies so that the integral remains convergent.

Let us consider a noncommutative Euclidean Dp-brane with an even number p+1 of world-volume directions. Given a collection of local operators $\mathrm{O}_I(x)$ on the brane world-volume which transform in the adjoint under gauge transformations, one can obtain a natural gauge-invariant operator of fixed momentum k^i by smearing the locations of these operators along a straight contour given by $\xi^i(\tau) = \theta^{ij} k_j \tau$ with $0 \le \tau \le 1$, and multiplying the product by a Wilson line $W(x,C)$ along the same contour,

$$W(x,C)\equiv \exp\left(i\int_0^1 d\tau \frac{\partial \xi^i(\tau)}{\partial \tau}\hat{A}_i(x+\xi(\tau))\right), \quad (188)$$

where $\hat{A}$ denotes the noncommutative gauge field, and $\hat{F}$ the corresponding field strength. The resulting formula for the gauge-invariant operator is:

$$\mathcal{Q}(k)=\int \frac{1}{(2\pi)^{p+1}} d^{p+1}x\left(\prod_{I=1}^{n}\int_0^1 d\tau_I\right)P_*\left[W(x,C)\prod_{I=1}^{n}\mathrm{O}_I(x+\xi(\tau_I))\right]*e^{ik.x}=$$
$$=\int \frac{1}{(2\pi)^{p+1}} d^{p+1}xL_*\left[W(x,C)\prod_{I=1}^{n}\mathrm{O}_I(x)\right]*e^{ik.x} \quad (189)$$

where P_* denotes path-ordering with respect to the $*$-product, while L_* is an abbreviation for the combined path-ordering and integrations over τ_I. In this formula the operators O_I are smeared over the straight contour of the Wilson line. This prescription arises by starting with the symmetrised-trace action for infinitely many D-instantons and expanding it around the configuration describing a noncommutative Dp-brane.
Expanding the Wilson line, we get

$$\mathcal{Q}(k)=\sum_{m=0}^{\infty}\int \frac{1}{(2\pi)^{p+1}} d^{p+1}x\mathcal{Q}_m(x)e^{ik.x} \quad (190)$$

where

$$\mathcal{Q}_m(x)=\frac{1}{m!}(\theta\partial)^{i_1}...(\theta\partial)^{i_m}\left\langle \mathrm{O}_1(x),...,\mathrm{O}_n(x),\hat{A}_{i_1}(x),...,\hat{A}_{i_m}(x)\right\rangle_{*m+n}. \quad (191)$$

Thence, we obtain the following expression:

$$\mathcal{Q}(k)=\sum_{m=0}^{\infty}\int \frac{1}{(2\pi)^{p+1}} d^{p+1}x\ \frac{1}{m!}(\theta\partial)^{i_1}...(\theta\partial)^{i_m}\left\langle \mathrm{O}_1(x),...,\mathrm{O}_n(x),\hat{A}_{i_1}(x),...,\hat{A}_{i_m}(x)\right\rangle_{*m+n}\cdot e^{ik.x}. \quad (191b)$$

Here we have introduced the notation $\langle f_1(x), f_2(x),..., f_p(x)\rangle_{*p}$ for the $*_p$ product of p functions. We note here the simple formula for $*_2$:

$$\langle f(x), g(x)\rangle_{*_2} \equiv f(x)\frac{\sin\left(\frac{1}{2}\overleftarrow{\partial}_p\theta^{pq}\overrightarrow{\partial}_q\right)}{\frac{1}{2}\overleftarrow{\partial}_p\theta^{pq}\overrightarrow{\partial}_q}g(x). \quad (192)$$

If the zero-momentum coupling is

$$\tilde{\Sigma}(0)\int \frac{1}{(2\pi)^{p+1}} d^{p+1}x\frac{1}{Pf\theta}\mathrm{O}_\Sigma(A,X) \quad (193)$$

where A is the gauge field and X are the transverse scalars, then the coupling at nonzero momentum is given by

$$\tilde{\Sigma}(-k)\int \frac{1}{(2\pi)^{p+1}} d^{p+1}x \frac{1}{Pf\theta} L_*[\mathrm{O}_\Sigma(A,X)W(x,C)]*e^{ik.x} \,. \quad (194)$$

The constant factor $\mathrm{Pf}\,\theta \equiv \sqrt{\det\theta^{ij}}$ has been written explicitly, instead of absorbing it into the definition of O_Σ, for convenience. In our case the relevant closed string mode is the RR gauge potential $\tilde{C}^{p+1}(k)$, and, the role of O_Σ is played by the operator $\mu_p PfQ$ where μ_p is the brane tension and

$$Q^{ij} \equiv \theta^{ij} - \theta^{ik}\hat{F}_{kl}\theta^{lj} \,. \quad (195)$$

Hence we deduce the coupling of this brane to the form $C^{(p+1)}$, in momentum space, to be:

$$\mu_p \varepsilon^{i_1 \dots i_{p+1}} \tilde{C}^{(p+1)}_{i_1 \dots i_{p+1}}(-k)\int \frac{1}{(2\pi)^{p+1}} d^{p+1}x L_*\left[\sqrt{\det(1-\theta\hat{F})}W(x,C)\right]*e^{ik.x} \,. \quad (196)$$

On the other hand, we know that the coupling of a Dp-brane to a $C^{(p+1)}$ form is given in the commutative description by

$$\mu_p \varepsilon^{i_1 \dots i_{p+1}} \tilde{C}^{(p+1)}_{i_1 \dots i_{p+1}}(-k)\delta^{(p+1)}(k) \,. \quad (197)$$

As equations (196) and (197) describe the same system in two different descriptions, they must be equal. Thus we predict the identity:

$$\delta^{(p+1)}(k) = \int \frac{1}{(2\pi)^{p+1}} d^{p+1}x L_*\left[\sqrt{\det(1-\theta\hat{F})}W(x,C)\right]*e^{ik.x} \,. \quad (198)$$

This amounts to saying that the right hand side is actually independent of $\hat{A}$, a rather nontrivial fact. Is interesting to express the eq. (198) in the operator formalism. We use the fact that

$$\int \frac{1}{(2\pi)^{\frac{p+1}{2}}} d^{p+1}x \frac{1}{Pf\theta} \to tr \quad (199)$$

to rewrite the left hand side of eq. (198) as follows:

$$\delta^{p+1}(k) = \int \frac{1}{(2\pi)^{p+1}} d^{p+1}x e^{ik.x} = \frac{1}{(2\pi)^{\frac{p+1}{2}}} tr\left(Pf\theta e^{ik.x}\right) . \quad (200)$$

The right hand side of eq. (198) can be converted to a symmetrised trace involving $X^i \equiv x^i + \theta^{ij}\hat{A}_j(x)$, and it becomes:

$$\frac{1}{(2\pi)^{\frac{p+1}{2}}} Str\left(PfQe^{ik.X}\right) \quad (201)$$

where Str denotes the symmetrised trace. Finally, we use $\left[x^i, x^j\right] = i\theta^{ij}$ and $\left[X^i, X^j\right] = iQ^{ij}(x)$. Then eq. (198) takes the elegant form:

$$tr\left(Pf\left[x^i, x^j\right]e^{ik.x}\right) = Str\left(Pf\left[X^i, X^j\right]e^{ik.X}\right). \quad (202)$$

In this form, it is easy to see that eq. (198) holds for constant $\hat{F}$, or equivalently for constant Q. In this special case it can be proved by pulling PfQ out of the symmetrised trace on the right hand side, and then using eq. (199) with θ replaced by Q.

Now let us turn to the coupling of a noncommutative p-brane to the RR form $C^{(p-1)}$. In the commutative case this form appears in a wedge product with the 2-form $B+F$. For the noncommutative brane in a constant RR background, $B+F$ must be replaced by the 2-form Q^{-1} with Q^{ij} given by eq. (195). It follows that the coupling in the general noncommutative case (with varying $C^{(p-1)}$) is:

$$\mu_p \varepsilon^{i_1 \ldots i_{p+1}} \tilde{C}^{(p-1)}_{i_1 \ldots i_{p-1}}(-k) \int \frac{1}{(2\pi)^{p+1}} d^{p+1}x L_* \left[\sqrt{\det\left(1-\theta\hat{F}\right)}\left(Q^{-1}\right)_{i_p i_{p+1}} W(x,C)\right] * e^{ik.x} . \quad (203)$$

For comparison, the coupling of a Dp-brane to the form $C^{(p-1)}$ in terms of commutative variables is given by:

$$\mu_p \varepsilon^{i_1 \ldots i_{p+1}} \tilde{C}^{(p-1)}_{i_1 \ldots i_{p-1}}(-k) \int \frac{1}{(2\pi)^{p+1}} d^{p+1}x (B+F)_{i_p i_{p+1}}(x) e^{ik.x} = \mu_p \varepsilon^{i_1 \ldots i_{p+1}} \tilde{C}^{(p-1)}_{i_1 \ldots i_{p-1}}(-k)\left[\delta^{p+1}(k) B_{i_p i_{p+1}} + \tilde{F}_{i_p i_{p+1}}(k)\right].$$

(204)

Next, rewrite Q^{-1} as

$$Q^{-1} = \theta^{-1}\left[1 + \theta\hat{F}\left(1-\theta\hat{F}\right)^{-1}\right] = B + \hat{F}\left(1-\theta\hat{F}\right)^{-1} \quad (205)$$

where we have used the relation $B = \theta^{-1}$. Using this relation and also eq. (198), we can rewrite eq. (203) as:

$$\mu_p \varepsilon^{i_1 \ldots i_{p+1}} \tilde{C}^{(p-1)}_{i_1 \ldots i_{p-1}}(-k)\left\{\delta^{p+1}(k) B_{i_p i_{p+1}} + \int \frac{1}{(2\pi)^{p+1}} d^{p+1}x L_* \left[\sqrt{\det\left(1-\theta\hat{F}\right)}\left(\hat{F}\left(1-\theta\hat{F}\right)^{-1}\right)_{i_p i_{p+1}} W(x,C)\right] * e^{ik.x}\right\}.$$

(206)

Equating this to eq. (204), we find that

$$\tilde{F}_{ij}(k) = \int \frac{1}{(2\pi)^{p+1}} d^{p+1}x L_* \left[\sqrt{\det\left(1-\theta\hat{F}\right)}\left(\hat{F}\left(1-\theta\hat{F}\right)^{-1}\right)_{ij} W(x,C)\right] * e^{ik.x} . \quad (207)$$

This relates the commutative field strength F to the non-commutative field strength $\hat{F}$, therefore it amounts to a closed-form expression for the Seiberg-Witten map.

The coupling of a noncommutative Dp-brane to the RR form $C^{(p-3)}$ for the case of constant RR field is:

$$\frac{1}{2}\mu_p \varepsilon^{i_1 \dots i_{p+1}} \tilde{C}^{(p-3)}_{i_1 \dots i_{p-3}}(0) \int \frac{1}{(2\pi)^{p+1}} d^{p+1}x \sqrt{\det\left(1-\theta\hat{F}\right)} \left(Q^{-1}\right)_{i_p - 2 i_{p-1}} \left(Q^{-1}\right)_{i_p i_{p+1}} \quad (208)$$

where the 2-form Q^{-1} is given in eq. (205). For spatially varying $C^{(p-3)}$ we can therefore write the coupling as:

$$\frac{1}{2}\mu_p \varepsilon^{i_1 \dots i_{p+1}} \tilde{C}^{(p-3)}_{i_1 \dots i_{p-3}}(-k) \int \frac{1}{(2\pi)^{p+1}} d^{p+1}x L_* [\sqrt{\det\left(1-\theta\hat{F}\right)} \left(Q^{-1}\right)_{i_p - 2 i_{p-1}} \left(Q^{-1}\right)_{i_p i_{p+1}} W(x,C)] * e^{ik.x} \ . \quad (209)$$

In the DBI approximation of slowly varying fields, the commutative coupling is:

$$\frac{1}{2}\mu_p \varepsilon^{i_1 \dots i_{p+1}} \tilde{C}^{(p-3)}_{i_1 \dots i_{p-3}}(-k) \int \frac{1}{(2\pi)^{p+1}} d^{p+1}x (B+F)_{i_p - 2 i_{p-1}} (B+F)_{i_p i_{p+1}} e^{ik.x} \ . \quad (210)$$

Inserting eq. (205) for Q^{-1} in eq. (209), and comparing with eq. (210), we find that in the DBI approximation we must have:

$$\int d^{p+1}k' \tilde{F}_{ij}(k') F_{kl}(k-k') = \int \frac{1}{(2\pi)^{p+1}} d^{p+1}x L_* \left[\sqrt{\det\left(1-\theta\hat{F}\right)} \left(\hat{F}\left(1-\theta\hat{F}\right)^{-1} \right)_{ij} \left(\hat{F}\left(1-\theta\hat{F}\right)^{-1} \right)_{kl} W(x,C) \right] * e^{ik.x}$$

(211)

To arrive at this expression we have made use of the identities eqs. (198) and (207).
The open Wilson line including transverse scalars is given by:

$$W'(x,C) = P_* \exp\left[i\int_0^1 d\tau \left(\frac{\partial \xi^i(\tau)}{\partial \tau} \hat{A}_i(x+\xi(\tau)) + q_a \hat{\Phi}^a(x+\xi(\tau)) \right) \right]. \quad (212)$$

Inserting this definition in place of $W(x,C)$ in eq. (190), and denoting the left hand side by $\mathcal{Q}'(k)$, one finds that the couplings to a general spatially varying supergravity mode are:

$$\mathcal{Q}'(k) = \sum_{m=0}^{\infty} \int \frac{1}{(2\pi)^{p+1}} d^{p+1}x \mathcal{Q}'_m(x) e^{ik.x} \quad (213)$$

where the $\mathcal{Q}'_m$ are given by

$$\mathcal{Q}'_m(x) = \frac{1}{m!} \sum_{k=0}^{m} \binom{m}{k} (\theta\partial)^{i_1} \dots (\theta\partial)^{i_k} (iq)_{a_{k+1}} \dots (iq)_{a_m} \left\langle \mathrm{O}_1(x), \dots, \mathrm{O}_n(x), \hat{A}_{i_1}(x), \dots, \hat{A}_{i_k}(x), \hat{\Phi}^{a_{k+1}}(x), \dots, \hat{\Phi}^{a_m}(x) \right\rangle_{*_{n+m}} \ .$$

(214)

Thence, we obtain that:

$$\mathcal{Q}'(k) = \sum_{m=0}^{\infty} \int \frac{1}{(2\pi)^{p+1}} d^{p+1}x \times$$

$$\times \frac{1}{m!} \sum_{k=0}^{m} \binom{m}{k} (\theta\partial)^{i_1} \ldots (\theta\partial)^{i_k} (iq)_{a_{k+1}} \ldots (iq)_{a_m} \left\langle \mathrm{O}_1(x), \ldots, \mathrm{O}_n(x), \hat{A}_{i_1}(x), \ldots, \hat{A}_{i_k}(x), \hat{\Phi}^{a_{k+1}}(x), \ldots, \hat{\Phi}^{a_m}(x) \right\rangle_{*_{n+m}} e^{ik.x} \quad .$$

(214b)

It is well-known that non-BPS branes in superstring theory also couple to RR forms. In commutative variables, these couplings for a single non-BPS brane are given by:

$$\hat{S}_{CS} = \frac{1}{2T_0} \mu_{p-1} \int dT \wedge \sum_{n} C^{(n)} \wedge e^{B+F} \quad (215)$$

where T is the tachyon field and T_0 is its value at the minimum of the tachyon potential.

Consider the coupling of a Euclidean non-BPS Dp-brane with an even number of world-volume directions, to the RR form $C^{(p)}$, in the commutative description:

$$\frac{1}{2T_0} \mu_{p-1} \int dT \wedge C^{(p)} = \frac{1}{2T_0} \mu_{p-1} \int d^{p+1}x \varepsilon^{i_1 i_2 \ldots i_{p+1}} \partial_{i_1} T(x) C^{(p)}_{i_2 \ldots i_{p+1}}(x) =$$

$$= \frac{1}{2T_0} \mu_{p-1} \int d^{p+1}k \varepsilon^{i_1 \ldots i_{p+1}} (-ik_{i_1}) \tilde{T}(k) \tilde{C}^{(p)}_{i_2 \ldots i_{p+1}}(-k). \quad (216)$$

The noncommutative generalisation of this coupling, for constant RR fields, is

$$\frac{1}{2T_0} \mu_{p-1} \varepsilon^{i_1 i_2 \ldots i_{p+1}} \tilde{C}^{(p)}_{i_2 \ldots i_{p+1}}(0) \int \frac{1}{(2\pi)^{p+1}} d^{p+1}x \sqrt{\det(1 - \theta\hat{F})} \mathcal{D}_{i_1} \hat{T}(x) \quad (217)$$

where

$$\mathcal{D}_i \hat{T}(x) = -iQ^{-1}_{ij} \left[X^j, \hat{T}(x) \right]. \quad (218)$$

Then, the same RR form $C^{(p)}$ couples to a noncommutative non-BPS Dp-brane through the following coupling for each momentum mode:

$$\frac{1}{2T_0} \mu_{p-1} \varepsilon^{i_1 i_2 \ldots i_{p+1}} \tilde{C}^{(p)}_{i_2 \ldots i_{p+1}}(-k) \int \frac{1}{(2\pi)^{p+1}} d^{p+1}x L_* \left[\sqrt{\det(1 - \theta\hat{F})} \mathcal{D}_{i_1} \hat{T}(x) W(x, C) \right] * e^{ik.x} \quad . \quad (219)$$

4. On some equations concerning the noncommutative quantum mechanics regarding the particle in a constant field and the noncommutative classical dynamics related to quadratic Lagrangians [8]

We consider here the most simple and usual noncommutative quantum mechanics (NCQM) which is based on the following algebra:

$$[\hat{x}_k, \hat{p}_j] = i\hbar\delta_{kj}, \quad [\hat{x}_k, \hat{x}_j] = i\hbar\theta_{kj}, \quad [\hat{p}_k, \hat{p}_j] = 0 \quad (220)$$

where $\Theta = (\theta_{kj})$ is the antisymmetric matrix with constant elements. To find elements $\Psi(x,t)$ of the Hilbert space in ordinary quantum mechanics (OQM), it is usually used the Schrodinger equation

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \hat{H}\Psi(x,t), \quad (221)$$

which realizes the eigenvalue problem for the corresponding Hamiltonian operator $\hat{H} = H(\hat{p}, x, t)$, where $\hat{p}_k = -i\hbar\left(\frac{\partial}{\partial x_k}\right)$.

There is another approach based on the Feynman path integral method

$$\mathrm{K}(x'',t'';x',t') = \int_{(x')}^{(x'')} \exp\left(\frac{i}{\hbar}S[q]\right)\mathcal{D}q, \quad (222)$$

where $\mathrm{K}(x'',t'';x',t')$ is the kernel of the unitary evolution operator $U(t)$ acting on $\Psi(x,t)$ in $L_2(R^D)$.

Path integral in its most general formulation contains integration over paths in the phase space $R^{2D} = \{(p,q)\}$ with fixed end points x' and x'', and no restrictions on the initial and final values of the momenta, i.e.

$$\mathrm{K}(x'',t'';x',t') = \int_{(x')}^{(x'')} \exp\left(\frac{2\pi i}{h}\int_{t'}^{t''}[p_k\dot{q}_k - H(p,q,t)]dt\right)\mathcal{D}q\mathcal{D}p, \quad (223)$$

The Feynman path integral for quadratic Lagrangians can be evaluated analytically and the exact expression for the probability amplitude is:

$$\mathrm{K}(x'',t'';x',t') = \frac{1}{(ih)^{\frac{D}{2}}}\sqrt{\det\left(-\frac{\partial^2\bar{S}}{\partial x_k''\partial x_j'}\right)} \times \exp\left(\frac{2\pi i}{h}\bar{S}(x'',t'';x',t')\right), \quad (224)$$

where $\bar{S}(x'',t'';x',t')$ is the action for the classical trajectory which is the solution of the Euler-Lagrange equation of motion.

Thence, we obtain that:

$$\mathrm{K}(x'',t'';x',t') = \int_{(x')}^{(x'')} \exp\left(\frac{2\pi i}{h}\int_{t'}^{t''}[p_k\dot{q}_k - H(p,q,t)]dt\right)\mathcal{D}q\mathcal{D}p =$$

$$= \frac{1}{(ih)^{\frac{D}{2}}}\sqrt{\det\left(-\frac{\partial^2\bar{S}}{\partial x_k''\partial x_j'}\right)} \times \exp\left(\frac{2\pi i}{h}\bar{S}(x'',t'';x',t')\right). \quad (224b)$$

If we known the following Lagrangian

$$L(\dot{x},x,t) = \langle\alpha\dot{x},\dot{x}\rangle + \langle\beta x,\dot{x}\rangle + \langle\gamma x,x\rangle + \langle\delta,\dot{x}\rangle + \langle\eta,x\rangle + \phi, \quad (225)$$

and algebra (220), we can obtain the corresponding effective Lagrangian

$$L_\theta(\dot{q},q,t)=\langle\alpha_\theta\dot{q},\dot{q}\rangle+\langle\beta_\theta q,\dot{q}\rangle+\langle\gamma_\theta q,q\rangle+\langle\delta_\theta,\dot{q}\rangle+\langle\eta_\theta,q\rangle+\phi_\theta \quad (226)$$

that is suitable for quantization with path integral in NCQM. Exploiting the Euler-Lagrange equations

$$\frac{\partial L_\theta}{\partial q_k}-\frac{d}{dt}\frac{\partial L_\theta}{\partial \dot{q}_k}=0, \quad k=1,2,\dots,D$$

one can obtain classical path $q_k=q_k(t)$ connecting given end points $x'=q(t')$ and $x''=q(t'')$. For this classical trajectory one can calculate action

$$\bar{S}_\theta(x'',t'';x',t')=\int_{t'}^{t''}L_\theta(\dot{q},q,t)dt .$$

Path integral in NCQM is a direct analog of (222) and its exact expression in the form of quadratic actions $\bar{S}_\theta(x'',t'';x',t')$ is

$$\mathrm{K}_\theta(x'',t'';x',t')=\frac{1}{(ih)^{\frac{D}{2}}}\sqrt{\det\left(-\frac{\partial^2\bar{S}_\theta}{\partial x_k''\partial x_j'}\right)}\times\exp\left(\frac{2\pi i}{h}\bar{S}_\theta(x'',t'';x',t')\right). \quad (227)$$

For a particle in a constant field, the Lagrangian on commutative configuration space is:

$$L(\dot{x},x)=\frac{m}{2}(\dot{x}_1^2+\dot{x}_2^2)-\eta_1x_1-\eta_2x_2. \quad (228)$$

The corresponding data in the matrix form are:

$$\alpha=\frac{m}{2}I, \quad \beta=0, \quad \gamma=0, \quad \delta=0, \quad \eta^\tau=(-\eta_1,-\eta_2), \quad \phi=0, \quad (229)$$

where I is 2×2 unit matrix. We now note that one can easily find

$$\alpha_\theta=\frac{m}{2}I, \quad \beta_\theta=0, \quad \gamma_\theta=0, \quad \eta_\theta=\eta, \quad \delta_\theta^\tau=\frac{m\theta}{2}(-\eta_2,\eta_1), \quad \phi_\theta=\frac{m\theta^2}{8}(\eta_1^2+\eta_2^2). \quad (230)$$

In this case, it is easy to find the classical action. The Lagrangian $L_\theta(\dot{q},q,t)$ is

$$L_\theta=\frac{1}{2}m(\dot{q}_1^2+\dot{q}_2^2)+\frac{1}{2}m\theta(\eta_1\dot{q}_2-\eta_2\dot{q}_1)-\eta_1q_1-\eta_2q_2+\frac{1}{8}m\theta^2(\eta_1^2+\eta_2^2). \quad (231)$$

The Lagrangian given by (231) implies the Euler-Lagrange equations

$$m\ddot{q}_1=-\eta_1, \qquad m\ddot{q}_2=-\eta_2. \quad (232)$$

Their solutions are:

$$q_1(t) = -\frac{\eta_1 t^2}{2m} + tC_2 + C_1, \qquad q_2(t) = -\frac{\eta_2 t^2}{2m} + tD_2 + D_1, \quad (233)$$

where C_1, C_2, D_1 and D_2 are constants which have to be determined from conditions:

$$q_1(0) = x_1^{'}, q_1(T) = x_1^{''}, q_2(0) = x_2^{'}, q_2(T) = x_2^{''}. \quad (234)$$

After finding the corresponding constants, we have

$$q_j(t) = x_j^{'} - \frac{\eta_j t^2}{2m} + t\left(\frac{1}{T}\left(x_j^{''} - x_j^{'}\right) + \frac{\eta_j T}{2m}\right), \quad \dot{q}_j(t) = -\frac{\eta_j t}{m} + \frac{1}{T}\left(x_j^{''} - x_j^{'}\right) + \frac{\eta_j T}{2m}, \quad j = 1,2. \quad (235)$$

Using (234) and (235), we finally calculate the corresponding action

$$\bar{S}_\theta(x'', T; x', 0) = \int_0^T L_\theta(\dot{q}, q, t)dt = \frac{1}{2T}m\left[\left(x_1^{''} - x_1^{'}\right)^2 + \left(x_2^{''} - x_2^{'}\right)^2\right] - \frac{1}{2}T\left[\eta_1\left(x_1^{''} + x_1^{'}\right) + \eta_2\left(x_2^{''} + x_2^{'}\right)\right] +$$
$$+ \frac{1}{2}m\theta\left[\eta_1\left(x_2^{''} - x_2^{'}\right) - \eta_2\left(x_1^{''} - x_1^{'}\right)\right] - \frac{1}{24m}T^3\left(\eta_1^2 + \eta_2^2\right) + \frac{1}{8}m\theta^2 T\left(\eta_1^2 + \eta_2^2\right). \quad (236)$$

According to (227) one gets

$$\mathrm{K}_\theta(x'', T; x', 0) = \frac{1}{ih}\frac{m}{T}\exp\left(\frac{2\pi i}{h}\bar{S}_\theta(x'', T; x', 0)\right) = \mathrm{K}_0(x'', T; x', 0)\exp\left(\frac{2\pi i}{h}\frac{m\theta}{2}\cdot\right.$$
$$\left.\cdot\left[\eta_1\left(x_2^{''} - x_2^{'}\right) - \eta_2\left(x_1^{''} - x_1^{'}\right) + \frac{1}{4}\theta T\left(\eta_1^2 + \eta_2^2\right)\right]\right), \quad (237)$$

where $\mathrm{K}_0(x'', T; x', 0)$ is related to the Lagrangian (228) for which $\theta = 0$. Hence, in this case there is a difference only in the phase factor. It is easy to see that the following connection holds:

$$\mathrm{K}_\theta(x'', T; x', 0) = \mathrm{K}_0\left(x'' + \frac{1}{2}\theta T J\eta, T; x', 0\right), \quad (238)$$

where

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Now taking $N \to \infty$ one can rewrite the following equation

$$\mathrm{K}(x'', t''; x', t') = \lim_{N\to\infty}\int_{R^{DN}}\prod_{n=1}^{N+1}\sqrt{\left(\frac{2}{ih\varepsilon}\right)^D \det(\alpha_n)} \times \exp\left(\frac{2\pi i\varepsilon}{h}\left[\langle\alpha_n\dot{q}_n, \dot{q}_n\rangle + \langle\beta_n q_n, \dot{q}_n\rangle +\right.\right.$$
$$\left.\left. + \langle\gamma_n q_n, q_n\rangle + \langle\delta_n, \dot{q}_n\rangle + \langle\eta_n, q_n\rangle + \phi_n\right]\right)\prod_{n=1}^{N} d^D q_n, \quad (239)$$

as (222):

$$\mathrm{K}(x'',t'';x',t')=\int \exp\left(\frac{2\pi i}{h}\int_{t'}^{t''} L(\dot{q},q,t)dt\right)\lim_{N\to\infty}\prod_{n=1}^{N}\left(\left(\frac{2}{ih\varepsilon}\right)^{D}\det(\alpha(t''))\right)^{\frac{1}{N}}\times\sqrt{\left(\frac{2}{ih\varepsilon}\right)^{D}\det(\alpha_n)dq_n} \ . \tag{240}$$

5. Mathematical connections [9]

Ramanujan's modular equations and Palumbo-Nardelli model

Now, we note that the number 8, and thence the numbers $64 = 8^2$ and $32 = 2^2 \times 8$, are connected with the "modes" that correspond to the physical vibrations of a superstring by the following Ramanujan function:

$$8=\frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos \pi t x w'}{\cosh \pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]} \ . \tag{241}$$

Furthermore, with regard the number 24 (12 = 24 / 2 and 32 = 24 + 8) they are related to the physical vibrations of the bosonic strings by the following Ramanujan function:

$$24=\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos \pi t x w'}{\cosh \pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]} \ . \tag{242}$$

Palumbo (2001) ha proposed a simple model of the birth and of the evolution of the Universe. Palumbo and Nardelli (2005) have compared this model with the theory of the strings, and translated it in terms of the latter obtaining:

$$-\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr(G_{\mu\nu}G_{\rho\sigma})f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]=$$
$$=\int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{\kappa_{10}^2}{g_{10}^2}Tr_\nu\left(\left|F_2\right|^2\right)\right], \tag{243}$$

A general relationship that links bosonic and fermionic strings acting in all natural systems.

p-adic, adelic and zeta-strings

Like in the ordinary string theory, the starting point of p-adic strings is a construction of the corresponding scattering amplitudes. Recall that the ordinary crossing symmetric Veneziano amplitude can be presented in the following forms:

$$A_\infty(a,b) = g^2 \int_R |x|_\infty^{a-1} |1-x|_\infty^{b-1} dx = g^2 \left[\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} + \frac{\Gamma(b)\Gamma(c)}{\Gamma(b+c)} + \frac{\Gamma(c)\Gamma(a)}{\Gamma(c+a)} \right] = g^2 \frac{\zeta(1-a)}{\zeta(a)} \frac{\zeta(1-b)}{\zeta(b)} \frac{\zeta(1-c)}{\zeta(c)} =$$

$$= g^2 \int DX \exp\left(-\frac{i}{2\pi} \int d^2\sigma \partial^\alpha X_\mu \partial_\alpha X^\mu \right) \prod_{j=1}^{4} \int d^2\sigma_j \exp\left(i k_\mu^{(j)} X^\mu \right), \quad (244)$$

where $\hbar = 1$, $T = 1/\pi$, and $a = -\alpha(s) = -1 - \frac{s}{2}$, $b = -\alpha(t)$, $c = -\alpha(u)$ with the condition $s + t + u = -8$, i.e. $a + b + c = 1$.
The p-adic generalization of the above expression

$$A_\infty(a,b) = g^2 \int_R |x|_\infty^{a-1} |1-x|_\infty^{b-1} dx,$$

is:

$$A_p(a,b) = g_p^2 \int_{Q_p} |x|_p^{a-1} |1-x|_p^{b-1} dx, \quad (245)$$

where $|\ldots|_p$ denotes p-adic absolute value. In this case only string world-sheet parameter x is treated as p-adic variable, and all other quantities have their usual (real) valuation.
Now, we remember that the Gauss integrals satisfy adelic product formula

$$\int_R \chi_\infty(ax^2 + bx) d_\infty x \prod_{p \in P} \int_{Q_p} \chi_p(ax^2 + bx) d_p x = 1, \quad a \in Q^\times, \quad b \in Q, \quad (246)$$

what follows from

$$\int_{Q_v} \chi_v(ax^2 + bx) d_v x = \lambda_v(a) |2a|_v^{-\frac{1}{2}} \chi_v\left(-\frac{b^2}{4a} \right), \quad v = \infty, 2, \ldots, p \ldots . \quad (247)$$

These Gauss integrals apply in evaluation of the Feynman path integrals

$$K_v(x'', t''; x', t') = \int_{x',t'}^{x'',t''} \chi_v\left(-\frac{1}{h} \int_{t'}^{t''} L(\dot{q}, q, t) dt \right) D_v q, \quad (248)$$

for kernels $K_v(x'', t''; x', t')$ of the evolution operator in adelic quantum mechanics for quadratic Lagrangians. In the case of Lagrangian

$$L(\dot{q},q)=\frac{1}{2}\left(-\frac{\dot{q}^2}{4}-\lambda q+1\right),$$

for the de Sitter cosmological model one obtains

$$K_\infty(x'',T;x',0)\prod_{p\in P}K_p(x'',T;x',0)=1,\quad x'',x',\lambda\in Q\,,\; T\in Q^\times,\quad (249)$$

where

$$K_v(x'',T;x',0)=\lambda_v(-8T)|4T|_v^{-\frac{1}{2}}\chi_v\left(-\frac{\lambda^2T^3}{24}+[\lambda(x''+x')-2]\frac{T}{4}+\frac{(x''-x')^2}{8T}\right).\quad (250)$$

Also here we have the number 24 that correspond to the Ramanujan function that has 24 "modes", i.e., the physical vibrations of a bosonic string. Hence, we obtain the following mathematical connection:

$$K_v(x'',T;x',0)=\lambda_v(-8T)|4T|_v^{-\frac{1}{2}}\chi_v\left(-\frac{\lambda^2T^3}{24}+[\lambda(x''+x')-2]\frac{T}{4}+\frac{(x''-x')^2}{8T}\right)\Rightarrow$$

$$\Rightarrow\frac{4\left[anti\log\frac{\int_0^\infty\frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}.\quad (250b)$$

The adelic wave function for the simplest ground state has the form

$$\psi_A(x)=\psi_\infty(x)\prod_{p\in P}\Omega(|x|_p)=\begin{cases}\psi_\infty(x), x\in Z\\ 0, x\in Q\setminus Z\end{cases},\quad (251)$$

where $\Omega(|x|_p)=1$ if $|x|_p\le 1$ and $\Omega(|x|_p)=0$ if $|x|_p>1$. Since this wave function is non-zero only in integer points it can be interpreted as discreteness of the space due to p-adic effects in adelic approach. The Gel'fand-Graev-Tate gamma and beta functions are:

$$\Gamma_\infty(a)=\int_R|x|_\infty^{a-1}\chi_\infty(x)d_\infty x=\frac{\zeta(1-a)}{\zeta(a)},\quad \Gamma_p(a)=\int_{Q_p}|x|_p^{a-1}\chi_p(x)d_px=\frac{1-p^{a-1}}{1-p^{-a}},\quad (252)$$

$$B_\infty(a,b)=\int_R|x|_\infty^{a-1}|1-x|_\infty^{b-1}d_\infty x=\Gamma_\infty(a)\Gamma_\infty(b)\Gamma_\infty(c),\quad (253)$$

$$B_p(a,b)=\int_{Q_p}|x|_p^{a-1}|1-x|_p^{b-1}d_px=\Gamma_p(a)\Gamma_p(b)\Gamma_p(c),\quad (254)$$

where $a,b,c\in C$ with condition $a+b+c=1$ and $\zeta(a)$ is the Riemann zeta function. With a regularization of the product of p-adic gamma functions one has adelic products:

$$\Gamma_\infty(u)\prod_{p\in P}\Gamma_p(u)=1,\quad B_\infty(a,b)\prod_{p\in P}B_p(a,b)=1,\quad u\neq 0,1,\quad u=a,b,c, \quad (255)$$

where $a+b+c=1$. We note that $B_\infty(a,b)$ and $B_p(a,b)$ are the crossing symmetric standard and p-adic Veneziano amplitudes for scattering of two open tachyon strings. Introducing real, p-adic and adelic zeta functions as

$$\zeta_\infty(a)=\int_R \exp(-\pi x^2)|x|_\infty^{a-1}d_\infty x=\pi^{-\frac{a}{2}}\Gamma\left(\frac{a}{2}\right), \quad (256)$$

$$\zeta_p(a)=\frac{1}{1-p^{-1}}\int_{Q_p}\Omega(|x|_p)|x|_p^{a-1}d_p x=\frac{1}{1-p^{-a}}, \quad \mathrm{Re}\,a>1, \quad (257)$$

$$\zeta_A(a)=\zeta_\infty(a)\prod_{p\in P}\zeta_p(a)=\zeta_\infty(a)\zeta(a), \quad (258)$$

one obtains

$$\zeta_A(1-a)=\zeta_A(a), \quad (259)$$

where $\zeta_A(a)$ can be called adelic zeta function. We have also that

$$\zeta_A(a)=\zeta_\infty(a)\prod_{p\in P}\zeta_p(a)=\zeta_\infty(a)\zeta(a)=\int_R \exp(-\pi x^2)|x|_\infty^{a-1}d_\infty x\cdot\frac{1}{1-p^{-1}}\int_{Q_p}\Omega(|x|_p)|x|_p^{a-1}d_p x. \quad (259b)$$

Let us note that $\exp(-\pi x^2)$ and $\Omega(|x|_p)$ are analogous functions in real and p-adic cases. Adelic harmonic oscillator has connection with the Riemann zeta function. The simplest vacuum state of the adelic harmonic oscillator is the following Schwartz-Bruhat function:

$$\psi_A(x)=2^{\frac{1}{4}}e^{-\pi x_\infty^2}\prod_{p\in P}\Omega(|x_p|_p), \quad (260)$$

whose the Fourier transform

$$\psi_A(k)=\int\chi_A(kx)\psi_A(x)=2^{\frac{1}{4}}e^{-\pi k_\infty^2}\prod_{p\in P}\Omega(|k_p|_p) \quad (261)$$

has the same form as $\psi_A(x)$. The Mellin transform of $\psi_A(x)$ is

$$\Phi_A(a)=\int\psi_A(x)|x|^a d_A^\times x=\int_R\psi_\infty(x)|x|^{a-1}d_\infty x\prod_{p\in P}\frac{1}{1-p^{-1}}\int_{Q_p}\Omega(|x|_p)|x|^{a-1}d_p x=\sqrt{2}\Gamma\left(\frac{a}{2}\right)\pi^{-\frac{a}{2}}\zeta(a) \quad (262)$$

and the same for $\psi_A(k)$. Then according to the Tate formula one obtains (259).
The exact tree-level Lagrangian for effective scalar field φ which describes open p-adic string tachyon is

$$\mathcal{L}_p = \frac{1}{g^2}\frac{p^2}{p-1}\left[-\frac{1}{2}\varphi p^{-\frac{\Box}{2}}\varphi + \frac{1}{p+1}\varphi^{p+1}\right], \quad (263)$$

where p is any prime number, $\Box = -\partial_t^2 + \nabla^2$ is the D-dimensional d'Alambertian and we adopt metric with signature $(-+...+)$. Now, we want to show a model which incorporates the p-adic string Lagrangians in a restricted adelic way. Let us take the following Lagrangian

$$L = \sum_{n\geq 1} C_n \mathcal{L}_n = \sum_{n\geq 1}\frac{n-1}{n^2}\mathcal{L}_n = \frac{1}{g^2}\left[-\frac{1}{2}\phi\sum_{n\geq 1} n^{-\frac{\Box}{2}}\phi + \sum_{n\geq 1}\frac{1}{n+1}\phi^{n+1}\right]. \quad (264)$$

Recall that the Riemann zeta function is defined as

$$\zeta(s) = \sum_{n\geq 1}\frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}, \quad s = \sigma + i\tau, \quad \sigma > 1. \quad (265)$$

Employing usual expansion for the logarithmic function and definition (265) we can rewrite (264) in the form

$$L = -\frac{1}{g^2}\left[\frac{1}{2}\phi\zeta\left(\frac{\Box}{2}\right)\phi + \phi + \ln(1-\phi)\right], \quad (266)$$

where $|\phi| < 1$. $\zeta\left(\frac{\Box}{2}\right)$ acts as pseudodifferential operator in the following way:

$$\zeta\left(\frac{\Box}{2}\right)\phi(x) = \frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk, \quad -k^2 = k_0^2 - \vec{k}^2 > 2+\varepsilon, \quad (267)$$

where $\tilde{\phi}(k) = \int e^{(-ikx)}\phi(x)dx$ is the Fourier transform of $\phi(x)$.

Dynamics of this field ϕ is encoded in the (pseudo)differential form of the Riemann zeta function. **When the d'Alambertian is an argument of the Riemann zeta function we shall call such string a "zeta string".** Consequently, the above ϕ is an open scalar zeta string. The equation of motion for the zeta string ϕ is

$$\zeta\left(\frac{\Box}{2}\right)\phi = \frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon} e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk = \frac{\phi}{1-\phi} \quad (268)$$

which has an evident solution $\phi = 0$.

For the case of time dependent spatially homogeneous solutions, we have the following equation of motion

$$\zeta\left(\frac{-\partial_t^2}{2}\right)\phi(t) = \frac{1}{(2\pi)}\int_{|k_0|>\sqrt{2}+\varepsilon} e^{-ik_0 t}\zeta\left(\frac{k_0^2}{2}\right)\tilde{\phi}(k_0)dk_0 = \frac{\phi(t)}{1-\phi(t)}. \quad (269)$$

With regard the open and closed scalar zeta strings, the equations of motion are

$$\zeta\left(\frac{\Box}{2}\right)\phi = \frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk = \sum_{n\geq 1}\theta^{\frac{n(n-1)}{2}}\phi^n, \quad (270)$$

$$\zeta\left(\frac{\Box}{4}\right)\theta = \frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{4}\right)\tilde{\theta}(k)dk = \sum_{n\geq 1}\left[\theta^{n^2} + \frac{n(n-1)}{2(n+1)}\theta^{\frac{n(n-1)}{2}-1}\left(\phi^{n+1}-1\right)\right], \quad (271)$$

and one can easily see trivial solution $\phi = \theta = 0$.

Now we take the eq. (87b) of Section 2. We note that are possible the following mathematical connections with the Palumbo-Nardelli model (243) and the zeta strings (268):

$$V(r) = \frac{\Gamma(N/2)}{2\pi^{N/2}(N-2)}\frac{\left(2R\sqrt{\pi}\right)^{N-3}\Gamma\left(\frac{N-3}{2}\right)}{\varepsilon_0}\int_0^{q'}\frac{1}{R^{N-2}}dq' \Rightarrow$$
$$\Rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right] =$$
$$= \int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2}Tr_\nu\left(\left|F_2\right|^2\right)\right], \quad (272)$$

$$V(r) = \frac{\Gamma(N/2)}{2\pi^{N/2}(N-2)}\frac{\left(2R\sqrt{\pi}\right)^{N-3}\Gamma\left(\frac{N-3}{2}\right)}{\varepsilon_0}\int_0^{q'}\frac{1}{R^{N-2}}dq' \Rightarrow$$
$$\Rightarrow \frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon} e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk = \frac{\phi}{1-\phi}. \quad (273)$$

Now, we take the right hand side of eq. (100c) of Section 3. We have the following mathematical connections with the eqs. (129), (137) and (138) and with Palumbo-Nardelli model:

$$\hat{S}_{DBI}\Big|_{SW} = \int\frac{\sqrt{\det(G+2\pi\alpha'\Phi)}}{G_s}\left[1 + \theta^{ij}\partial_j\hat{A}_i + \frac{1}{2}\theta^{ij}\theta^{kl}\left\langle\partial_j\hat{A}_k, \partial_l\hat{A}_i\right\rangle_{*2} + \right.$$
$$\left. + \frac{1}{2}\theta^{ba}\theta^{kl}\left\langle\hat{A}_k, \partial_l\hat{F}_{ab}\right\rangle_{*2} + \frac{1}{8}\theta^{ij}\theta^{kl}\left\langle\hat{F}_{ji}, \hat{F}_{lk}\right\rangle_{*2}\right] \Rightarrow$$
$$\Rightarrow \approx \oint_C B\cdot dl = \mu_0 I_{enc} + \mu_0\varepsilon_0\frac{d}{dt}\int_S E\cdot\hat{n}dA \Rightarrow$$
$$\Rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right] =$$
$$= \int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2}Tr_\nu\left(\left|F_2\right|^2\right)\right], \quad (274)$$

$$\hat{S}_{DBI} = \frac{1}{G_s}\int L_*\left[\sqrt{\det\left(G + 2\pi\alpha'\left(\hat{F} + \Phi\right)\right)}W(x,C)\right] * e^{ik.x} =$$

$$= \frac{1}{G_s}\int \sqrt{\det G} L_*\left[\left(1 - \frac{1}{4}(2\pi\alpha')^2 trG^{-1}\left(\hat{F} + \Phi\right)G^{-1}\left(\hat{F} + \Phi\right)\right)W(x,C)\right] * e^{ik.x} + ... \Rightarrow$$

$$\Rightarrow \approx \oint_C B \cdot dl = \mu_0 I_{enc} + \mu_0 \varepsilon_0 \frac{d}{dt}\int_S E \cdot \hat{n} dA \Rightarrow$$

$$\Rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right] =$$

$$= \int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2}Tr_\nu\left(\left|F_2\right|^2\right)\right], \quad (275)$$

$$\hat{S}_{DBI} = \frac{1}{G_s}\int \sqrt{\det(G + 2\pi\alpha'\Phi)}\left[L_*(W(x,C)) + \frac{2\pi\alpha'}{2}\left\{trMF + M^{kl}\theta^{ij}\left\langle\partial_j F_{lk}, A_i\right\rangle_{*2} + \frac{1}{2}\left\langle trMF, tr\theta F\right\rangle_{*2}\right\} +\right.$$

$$\left. - \frac{(2\pi\alpha')^2}{4}tr\left\langle MF, MF\right\rangle_{*2} + \frac{(2\pi\alpha')^2}{8}\left\langle trMF, trMF\right\rangle_{*2} + ...\right] \Rightarrow$$

$$\Rightarrow \approx \oint_C B \cdot dl = \mu_0 I_{enc} + \mu_0 \varepsilon_0 \frac{d}{dt}\int_S E \cdot \hat{n} dA \Rightarrow$$

$$\Rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right] =$$

$$= \int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2}Tr_\nu\left(\left|F_2\right|^2\right)\right], \quad (276)$$

thence, the mathematical connections between noncommutative DBI actions, Maxwell's equations and Palumbo-Nardelli model.

Now we take the eqs. (164b) e (176c) of Section 3. We note that are possible the following mathematical connections with the eq. (268) concerning the zeta strings:

$$T_{(1)}^{a_1 \dots a_{2p+1}; b_1 \dots b_{2p+1}} = \frac{(-1)^p}{2}\frac{1}{(2p)!}\frac{1}{(2\pi)^{2p+1}}\frac{(\theta g \theta)^{a_1 b_1}}{2\pi\alpha'}\theta^{a_2 b_2}\dots\theta^{a_{2p+1}b_{2p+1}} \times$$

$$\times \int_0^{2\pi}\frac{d\sigma}{2\pi}(\sigma - \pi)^{2p}\ln\left|1 - e^{i\sigma}\right|^2 = 2(-1)^p(2p)!\sum_{j=0}^{p-1}(-1)^j\frac{\pi^{2j}}{(2j+1)!}\zeta(2p - 2j + 1) \Rightarrow$$

$$\Rightarrow \frac{1}{(2\pi)^D}\int_{k_0^2 - \vec{k}^2 > 2+\varepsilon} e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk = \frac{\phi}{1-\phi}, \quad (277)$$

$$S_{CS}+\Delta S_{CS}=\frac{1}{2}\int C^{(6)}\wedge F\wedge F+\int C^{(6)}\wedge$$

$$\wedge\frac{\sin\left(\frac{k_1\times k_2}{2}\right)}{\frac{k_1\times k_2}{2}}\sum_{p=0}^{\infty}\frac{1}{2\pi\alpha'(2\pi)^{2p+3}}\zeta(2p+3)(k_1\times k_2)^{2p+2}\left(k_{1i}(\theta g\theta)^{ij}k_{2j}\right)\tilde{F}(k_1)\wedge\tilde{F}(k_2)\Rightarrow$$

$$\Rightarrow\frac{1}{(2\pi)^D}\int_{k_0^2-\bar{k}^2>2+\varepsilon}e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk=\frac{\phi}{1-\phi}. \quad (278)$$

Now we take the eqs. (211) and (219) of Section 3. Also these equations can be connected with the Maxwell's equations and with Palumbo-Nardelli model:

$$\int d^{p+1}k'\tilde{F}_{ij}(k')F_{kl}(k-k')=\int\frac{1}{(2\pi)^{p+1}}d^{p+1}xL_*\left[\sqrt{\det\left(1-\theta\hat{F}\right)}\left(\hat{F}\left(1-\theta\hat{F}\right)^{-1}\right)_{ij}\left(\hat{F}\left(1-\theta\hat{F}\right)^{-1}\right)_{kl}W(x,C)\right]*e^{ik.x}$$

$$\Rightarrow\approx\oint_C B\cdot dl=\mu_0 I_{enc}+\mu_0\varepsilon_0\frac{d}{dt}\int_S E\cdot\hat{n}dA\Rightarrow$$

$$\Rightarrow-\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]=$$

$$=\int_0^\infty\frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{\kappa_{10}^2}{g_{10}^2}Tr_v\left(\left|F_2\right|^2\right)\right], \quad (279)$$

$$\frac{1}{2T_0}\mu_{p-1}\varepsilon^{i_1i_2\dots i_{p+1}}\tilde{C}^{(p)}_{i_2\dots i_{p+1}}(-k)\int\frac{1}{(2\pi)^{p+1}}d^{p+1}xL_*\left[\sqrt{\det\left(1-\theta\hat{F}\right)}\mathfrak{D}_{i_1}\hat{T}(x)W(x,C)\right]*e^{ik.x}\Rightarrow$$

$$\Rightarrow\approx\oint_C B\cdot dl=\mu_0 I_{enc}+\mu_0\varepsilon_0\frac{d}{dt}\int_S E\cdot\hat{n}dA\Rightarrow$$

$$\Rightarrow-\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]=$$

$$=\int_0^\infty\frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{\kappa_{10}^2}{g_{10}^2}Tr_v\left(\left|F_2\right|^2\right)\right]. \quad (280)$$

In conclusion, we take the eq. (236). We note that are possible the following mathematical connections with eqs. (129), (138), i.e. the DBI noncommutative actions, eqs. (179), (186), i.e. the Chern-Simons theory , the Ramanujan's modular equations and the Palumbo-Nardelli model:

$$\bar{S}_\theta(x'',T;x',0)=\int_0^T L_\theta(\dot{q},q,t)dt=\frac{1}{2T}m\left[\left(x_1''-x_1'\right)^2+\left(x_2''-x_2'\right)^2\right]-\frac{1}{2}T\left[\eta_1\left(x_1''+x_1'\right)+\eta_2\left(x_2''+x_2'\right)\right]+$$

$$+\frac{1}{2}m\theta\left[\eta_1\left(x_2''-x_2'\right)-\eta_2\left(x_1''-x_1'\right)\right]-\frac{1}{24m}T^3\left(\eta_1^2+\eta_2^2\right)+\frac{1}{8}m\theta^2T\left(\eta_1^2+\eta_2^2\right)\Rightarrow$$

$$\Rightarrow\int\frac{\sqrt{\det(G+2\pi\alpha'\Phi)}}{G_s}\left[1+\theta^{ij}\partial_j\hat{A}_i+\frac{1}{2}\theta^{ij}\theta^{kl}\left\langle\partial_j\hat{A}_k,\partial_l\hat{A}_i\right\rangle_{*2}+\right.$$

$$\left.+\frac{1}{2}\theta^{ba}\theta^{kl}\left\langle\hat{A}_k,\partial_l\hat{F}_{ab}\right\rangle_{*2}+\frac{1}{8}\theta^{ij}\theta^{kl}\left\langle\hat{F}_{ji},\hat{F}_{lk}\right\rangle_{*2}\right]\Rightarrow$$

$$\Rightarrow\frac{t}{8\pi}\int_M Tr\left(\mathrm{A}\wedge d\mathrm{A}+\frac{2}{3}\mathrm{A}\wedge\mathrm{A}\wedge\mathrm{A}\right)+\frac{\tilde{t}}{8\pi}\int_M Tr\left(\tilde{\mathrm{A}}\wedge d\tilde{\mathrm{A}}+\frac{2}{3}\tilde{\mathrm{A}}\wedge\tilde{\mathrm{A}}\wedge\tilde{\mathrm{A}}\right)\Rightarrow$$

$$\Rightarrow\frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty\frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2w'}dx}{e^{-\frac{\pi^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}\Rightarrow$$

$$\Rightarrow-\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]=$$

$$=\int_0^\infty\frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{\kappa_{10}^2}{g_{10}^2}Tr_\nu\left(\left|F_2\right|^2\right)\right].\quad(281)$$

Also in these expressions is very evident the link between π and $\phi=\frac{\sqrt{5}-1}{2}$, i.e. the Aurea ratio, by the simple formula

$$\arccos\phi=0{,}2879\pi\ .\quad(282)$$

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