The solitary and cnoidal waves in shallow water

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Summary. — The problems associated with shallow-water waves has received considerable attention in the recent years. In the case of solitary and cnoidal waves, it is often assumed that the nonlinear and dispersion effects balance whilst dissipation is slight or totally neglected in the areas of the shallow water concerned. As a follow-up, this model attempts to provide a description of the classical solitary waves and the related cnoidal oscillations using the traditional shallow-water equations (WHITHAN G. B., *Linear and Non-Linear Waves* (Wiley and Sons) 1973, pp. 460-470). The derivations in this case do not involve any series expansion; and thus, they differ significantly from the approach using the Korteweg-de Vries (KDV) equations (WHITHAN G. B., *Linear and Non-Linear Waves* (Wiley and Sons) 1973, pp. 460-470; ZABUSKY N. J. and GALVIN C. J., *Shallow waves. The KDV equation and solitons, J. Fluid Mech.*, **47** (1971) 811-824). Consequently, this model attempts to describe the wave pattern even when the wave height grows as high as very close in magnitude to the depth of the corresponding undisturbed water layer or before breaking begins. In this consideration, realistic shallow-water parameters are used to examine the structure of the wave processes concerned.

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1. – Introduction

The transformations of the finite-amplitude waves in shallow water are to a considerable extent governed by the nonlinear Korteweg-de Vries equation [6, 8]. However, this is the case only in the shallow-water zone with the depth small compared with the typical wavelength and the associated wave height less than the depth of undisturbed water level [3, 4]. In this consideration, nonlinearity and dispersion compete effectively; at the same time, dissipation is neglected [7].

Two types of waveforms appear to be prominent under these conditions. They are a) the solitary wave [9] characterized by a simple hump [1] and b) the cnoidal oscillation which is periodic in form [1, 7]. The existence of these waves as shallow-water phenomena has been noted for some time. Investigation by Zabusky and Galvin [8, 6] conclusively established the soliton concept as an observable shallow-water phenomenon. At geophysical scale, Tsunami waves [9] possess identical characteristics with solitary waves. On the other hand, the profiles of low swell are often periodic and thus identical with cnoidal oscillation.

Further, the KDV equation is an approximation to the traditional shallow-water equations. This is so because, in the derivation, the terms of order η_0^2/h_0^2 are neglected; here η_0 and h_0 are, respectively, the peak wave height and the associated undisturbed depth of water layer. This is aimed at ensuring the convergence of the binomial expansion arising from the formulation. Thus, this approach suggests that the solutions are restricted in application to waves with peak amplitude strictly less than the depth of the associated water layer in the shallow water.

In this analysis, we shall use rather a different approach which does not involve series expansion. Consequently, the method admits higher numerical value of η_0/h_0 in the subsequent calculations and, also, the derived steady wave profile appears to remain realistic even when $\eta_0 = 0.78\, h_0$; suggesting the stage when the wave begins to break. This value appears to be near to the usual theoretical prediction of $\eta_0 = 0.85\, h_0$.

2. – Specifications and the governing equations

In this model, the origin is taken as the shoreline with the x-axis normal to it; the y-axis being the vertical coordinate. The horizontal bottom of the water layer is defined by $y = -h_0$; $t > 0$ represents the time. The elevation and depression of the free surface defined by $y = \eta(x, t)$ present the appearance of a series of ridges and furrows, assumed in this consideration to be parallel to the shoreline.

The governing equations are the system [3, 7]:

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}\{u(h_0 + \eta)\} = 0\,, \tag{2.1}$$

$$\frac{\partial u}{\partial t} + \frac{u\,\partial u}{\partial x} + g\frac{\partial \eta}{\partial x} + \gamma\frac{\partial^3 \eta}{\partial x^3} = 0\,. \tag{2.2}$$

g is the constant gravitational acceleration, $u(x, t)$ is the flow velocity. In the absence of the dispersion term $\gamma(\partial^3\eta/\partial x^3)$, $u(x, t)$ is defined by

$$\begin{cases} u(x, t) = 2\sqrt{g(h_0 + \eta)} - 2\sqrt{gh_0} = C_0(\eta/h_0) + O(\eta^2/h_0^2)\,, \\ \gamma = \dfrac{C_0^3}{3}\, h_0\,, \qquad C_0^2 = gh_0\,. \end{cases} \tag{2.3}$$

The process of elimination of $u(x, t)$ in (2.1) using (2.3) introduces the binomial series and the subsequent approximations in the KDV equation formulation as already mentioned.

3. – Formal development

In this approach, we introduce χ defined by $\chi = x - u_0 t$, $u_0 = \sqrt{g(h_0 + \eta_0)}$ corresponding to progressive wave train solution. Thus, (2.1) becomes

$$u = \frac{u_0\eta + A}{h_0 + \eta}\,, \tag{3.1}$$

A being an arbitrary constant of integration, and from (2.2)

$$-u_0 \frac{du}{d\chi} + u \frac{du}{d\chi} + g \frac{d\eta}{d\chi} + \gamma \frac{d^3\eta}{d\chi^3} . \tag{3.2}$$

A first integration of (3.2) gives

$$-u_0 u + \frac{u^2}{2} - g\eta + \gamma \frac{d^2\eta}{d\chi^2} = B . \tag{3.3}$$

B is also an arbitrary constant of integration. Eliminating $u(\chi)$ between (3.1) and (3.3), then

$$\gamma(h_0+\eta)^2 \frac{d^2\eta}{d\chi^2} + g\eta^3 + C_3\eta^2 + C_2\eta + C_1 = 0 , \tag{3.4}$$

where

$$C_1 = \frac{A^2}{2} - u_0 A h_0 - B h_0^2 , \qquad C_2 = g h_0^2 - u_0^2 h_0 - 2B h_0 , \qquad C_3 = 2 g h_0 - B - \frac{u_0^2}{2} .$$

4. – Solitary waves derivable from (3.4)

In the special case of solitary waves, $u(\chi)$, $\eta(\chi)$ and $d\eta/d\chi$ are usually assumed to vanish at infinity for a choice of h_0. Further, the same suitable choice of h_0 will also make $d^2\eta/d\chi^2 = 0$ at infinity; thus, $A = B = 0$. Equation (3.4) now becomes

$$\gamma(h_0+\eta)^2 \frac{d^2\eta}{d\chi^2} + g\eta^3 + \eta^2\left(2gh_0 - \frac{u_0^2}{2}\right) + \eta(gh_0^2 - u_0^2 h_0) = 0 . \tag{4.1}$$

Let $\eta = \xi - h_0$; (4.1) becomes

$$\frac{\gamma}{2}\frac{d^2\xi}{d\chi^2} + g\xi - \left(gh_0 + \frac{u_0^2}{2}\right) + \frac{u_0^2 h_0^2}{2\xi^2} = 0 . \tag{4.2}$$

The absence of the term involving $1/\xi$ in (4.1) and the associated logarithmic term in the subsequent integration imply that no series expansion is required and hence, the advantage of this approach.

Integrating (4.2) again

$$\frac{\gamma}{2}\left(\frac{d\xi}{d\chi}\right)^2 + g\frac{\xi^2}{2} - \left(gh_0 + \frac{u_0^2}{2}\right)\xi - \frac{u_0^2 h_0^2}{2\xi} = \mathbf{C}_0 . \tag{4.3}$$

$\mathbf{C}_0$ is also constant of integration easily determined by same choice of $\xi' = \eta' = \eta = 0$ corresponding to $\xi = h_0$. Therefore, $\mathbf{C}_0 = -(h_0/2)(gh_0 + u_0^2)$ (4.3) now becomes

$$\frac{\gamma}{2}\left(\frac{d\xi}{dz}\right)^2 + \xi^3 - (2h_0 + f^2)\,\xi^2 + h_0(h_0 2f^2)\,\xi - f^2 h_0^2 = 0 , \tag{4.4a}$$

where $f^2 = u_0^2/g$. We now assume that

(4.4*b*) $$\mathrm{d}\chi = \sqrt{\xi}\,\mathrm{d}z\,,$$

(4.4*b*) is valid provided that the water surface never touches the bottom. That is, $\eta > -h_0$; this is the usual shallow-water assumption.

Equation (4.4*a*) takes the form

(4.5) $$\frac{\gamma}{g}\left(\frac{\mathrm{d}\xi}{\mathrm{d}z}\right)^2 = F(\xi)\,,$$

(4.6) $$F(\xi) = -\xi^3(2h_0+f^2)\,\xi^2 - h_0(h_0+2f^2)\,\xi + f^2h_0^2\,.$$

We now obtain the solution of (4.5) in the form of an elementary function corresponding to the solitary wave. In this case, it is usual to assume that the roots of the equation

$$F(\xi) = 0 \qquad \text{are} \ -a\,,\ \ -a\,,\ b\,.$$

In the case of the KDV equation [1], these are 0, 0, b. We now have

(4.7) $$F(\xi) = (a+\xi)^2(b-\xi)\,.$$

By comparing coefficients in (4.6) and (4.7), a and b satisfy the equations

(4.8) $$a^2(2h_0+f^2) + 2ah_0(h_0+2f^2) - 3f^2h_0^2 = 0\,,$$

(4.9) $$b = 2a + h_0 + f^2\,.$$

Separating variables in (4.5) and using (4.7), then

(4.10) $$\int\frac{\mathrm{d}\xi}{(\xi+a)(b-\xi)^{1/2}} = \frac{zg}{\gamma} + \beta_0 = \frac{1}{\sqrt{a+b}}\ \ln\left\{\frac{(b+\xi)^{1/2}+(b+a)^{1/2}}{(a-\xi)^{1/2}-(b+a)^{1/2}}\right\},$$

where β_0 is an arbitrary constant. Choose

$$\beta_0 = \frac{\pi i}{\sqrt{a+b}}\,,$$

then

(4.11) $$\xi = -a + (a+b)\,\mathrm{sech}^2\left\{\frac{g}{\gamma}\,(a+b)^{1/2}z\right\}$$

and

(4.12) $$\sqrt{a+2h_0}\sinh\left[h_0^{1/2}\left\{\frac{g}{\gamma}\,(a+b)\right\}^{1/2}\chi\right] = \sqrt{a+b}\sinh\left[\left\{\frac{g}{\gamma}\,(a+b)\right\}^{1/2}z\right].$$

Equation (4.11) represents a solitary wave with b as the amplitude but tails off to $-a$ as $z\to\pm\infty$. This is shown in fig. 1. The parameter $a = 0$ in the usual solution arising from the KDV equation for shallow-water waves. In this case, (4.10) thus suggests a more complete solution. Interestingly, the constant a seems to represent a measure of depth

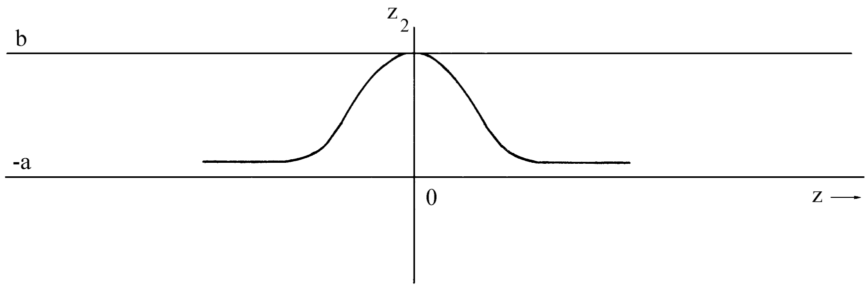

Fig. 1. – Solitary wave.

for which the undisturbed level of the surrounding shallow water is depressed during the passage of the solitary wave.

Numerically, $\eta_0 \in (0.5h_0, 0.85h_0)$, then $a \in (1.1h_0, 1.31h_0)$.

Consequently, since the origin of the co-ordinate system is taken at the undisturbed water level, this analysis conclusively suggests that with this range of values of η_0 and a, the observed surrounding sea level will be depressed to a depth between $0.1h_0$ and $0.31h_0$ from the undisturbed level during the passage of the solitary wave.

5. – Cnoidal oscillation derivable from (3.4)

Solution of (3.4) can be expressed in the form of periodic or cnoidal wave. In this consideration, the oscillation does not need to vanish at infinity before the constants of integration involved in (3.4) are fixed. Instead, h_0 can be suitably chosen such that $\eta = \mathrm{d}^2\eta/\mathrm{d}\chi^2 = 0$ and, consequently, in (3.1) and (3.3), $A = B = 0$. This is achieved if we assume that the uniform depth h_0 extends to the seaward end of the shallow water beyond which all oscillations vanish owing to probable introduction of reverse current. In this case, (4.5) still holds.

However, we firstly let

$$F^2 = \frac{u_0^2}{gh_0} = \frac{f^2}{h_0}\,, \qquad R_0 = \frac{\gamma}{gh_0} \qquad \text{and} \qquad \xi = h_0\bar{\xi}\,;$$

then, dropping the bar, (4.4a) becomes

$$R_0\dot{\xi}^2 = G(\xi) = R_0\left(\frac{\mathrm{d}\xi}{\mathrm{d}z}\right)^2 = R_0\dot{\xi}^2 \tag{5.1}$$

$(\cdot = \mathrm{d}/\mathrm{d}z)$, where

$$G(\xi) = -\xi^3 + (2+F^2)\,\xi^2 - (1+2F^2)\,\xi + F^2\,. \tag{5.2}$$

Express $G(\xi)$ in the form

$$G(\xi) = 4A_1(\xi_0 - \xi)^3 + 6A_2(\xi - \xi_0)^2 + 4A_3(\xi_0 - \xi)\,. \tag{5.3}$$

TABLE I. – *Variation of non-breaking wave parameters with wave height.*

η_0/h_0	$-\xi_0$	k	θ	$\omega R_0^{1/2}$
0.60	0.20075	0.4925	20.51°	1.0112
0.65	0.20103	0.4910	29.41°	1.0207
0.71	0.20130	0.4905	29.38°	1.0212
0.78	0.20160	0.4898	29.32°	1.0222
0.80	0.20207	0.4895	29.31°	1.0237
0.83	0.20401	0.4876	29.18°	1.0253

By comparing coefficients of ξ in (5.2) and (5.3), we have

$$A_1 = \frac{1}{4}\,, \qquad A_2 = \frac{1}{6}\,(2 + F^2 - 3\xi_0)\,, \qquad A_3 = \frac{1}{4}\,\{2\xi_0(2+F^2) - 3\xi_0^2 - (1+2F^2)\}$$

and ξ_0 satisfies the cubic equation

$$5\xi_0^3 - 3\xi_0^2(2+F^2) + \xi_0(1+F^2) + F^2 = 0\,. \tag{5.4}$$

Numerical calculations of the only real root of (5.4) as a function of η_0/h_0 is shown in table I. This is computed to an acceptable error of -1.5×10^{-9}.

Again, let $P^2 = \xi_0 - \xi$, then (5.1) simplifies to

$$\dot{P}^2 = \frac{1}{4R_0}\,(P^2 - P_2^2)(P^2 - P_1^2)\,, \tag{5.5}$$

where

$$P_1^2 = R - 3A_2\,, \qquad P_2^2 = R + 3A_2 \qquad \text{and} \qquad R = (9A_2^2 - 4A_3)^{1/2}\,.$$

Further, let $P = P_1 y$; then (5.5) takes the standard form

$$\dot{y}^2 = \omega^2(1-y^2)(1-k^2y^2)\,, \tag{5.6}$$

where $\omega^2 = P_2^2/4R_0$, $k = P_1/P_2$; (5.6) has solution

$$y = \mathrm{Sn}\,(\omega z) \;\; \text{and} \;\; \eta/h_0 = A_0 + P_n^2\,\mathrm{Cn}^2\,(\omega z)\,, \qquad A_0 = \xi_0 - P_1^2\,. \tag{5.7}$$

Sn and Cn are Jacobian elliptic functions of first kind with modulus k; $k = \sin\theta$, θ being modular angle. From the properties of Cn, ξ oscillates between ξ_0 and A_0 as shown in fig. 2.

Solution (5.7) thus, describes a periodic or cnoidal wave of period T given by

$$T = \frac{4K(k^2)}{\omega}\,,$$

where $K(k^2)$ is a complete Jacobian elliptic function of the first kind expressible in

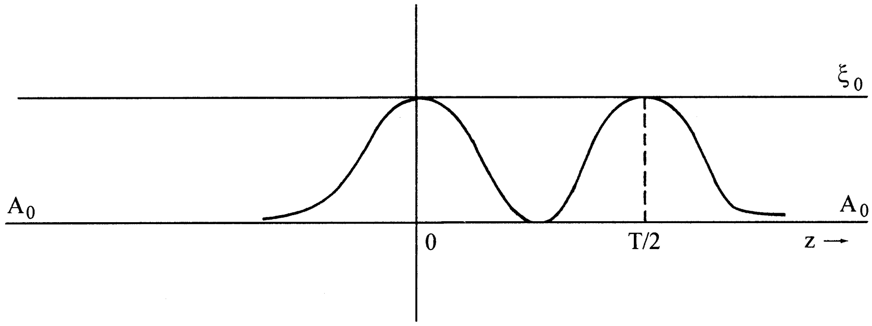

Fig. 2. – Cnoidal wave.

terms of hypergeometric function; that is

$$K(k^2) = \frac{\pi}{2}\, {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right) \tag{5.8}$$

and

$$T = \frac{2\pi}{\omega}\, {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right).$$

Numerically, $0 < k < 1$ as shown in table I.

Using the properties of the hypergeometric functions,

$$ {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right) = +1\frac{k^2}{4} + o(k^4); \tag{5.9}$$

then,

$$T = \frac{2\pi}{\omega}\left[1 + \frac{k^2}{4} + O(k^4)\right]. \tag{5.9}$$

Equation (5.9) can be used to estimate the period T given η_0 and h_0. For example, if $0.70h_0 \leqslant \eta(\chi) \leqslant 0.85h_0$, and $h_0 = 2(1/2)$ m, $\gamma = 5/6$ (in metre unit); then $T = 8$ seconds. This period is the one usually associated with locally generated short swell on very shallow beach in steady state.

6. – Conclusions

This theoretical model illustrates that with the suitable change of variables, the solutions of the shallow-water wave equations can be completely obtained. In particular, eq. (4.2) applies even in the case of the arbitrary depth distribution, and so, can be integrated numerically. The equation is further simplified in all the cases involving uniformly sloping bottom [4].

Equation (5.9) gives a quantitative estimate of the wave period in the shallow water with constant depth distribution. A quantitative estimate of 8 second period when the undisturbed water depth is 2(1/2) m suggests that the solution (5.7) can reasonably describe the profile of a short swell propagating towards a shoreline (see fig. 2). Consequently, matching of the Fourier analysis of the solution (5.7) with the corresponding microseismic frequency spectrum may give the improved estimate of microseismic [5] activities in the locality concerned.

Finally, the variation of wave parameters such as k, θ and ω are shown as functions of relative wave amplitude peak. The gradual variations in these parameters are explainable by the slow evolution of wave profile relative to its speed in the shallow water.

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