Growth and equilibrium size of water droplets in air (*)

D. GIUSTI, V. MOLINARI and D. MOSTACCI

Laboratorio di Ingegneria Nucleare di Montecuccolino, Università degli Studi di Bologna
Via dei Colli 16, I-40136 Bologna, Italy

Summary. — A model is presented to describe the growth in time of the average water drop in supersaturated air, and predict their radius at equilibrium. Many previous works consider the growth of an isolated drop, whereas in the present work the effect of the presence of a large number of drops, with the ensuing depletion in water content in the surrounding air, is considered: it is shown that the effect of depletion is crucial to obtain the equilibrium radius. Preliminary results, obtained under some simplifying assumptions, are presented: expressions accounting for this depletion effect are given for the time evolution of the liquid-water temperature and of the number of water molecules in the drop and drop radius near equilibrium, and for their asymptotic equilibrium values.

PACS 92.60.Nv – Cloud physics.

1. – Introduction

Consider an atmosphere of supersaturated air: if drops start forming, by some mechanism that will not be investigated here, they go through a growing process that brings them eventually to some equilibrium size. To reach this dynamic equilibrium two transport processes compete in a drop: condensation of water molecules onto the drop and evaporation of water molecules from the drop. When water molecules from the air hit the drop during their random motion, they have a probability of being absorbed (*i.e.* condensate) often described through a coefficient giving the average fraction of molecules that do condense upon striking the drop (condensation factor). On the other hand, molecules are retained within the drop by a potential: whatever the nature of this potential, its effect is that upon leaving the drop, molecules that do manage to escape loose an (average) energy, which can be described as $ms^2/2$ with an appropriate s [1].

The aim of this work is to describe the accretion phenomenon just discussed: how the average size of drops evolves in time and what is its equilibrium value. No attempt is made here to describe the birth of drops: the origin of time is set at some point when droplets, however small they may be, already exist and have started growing. No

---

(*) The authors of this paper have agreed to not receive the proofs for correction.

attempt either is made to investigate the nature of the potential inside the drop: it is simply recognised that there exists one, and that molecules therefore loose (on average) an energy $ms^2/2$ upon evaporating.

One last word needs to be spent on the condensation factor: there is no fundamental reason why it cannot be incorporated in the method presented here. However, the proper way to treat it is to consider its dependence on molecule impact velocity and include it in the integral in velocity space: as the present work gives a qualitative description of the phenomenon, it was chosen to ignore the condensation factor altogether (*i.e.* set it to 1). This said, consider then supersaturated air in which drops (already formed) are growing. Two main regimes can be recognised in the growth of a water drop in atmosphere: one in which a very small droplet grows in supersaturated atmosphere with little or no effect on the saturation state of the surrounding air, and one in which the drop size and the concentration of water vapor in the air are near their equilibrium values, *i.e.* when evaporation from the drop and condensation onto the drop are in balance. In both regimes, the growth of the drop happens at the expenses of the surrounding water vapor, of course, but at the onset of equilibrium, the surrounding vapor is very sensitive to the loss to the drop, as the equilibrium point is drawing near. Saturated air at 20 °C contains approximately 14.7 grams of water per every kilogram of air: a droplet of 100 μm of diameter contains 0.52 mg of water, that is, the water contained in 0.0296 litres of saturated air, corresponding to a sphere of air of radius 19.2 mm. Considering a temperature of 0 °C instead, the same drop would contain the same amount of water as 0.107 litres of saturated air, corresponding to a sphere of air of radius 29.4 mm. Pruppacher *et al.* [2] report for drops in atmosphere an average distance of 100 radii: calculating the volume of air surrounding each drop (a "cell"), one finds that it is approximately 2.5×10^5 times the volume of the drop; such a volume of saturated air at, say, 20 °C would contain barely 4 times the amount of water contained in the drop. From these numbers, however rough they may be, it is immediately clear that the depletion created in the air water vapor by the condensation onto the drop cannot be neglected in any meaningful way. The approximation often made of a single drop in an infinite space of wet air [2, 3] is well justified as long as the drops are very far apart compared to their size, *i.e.* when the volume of air available to each drop is infinitely large compared to the drop itself, or better said, the volume of air available to each drop contains an amount of water essentially infinite compared to that contained in the drop itself. With this assumption, there is at all times a current of molecules flowing toward the drop, and the drop grows forever. It is clear that such a model is only applicable with the limitations discussed above. As the drop grows, it is necessary to account for the amount of water that the drop subtracts to the air in its cell: as the drop grows, the saturation state of the surrounding air changes, the air becomes dryer, thus favoring evaporation. The model presented in this work attempts to address this issue, and to predict the equilibrium radius of the drop. As will be discussed in the next section, a drop will be assumed to interact only with the air contained in a certain "cell" of moist air around it: a cell which will be thought of as "closed", that is not interacting with each other. It goes without saying, then, that drops will not interact with each other either. This model neglects the space dependence of the water molecule density in air [4]: this is appropriate at equilibrium, when no net current of particles is present any more; away from equilibrium the current of particles sustains a density gradient, and the associated gradient of temperature, and these need to be calculated, although near equilibrium (which is the regime of interest for the present model) these gradients are presumably small. It is a problem that has been studied widely, by many

authors (see, *e.g.*, ref. [4] and its bibliography): in a forthcoming work it will be addressed, in the context of the problem at hand, with a method developed by the authors in a previous paper [5].

2. – Modelling of the problem and balance equation

Consider a volume of supersaturated air-water mixture at some initial temperature T_0: droplets will form around condensation centers. If the average distance between such droplets is known, a certain "cell" of air can be associated to each droplet, such that all the moisture contained in the cell interacts with the droplet (a similar method has been adopted to describe liquids [6]) as shown in fig. 1. Let V_0 be the volume of this typical cell, including the volume occupied by the drop itself, which, in turn, will be denoted $V(t)$, as it varies in time. The volume V_0 can be thought of as an "equivalent sphere" of radius R_0, whereas the droplet is indeed a sphere of radius $R(t)$, with an outer surface $S(t)$. Finally, let N_0 be the total number of water molecules in a cell, and $N(t)$ the number thereof that are in the drop: $N_0 - N(t)$ will be in the air in the cell as water vapor. The volume and surface of the drop can be expressed in terms of $N(t)$:

$$(1a) \qquad R(t) = \left(\frac{3}{4\pi} V(t) \right)^{1/3} = \left(\frac{3}{4\pi} \frac{m}{\varrho} N(t) \right)^{1/3} = R_{\mathrm{w}} \left(\frac{N(t)}{N_0} \right)^{1/3},$$

$$(1b) \qquad S(t) = 4\pi \left(\frac{3}{4\pi} \frac{m}{\varrho} N(t) \right)^{2/3} = S_{\mathrm{w}} \left(\frac{N(t)}{N_0} \right)^{2/3},$$

where ϱ is the density of the liquid water and m the mass of a water molecule and

$$(2a) \qquad R_{\mathrm{w}} = \left(\frac{3}{4\pi} \frac{m}{\varrho} N_0 \right)^{1/3}$$

and

$$(2b) \qquad S_{\mathrm{w}} = 4\pi \left(\frac{3}{4\pi} \frac{m}{\varrho} N_0 \right)^{2/3}$$

are the radius and surface of a drop of liquid water that would contain all of the molecules of the cell.
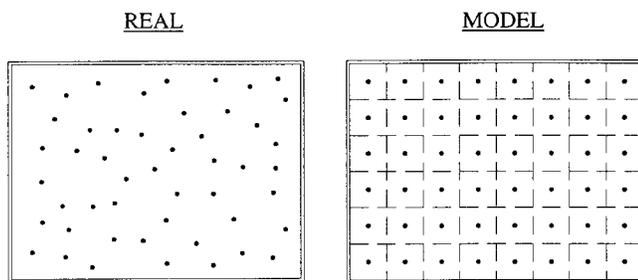

Fig. 1. – Sketch of the "Drop in Cell" model.

The rates of condensation and evaporation can be calculated from kinetic theory [5, 6]: the main results are summarized as follows.

2'1. *Condensation.* – If the water molecules in the air immediately surrounding the drop can be assumed to have a Maxwellian distribution function at some local temperature T_G (generally a function of time), every element dS of the droplet surface receives a current of impinging water molecules from the surrounding volume given by

$$(3) \qquad J^- = \frac{\varrho}{m} \int_{-\infty}^{+\infty} dV_\varphi \int_{-\infty}^{+\infty} dV_\vartheta \int_{-\infty}^{0} V_r \left(\frac{m}{2\pi K T_G} \right)^{3/2} e^{-mv^2/2KT_G} \, dV_r = n_v \sqrt{\frac{KT_G}{2\pi m}} ,$$

where n_v is the density of water vapor molecules in the cell. Assuming that this density is uniform, it can be calculated as

$$(4) \qquad n_v = \frac{N_0 - N(t)}{V_0 - V(t)} .$$

In fact, the water molecule density in the air is not uniform, in all likelihood: the effect of this disuniformity will be investigated in a future work (see sect. **4**).

As it is, the above expression is awkward to use; it can be reexpressed in terms of $N(t)$, and after some algebra:

$$(5) \qquad n_v = \frac{N_0 - N(t)}{V_0 - V(t)} = \frac{1}{V_0} \frac{N_0 - N(t)}{1 - (m/\varrho V_0) N(t)} = \frac{N_0}{V_0} \left(1 - \frac{N(t)}{N_0} \right) \left(1 - \frac{mN_0}{\varrho V_0} \frac{N(t)}{N} \right)^{-1} .$$

The number of water molecules impinging per unit time on the unit surface of the droplet is therefore

$$(6) \qquad J^- = \frac{N_0}{V_0} \sqrt{\frac{KT_G}{2\pi m}} \left(1 - \frac{N(t)}{N_0} \right) \left(1 - \frac{mN_0}{\varrho V_0} \frac{N(t)}{N_0} \right)^{-1} .$$

The energy carried by this flow of molecules can be calculated as

$$(7) \qquad Q^- = n_v \int_{-\infty}^{+\infty} dV_\varphi \int_{-\infty}^{+\infty} dV_\vartheta \int_{-\infty}^{0} V_r \frac{mv^2}{2} \left(\frac{m}{2\pi K T_G} \right)^{3/2} e^{-mv^2/2KT_G} \, dV_r =$$

$$= 2KT_G \frac{N_0}{V_0} \sqrt{\frac{KT_G}{2\pi m}} \left(1 - \frac{N(t)}{N_0} \right) \left(1 - \frac{mN_0}{\varrho V_0} \frac{N(t)}{N_0} \right)^{-1} = 2KT_G \times J^- .$$

2'2. *Evaporation.* – Again, the water molecules are assumed to have a Maxwellian distribution of energy, at the uniform liquid-water temperature T_L (unlike the temperature of the water molecules in the surrounding air, the temperature in the drop can safely be assumed to be uniform), and the flow of molecules impinging on the surface can be calculated in the same way as above. However, not all molecules impinging on the surface from within can escape: in leaving the drop, molecules loose an average amount of energy: the latent evaporation heat; for reasons of convenience, and with no loss of generality, this energy will be represented here in the form of a kinetic energy $ms^2/2$ [1]. Therefore, only those molecules possessing velocity along the

radius no less than s can escape: the outgoing flow per unit surface is therefore

$$(8) \qquad J^+ = \frac{\varrho}{m} \int\limits_{-\infty}^{+\infty} dv_\varphi \int\limits_{-\infty}^{+\infty} dv_\vartheta \int\limits_{S}^{+\infty} v_r \left(\frac{m}{2\pi K T_L}\right)^{3/2} e^{-mv^2/2KT_G} dv_r = \frac{\varrho}{m} \sqrt{\frac{KT_L}{2\pi m}} \, e^{-ms^2/2KT_G}.$$

In the same vein, the energy carried away by the evaporating molecules is given by

$$(9) \qquad Q^+ = \frac{\varrho}{m} \int\limits_{-\infty}^{+\infty} dv_\varphi \int\limits_{-\infty}^{+\infty} dv_\vartheta \int\limits_{S}^{+\infty} v_r \left(m\frac{v^2 - s^2}{2}\right)\left(\frac{m}{2\pi K T_L}\right)^{3/2} e^{-mv^2/2KT_L} dv_r =$$

$$= 2\,KT_a \frac{\varrho}{m} \sqrt{\frac{KT_L}{2\pi m}} \, e^{-ms^2/2KT_L} = 2\,KT_L \times J^+.$$

2'3. *Dimensionless equations.* – The above equations can be reduced to the following dimensionless form. To this end, some constants will be defined:

$$(10) \qquad \begin{cases} T_0 = T_L(0); \quad v_0 = \sqrt{\dfrac{KT_0}{2\pi m}}; \quad \dfrac{1}{t_0} = \dfrac{S_w}{V_0} v_0, \\[3mm] J_0 = \dfrac{N_0}{V_0} v_0; \quad Q_0 = J_0 2\,KT_0; \quad \alpha = \dfrac{mN_0}{\varrho V_0} \end{cases}$$

together with the following dimensionless variables:

$$(11) \qquad \begin{cases} \tilde{N} = \dfrac{N}{N_0}; \quad \tilde{T} = \dfrac{T}{T_0}; \quad \tilde{v} = \dfrac{v}{V_0}; \quad \tilde{s} = \dfrac{S}{V_0}; \\[3mm] \tilde{t} = \dfrac{t}{t_0}; \quad \tilde{J} = \dfrac{J}{J_0}; \quad \tilde{Q} = \dfrac{Q}{Q_0}. \end{cases}$$

Then, eqs. (6) to (9) can be rewritten as

$$(12) \qquad \tilde{J}^- = \sqrt{\tilde{T}_G}(1 - \tilde{N})(1 - \alpha\tilde{N})^{-1},$$

$$(13) \qquad \tilde{Q}^- = \tilde{T}_G \tilde{J}^-,$$

$$(14) \qquad \tilde{J}^+ = \sqrt{\tilde{T}_L} \frac{1}{\alpha} e^{-\tilde{s}^2/4\pi\tilde{T}_L},$$

$$(15) \qquad \tilde{Q}^+ = \tilde{T}_L \tilde{J}^+.$$

2'4. *The balance equations.* – The balance equations can be written with the help of the relations above. For the number of molecules in the drop, mass conservation requires that

$$(16) \qquad \frac{dN(t)}{dt} = S(t) \times (J^- - J^+)$$

or, in dimensionless form,

$$(17) \qquad \frac{d\widetilde{N}}{d\tilde{t}} = \widetilde{N}^{2/3}\left\{\sqrt{\widetilde{T}_G}(1-\widetilde{N})(1-\alpha\widetilde{N})^{-1} - \sqrt{\widetilde{T}_L}\frac{1}{\alpha}e^{-\tilde{s}^2/4\pi\widetilde{T}_L}\right\}.$$

An approximation can be made, considering that α is small compared to 1 (it represents the ratio of the actual mass of water in the cell to the mass of water that the cell would contain if it were filled entirely with liquid water):

$$(18) \qquad (1-\alpha\widetilde{N})^{-1} \approx (1+\alpha\widetilde{N}).$$

Making use of this approximation, eq. (17) can be rewritten in the following, more convenient form:

$$(19) \qquad \frac{d\widetilde{N}}{d\tilde{t}} = \widetilde{N}^{2/3}(\xi_1 - \xi_2\widetilde{N} - \xi_3\widetilde{N}^2),$$

where the quantities ξ_1, ξ_2 and ξ_3 are defined as

$$(20a) \qquad \xi_1 = \sqrt{\widetilde{T}_G} - \sqrt{\widetilde{T}_L}\frac{1}{\alpha}e^{-\tilde{s}^2/4\pi\widetilde{T}_L},$$

$$(20b) \qquad \xi_2 = \sqrt{\widetilde{T}_G}(1-\alpha),$$

$$(20c) \qquad \xi_3 = \sqrt{\widetilde{T}_G}\,\alpha.$$

It must be noticed about these three coefficients that, albeit they do not contain $\widetilde{N}$, they still depend on time through $\widetilde{T}_L$ and $\widetilde{T}_G$ (however, $\widetilde{T}_G$ in this context is the dimensionless vapor temperature at the surface of the drop and not a space-dependent variable).

In the simplifying assumption that temperature gradients can be neglected (as discussed in the introduction) and that the temperature remains uniform everywhere (albeit time dependent), an equation for this common value T of the temperature can be derived from a simple energy balance:

$$(21) \qquad V_0\varrho_0 c_{pA}T_0 + N_0 mc_{pG}T_0 + N_{eq}\frac{ms^2}{2} = N_{eq}mc_{pL}T + (N_0 - N_{eq})mc_{pG}T + V_0\varrho_0 c_{pA}T,$$

where ϱ_0 is the density of dry air before the onset of condensation, and c_{pG}, c_{pL} and c_{pA} are the specific heat capacities of water vapor, liquid water and dry air, respectively, assumed constant throughout the temperature range encountered in this problem.

Dividing both sides of the above equation by $N_0 mc_{pG}T_0$, and after some algebra, a dimensionless equation can be obtained:

$$(22) \qquad \widetilde{T} = \frac{1 + (1/\varphi)(c_{pA}/c_{pG}) + (c_{pL}/c_{pG})(\beta\gamma/2)\widetilde{N}}{1 + (1/\varphi)(c_{pA}/c_{pG}) + (c_{pL}/c_{pG} - 1)\widetilde{N}} \cong$$

$$\cong 1 + \frac{(c_{pL}/c_{pG})(\beta\gamma/2) + (1 - c_{pL}/c_{pG})}{1 + (1/\varphi)(c_{pA}/c_{pG})}\widetilde{N} \cong 1 + \varphi\frac{c_{pL}}{c_{pA}}\frac{\beta\gamma}{2}\widetilde{N},$$

where $\varphi = mN_0/V_0\varrho_0$ is the mass ratio of vapor to dry air before the onset of condensation, and the dimensionless constants β and γ are defined as

(23)
$$\beta = \frac{2\,K}{mc_{pL}}\,, \qquad \gamma = \frac{ms^2}{2\,KT_0}\,.$$

It should be pointed out at this point that no separate equation for T_G has been derived: water vapor, in the problem at hand, is mixed with air. Actually, there is generally a small amount of vapor in a much larger amount of air (typical numbers are 10 or 20 grams of water per kilogram of air), and water molecules often can be regarded as test particles in a field of air molecules: it is out of the question to calculate T_G without reference to the air through which the vapor diffuses. Again, this aspect will be given appropriate space in the forthcoming work referred to in the introduction.

3. – Drop growth: transient and equilibrium solution

From the balance equations derived above both the time evolution and the final radius of the drop can be calculated.

3'1. *Equilibrium solution.* – At equilibrium $J^- = J^+$ and $Q^- = Q^+$, and both time derivatives of particle number and temperature must vanish. The temperature equation is simply

(24)
$$Q^- - Q^+ = 2(KT_G \times J^- - KT_L \times J^+) = 0\,;$$

since $J^- = J^+$, this yields $T_L = T_G$. Coming to the equation for the number of molecules in the drop, it must be noticed that for a drop to exist, the equilibrium number of molecules N_{eq} must be greater than zero. The density equation becomes

(25)
$$\sqrt{\widetilde{T}_G}(1 - \widetilde{N})(1 - \alpha\widetilde{N})^{-1} - \sqrt{\widetilde{T}_L}\frac{1}{\alpha}\,e^{-\tilde{s}^2/4\pi\widetilde{T}} = 0$$

or, in the form of eq. (19),

(26)
$$\xi_1 - \xi_2\,\widetilde{N} - \xi_3\,\widetilde{N}^2 = 0\,.$$

Consider first this latter: it can be seen from their definitions that both ξ_2 and ξ_3 are positive numbers. ξ_1 can be either positive or negative, but for negative values of ξ_1 there can be no positive solution to the above equation. For positive values of ξ_1 there are one positive and one negative real solutions, and discarding the latter, the equilibrium number of molecules n_{eq} is given by

(27)
$$\widetilde{N}_{eq} = \frac{\sqrt{\xi_2^2 + 4\xi_1\xi_3} - \xi_2}{2\xi_3} \xrightarrow[\xi_3\to 0]{} \frac{\xi_1}{\xi_2}\,.$$

The condition $\xi_1 > 0$ stated above gives a condition on T_L and s:

(28)
$$e^{-ms^2/2\,KT_L} < \frac{mN}{\varrho V_0}\,.$$

Simply stated, this means that if the water is too hot compared to the energy keeping the drop bounded, it will evaporate and there will be no drop at all.

However, to find the equilibrium solution the approximation leading to eq. (19) is not necessary, and eq. (17) can be solved to yield (in dimensional quantities)

$$(29) \qquad \frac{N_{\text{eq}}}{N_0} = \frac{1 - (\varrho V_0/mN_0)\, e^{-ms^2/2KT_{\text{L}}}}{1 - e^{-ms^2/2KT_{\text{L}}}} \,.$$

Again, the positivity of the solution requires eq. (28) to be satisfied: moreover, it can be seen that the solution is always < 1, consistently with the model chosen.

From this solution, the equilibrium vapor pressure can be derived. Calculate first the number of molecules in the vapor phase:

$$(30) \qquad N_0 - N(t) = N_0 \left(1 - \frac{1 - (\varrho V_0/mN_0)\, e^{-ms^2/2KT_{\text{L}}}}{1 - e^{-ms^2/2KT_{\text{L}}}} \right) = N_0 \, \frac{(\varrho V_0/mN_0 - 1)\, e^{-ms^2/2KT_{\text{L}}}}{1 - e^{-ms^2/2KT_{\text{L}}}} \,.$$

The volume left to the vapor is

$$(31) \qquad V_0 - V(t) = V_0 \left(1 - \frac{mN_0}{\varrho V_0} \, \frac{N(t)}{N_0} \right) = V_0 \, \frac{1 - mN_0/\varrho V_0}{1 - e^{-ms^2/2KT_{\text{L}}}} \,.$$

The (uniform) number density n_{v} of molecules in the vapor phase is therefore

$$(32) \qquad n_{\text{v}} = \frac{N_0 - N(t)}{V_0 - V(t)} = \frac{N_0}{V_0} \, \frac{(\varrho V_0/mN_0 - 1)\, e^{-ms^2/2KT_{\text{L}}}}{1 - mN_0/\varrho V_0} = \frac{\varrho}{m} \, e^{-ms^2/2KT_{\text{L}}}$$

or, in terms of vapor mass density ϱ_{G},

$$(33) \qquad \varrho_{\text{G}} = n_{\text{v}} m = \varrho e^{-ms^2/2KT_{\text{L}}}$$

which is a well-known result of thermodynamics [5, 7].

The equilibrium value of the temperature (necessarily uniform, at equilibrium) can be found from eq. (22), now with no assumptions on the temperature gradients:

$$(34) \qquad \widetilde{T}_{\text{eq}} \cong 1 + \frac{(c_{p\text{L}}/c_{p\text{G}})(\beta\gamma/2) + (1 - c_{p\text{L}}/c_{p\text{G}})}{1 + (1/\varphi)(c_{p\text{A}}/c_{p\text{G}})} \, \widetilde{N}_{\text{eq}} \cong 1 + \varphi \, \frac{c_{p\text{L}}}{c_{p\text{A}}} \, \frac{\beta\gamma}{2} \, \widetilde{N}_{\text{eq}}.$$

Using typical values of the quantities involved, the equilibrium temperature $\widetilde{T}_{\text{eq}}$ is only a few percent larger than one.

3'2. *Transient solution.* – The transient solution for the number of molecules presents a greater degree of difficulty, as the coefficients are themselves time dependent. As a first approximation, it can be solved in the simplifying hypothesis that the temperature is essentially constant, and therefore so are the coefficients: this assumption is well justified in light of eq. (34), showing that in the cases of interest, the final temperature T_{eq} is only a few percent higher that the initial temperature T_0. Furthemore, the coefficients only depend on the square root of temperature. Then the

equation for the number of molecules, eq. (19), can be rewritten as

$$(36) \qquad \frac{d\widetilde{N}}{\widetilde{N}^{2/3}(\xi_1 - \xi_2\widetilde{N} - \xi_3\widetilde{N}^2)} = d\tilde{t}$$

yielding the solution in implicit form (valid for $\widetilde{N} = 0$ at $\tilde{t} = 0$)

$$(37) \qquad \tilde{t} = \frac{1}{\xi_2}\frac{\xi_2^{2/3}}{\xi_1^{2/3}}\left[\frac{1}{2}\ln\frac{(\xi_1/\xi_2)^{2/3} + (\xi_1/\xi_2)^{1/3}\widetilde{N}^{1/3} + \widetilde{N}^{2/3}}{[(\xi_1/\xi_2)^{1/3} - \widetilde{N}^{1/3}]^2} + \right.$$

$$\left. + \sqrt{3}\left(\text{arctg}\,\frac{(\xi_1/\xi_2)^{1/3} + 2\widetilde{N}^{1/3}}{\sqrt{3}(\xi_1/\xi_2)^{1/3}} - \text{arctg}\,\frac{1}{\sqrt{3}}\right)\right].$$

In terms of the ratio of radius $R(t)$ to final radius at equilibrium R_{eq}, $\widetilde{R} = R/R_{\text{eq}}$, the above equation can be rewritten as

$$(38) \qquad \tilde{t} = \frac{1}{\sqrt{\widetilde{T}_{\text{G}}}(\widetilde{N}_{\text{eq}})^{2/3}}\left[\frac{1}{2}\ln\frac{1 + \widetilde{R} + \widetilde{R}^2}{[1 - \widetilde{R}]^2} + \sqrt{3}\left(\text{arctg}\,\frac{1 + 2\widetilde{R}}{\sqrt{3}} - \text{arctg}\,\frac{1}{\sqrt{3}}\right)\right].$$
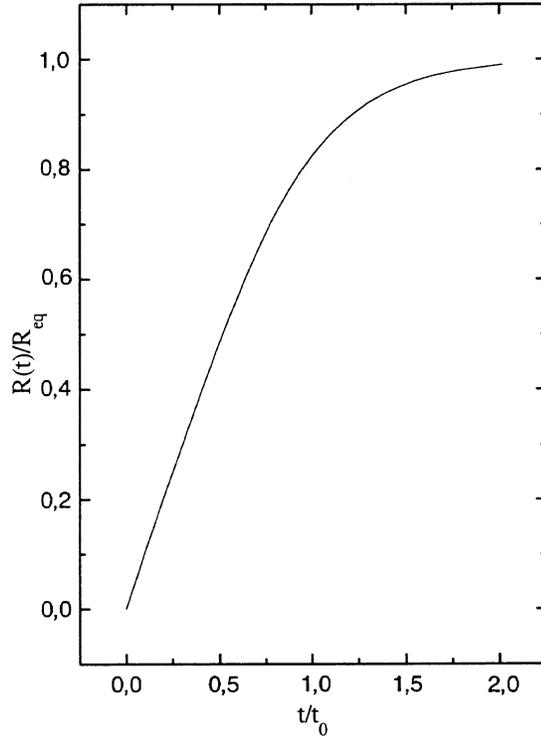

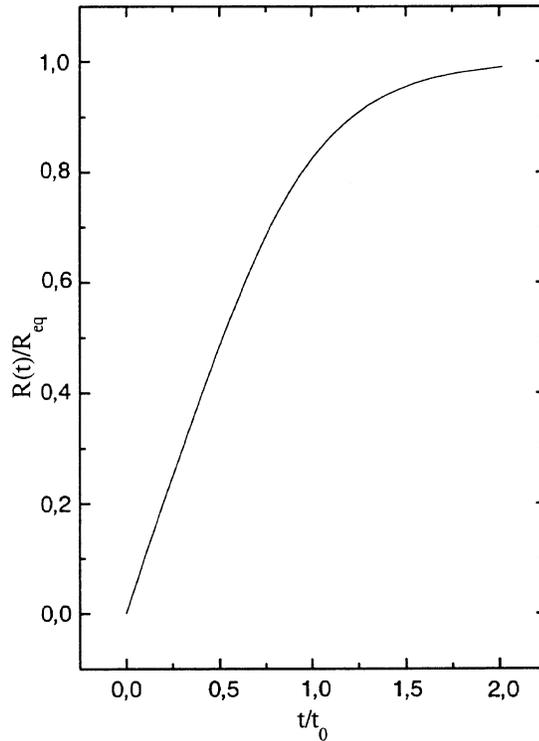Fig. 2. – Radius *vs.* time (in dimensionless units).

Fig. 3. – Temperature growth in time (in dimensionless units).

A plot of this dimensionless radius *vs.* dimensionless time can be seen in fig. 2. Under the assumptions mentioned earlier, the common temperature $T = T_G = T_L$ has been calculated in eq. (22). A plot of the temperature is shown in fig. 3.

4. – Conclusions and future work

As was mentioned in the introduction, the main point of the model presented is to account for the depletion of vapor in the air as the drop grows. In fact, this depletion effect is what leads to an equilibrium size of the droplets; if it were neglected as in some other models [1, 8], this would lead to the unphysical result of drops growing forever. The model presented demonstrates how the final average radius attained by the drops depends on the amount of supersaturation and on the average volume of the cell attributed to every drop, that is on the average distance between drops. Drops do not interact directly with each other, but they do indirectly, as each drop "captures" a volume of supersaturated air: the closer the drops, the smaller the cell volume. This model gives also, in a simplified way, the growth in time of the average radius of the drops, and the dependence of this growth on cell size and on the amount of supersaturation.

However, the above results are to be taken with care, as they contain many approximations: eq. (33), *e.g.*, can be arrived at from Clapeyron equation, which inbeds the perfect-gas approximation. Several points remain to be studied in greater depth: first and foremost, evaporation from the drop, for which a very rough approximation

has been made. Which is exactly the relationship between the escape velocity s and the latent heat of vaporization? What role does surface tension play? A quantitative understanding of those phenomena is required, as the model presented has shown clearly that both the growth in time and the equilibrium radius of drops are very sharply dependent on the evaporation mechanism. Another point of interest is the behaviour in time of vapor in the air: the equilibrium profiles of density and temperature are expected to be flat and little more can be added to that, but while the drop is still growing (and both particle and heat flows are present) there are temperature and density gradients [4]: it will be an interesting addition to the present model, to get a clearer picture of the accretion phenomenon, to include a better treatment of these gradients and to account for the interaction with the air, and to obtain also the space-dependent temperature of the water vapor.

More work needs to be done, and some is already under way: however, even with the limitations discussed, the model and some preliminary results seemed worth presenting, as they provide information on the general trend of the phenomenon: further refinements will be presented in a future paper.

REFERENCES

[1] LOEB L. B., *The Kinetic Theory of Gases* (Dover, New York) 1961.
[2] PRUPPACHER H. R. and KLETT J. D., *Microphysics of Clouds and Precipitations* (D. Reidell Publishing Co., Dordrecht) 1978.
[3] MASON B. J., *The Physics of Clouds* (Clarendon Press, Oxford) 1971.
[4] HARVEY J. and LORD G. (Editors), *Rarefied Gas Dynamics, Proceedings of the 19th International Symposium* (Oxford University Press, Oxford) 1995.
[5] MOLINARI V. and MOSTACCI D., *Diffusion near an interface via Onsager-type equations, Nuovo Cimento D*, **18** (1996) 689.
[6] HIRSCHFELDER J. O., CURTISS C. F. and BYRD R. B., *Molecular Theory of Gases and Liquids* (Wiley, New York) 1967.
[7] CHAPMAN S. and COWLING T. G., *The Mathematical Theory of Non-Uniform Gases* (Cambridge University Press, Cambridge) 1961.
[8] FERMI E., *Thermodynamics* (Dover, New York) 1936.
[9] ROGERS R. R. and YAU M. K., *A Short Course in Cloud Physics* (Pergamon Press, Oxford) 1989.