Physically consistent diffusion in media with memory (*)

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Summary. — Some data on the flow of fluids in rocks exhibit properties which cannot be interpreted with the classic theory of propagation of pressure and of fluids in media (ROELOFFS E. A., *Fault Stability Changes Induced Beneath a Reservoir with Cyclic Variations in Water Level, J. Geophys. Res.*, **93** (1988) 2107-2124) based on the classic D'Arcy's law which states that the flux is proportional to the pressure gradient. In order to obtain a better representation of the flow and pressure of fluids in media, the law of D'Arcy was modified introducing instead of the pressure gradient its time derivative of fractional order z (CAPUTO M., *The Green function of diffusion of fluids in porous media with memory*, *Rend. Fis. Acc. Lincei*, **9** (1996) 243-250). However, the new law implies a filtering of the pressure gradient with response curves which are singular or nil at zero or infinite frequency. In this note we extend the law of D'Arcy introducing a memory formalism operating on the flow as well as on the pressure gradient which implies a filtering of the pressure gradient without singularities; the properties of the filtering are also described.

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PACS 92.40.Cy – Modeling, general theory.
PACS 92.40.Kf – Groundwater.
PACS 91.10.Rn – Rheology of litosphere and mantle.
PACS 01.30.Cc – Conference proceedings.

Glossary

$1/A$ ($m^2 s^{-2}$) see formula (6),
$p(x, t)$ (kg $s^{-2} m^{-1}$) pressure of the fluid,
q (kg $s^{-1} m^{-2}$) fluid mass flow rate in the medium per unit area in the x-direction,
$p(x, 0)$ initial pressure in the medium,
t (s) time,
x (m) distance from the plane bounding the medium,
z (dimensionless) order of differentiation,
κ ($kg^{-1} m^3 s^1$) ratio of the permeability of the medium to the viscosity of the fluid (see formula (1)),

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η ($kg^{-1} m^3 s^{1+z}$) ratio of the pseudopermeability of the medium (see formula (2)) to the viscosity of the fluid,
ϱ_0 density of the fluid in the initial condition,
$\varrho(x, t)$ (kg, m^{-3}) variation of the density of the fluid from its initial condition.

1. – Introduction

The basic equations needed to study the flow of fluids in media have been set through this century perhaps beginning with Terzaghi (1923, 1936); since then a great progress has been made through the work of Biot (1941, 1956a, 1956b, 1973), Biot and Willis (1957), Boley and Tolins (1962), Nowacki (1964), McNamee and Gibson (1960), Booker (1974), Rice and Cleary (1976), Bell and Nur (1978) and Roeloffs (1988) who contributed in different ways to set the equations rigorously representing the interaction between the medium and the flow of the fluid through it and to obtain solutions of these equations in many interesting cases.

Most authors who studied diffusion problems in media used the classic empirical law of D'Arcy stating proportionality between the fluid mass flow rate and the gradient of the pore pressure in the same direction.

More general problems than these of diffusion with memory are those of the classical convection-dispersion model (CDM), where the transport coefficients are time dependent and which are also based on D'Arcy's law; they are well reviewed in Barry and Sposito (1989). Instead of introducing the time-dependent transport coefficients, as done in previous work of Wyss (1986), Mainardi (1993) and Caputo (1976, 1996a, 1998a, 1998b), in this note we shall introduce in the diffusion equation a memory formalism represented by derivatives of fractional order operating on the pressure gradient as well as on the flux. Specifically in this note we shall discuss the diffusion with memory in the frequency domain.

2. – The model

Many models, including the CDM type, imply that the permeability of the medium is variable with time and this phenomenon is taken into account when writing the law of D'Arcy which states that proportionality between the fluid mass flow rate q_i per unit area in the x_i direction and the gradient of the pore pressure p:

$$q_i = -\varrho_0 \kappa \partial p / \partial x_i , \tag{1}$$

where κ, with dimension $m^3 kg^{-1} s$, is the ratio of the permeability of the medium to the viscosity of the fluid and ϱ_0 is the density in the undisturbed condition.

One way to take into account the variability of the flow is to introduce a memory formalism in D'Arcy's law and consider that the flow depends on the history of the pressure gradient. A simple way of doing it is to write eq. (1) as follows:

$$q_i = -hr_0 (\partial^z / \partial t^z)(\partial p / \partial x_i) , \tag{2}$$

where η, with dimensions $kg^{-1} m^3 s^{1+z}$, is the ratio of the pseudopermeability of the medium to the viscosity of the fluid, ϱ_0 is the density $\varrho(x, t)$ of the fluid in its undisturbed condition and the definition of derivative of order z is

$$\partial^z p(x, t)/\partial t^z = (1/\Gamma(1-z))\int_0^t (t-u)^{-z}(\partial p(x, u)/\partial u)\,\mathrm{d}u \tag{3}$$

with $0 \leqslant z < 1$. In the definition (3) there is convergence at $t = u$ for any value of t since $0 \leqslant z < 1$ is assumed. It is clear that the memory formalism introduced here to describe the flow of the fluid implies the use of more than one parameter, namely z and η, instead of the only one parameter κ as in D'Arcy's law.

The extensions of the diffusion equation of fluids in media can also be done in the time domain, for instance by assuming that the diffusivity depends on time and/or on position (*e.g.*, Boltzman 1894, Barry and Sposito 1989) or in the frequency domain by assuming a memory formalism in the law of D'Arcy (*e.g.*, Wyss 1986, Mainardi 1993, Caputo 1996a).

The extension (2) of the diffusion equation used by Wyss (1986), Schneider and Wyss (1989), Mainardi (1993) and Caputo (1996a), although mathematically consistent, does not give satisfactory results, when $0 < z < 1$, since it implies a filter acting on the pressure gradient whose response curve is nil at zero frequency and infinite at infinite frequency. The case $-1 < z < 0$ implies again a filter whose response curve is infinite at zero frequency and nil at infinite frequency.

Although the estension (2) of the diffusion equations may be useful in limited frequency ranges, a physically more satisfactory extension of the diffusion equation based on D'Arcy's law can be obtained by writing it in the following form:

$$aq + b\partial^{z_1} q/\partial t^{z_1} = -(cp_x + d\partial^{z_2} p_x/\partial t^{z_2}), \tag{4}$$

where we consider the problem in one dimension and in an infinite medium, limited by a plane, where the coordinate x is normal to the boundary plane oriented positive towards the medium; $q(x, t)$ is the flux, $p(x, t)$ is the fluid pressure in the pores of the medium and the subscript x indicates the derivative with respect to x and therefore p_x is the gradient of p in the x-direction. The parameter a is dimensionless while b, c and d have dimensions s^{z_1}, s and s^{1+z_2}, respectively.

We shall see here that the introduction of (4) eliminates the singularities at zero and infinite frequencies.

The LT of (4) is of interest in the discussion of the filtering acting on the gradient of the pressure

$$Q = -(c + ds^{z_2})P_x/(a + bs^{z_1}) + (s^{z_1-1}bq(x, 0) + s^{z_2-1}dp_x(x, 0))/(a + bs^{z_1}), \tag{5}$$

where Q is the LT (Laplace Transform) of q, P is the LT of p with s LT variable and $0 < z_1 < 1$, $0 < z_2 < 1$.

We shall discuss here the more general form (4) of D'Arcy's law whose LT (5) readily shows that, when the initial pressure gradient and flow are nil, that is when the initial pressure is independent of time and position, and a, b, c and d are positive, then the response of the filter at zero and infinite frequency is not nil or singular.

In order to find the pressure distribution in the medium due to a given pressure at the boundary $x = 0$, let us associate to (1) the continuity equation

$$\partial q/\partial x + \partial r/\partial t = 0\,; \tag{6}$$

assuming that the variation of the density $\varrho(x, t)$ is related to the pressure by

$$\varrho(x, t) = Ap(x, t), \tag{7}$$

where A is a parameter with dimensions $m^{-2}s^2$, we find

$$\partial q/\partial x + A\partial p/\partial t = 0, \tag{8}$$

whose LT form is

$$Q_x + sAP - Ap(x, 0) = 0. \tag{9}$$

Substituting (7) in (5) we find the equation governing the propagation of the pressure in the medium,

$$\begin{aligned}P_{xx}(x, s) = A[s(a + bs^{z_1})/(c + ds^{z_2})]\, P(x, s) + (bs^{z_1 - 1} q_x(x, 0) + \\ + ds^{z_2 - 1} p_{xx}(x, 0))/(c + ds^{z_2}) - [(a + bs^{z_1})/(c + ds^{z_2})\, Ap(x, 0).\end{aligned} \tag{10}$$

3. – The Green function

As usual in this problem we assume that the pressure in the medium is initially independent of x and therefore

$$p_x(x, 0) = 0, \quad p_{xx}(x, 0) = 0, \quad q(x, 0) = 0, \quad q_x(x, 0) = 0. \tag{11}$$

We can then simplify (10) as follows:

$$P_{xx}(x, s) = A[s(a + bs^{z_1})/(c + ds^{z_2})]\, P(x, s) - [(a + bs^{z_1})/(c + ds^{z_2})]\, Ap(0), \tag{12}$$

where $p(0)$ is the initial pressure in the medium independent of x.

It is verified that a particular solution of (12) is $s^{-1}p(0)$.

The formal solution of (10), converging at infinite distance, is then

$$P(x, s) = B(s) \exp[-[As(a + bs^{z_1})/(c + ds^{z_2})]^{1/2} x] + s^{-1}p(0), \tag{13}$$

where $B(s)$ is an arbitrary function of s with LT^{-1} $B(s) = f(t)$ limited which will be defined with the condition at the boundary $x = 0$.

From here, in order to simplify the computations we shall assume that $z_1 = z_2 = z$ and seek the LT^{-1} of (13) following the same procedure previously used (Caputo 1996a).

By assuming

$$s = r \exp[i\theta], \tag{14}$$

the exponential factor of $B(s)$ appearing in (13) may be written as

$$\exp[-[As(a + bs^{z})/(c + ds^{z})]^{1/2} x] = \exp[-xT \exp[\Theta]], \tag{15}$$

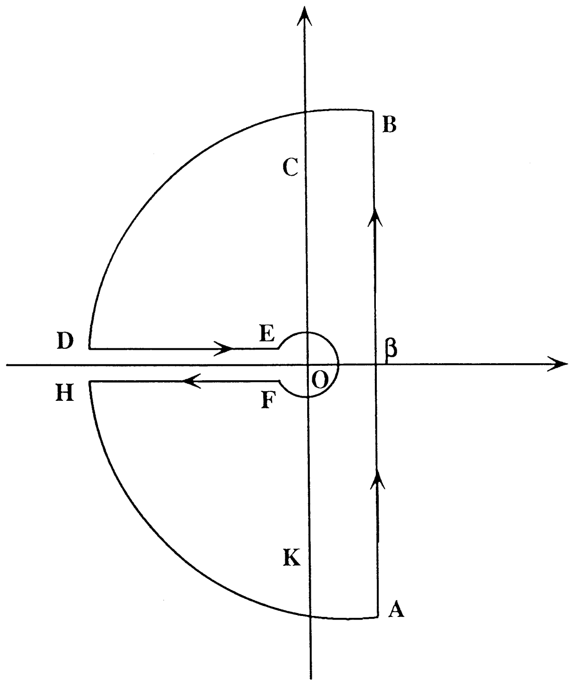

Fig. 1. – Path of integration of formula (14).

where

$$\text{(16)}\qquad \begin{cases} T(r,\,\theta) = (Kr)^{1/2}[(L^2 + r^{2z} + 2Lr\cos z\theta)/(M^2 + r^{2z} + 2Mr\cos z\theta)]^{1/4}, \\ Q(r,\,\theta) = 0.5\big(\tan^{-1}(r^z \sin z\theta/(L + r^z\cos z\theta)) - \tan^{-1}(r^z\sin z\theta/(M + r^z\cos z\theta))\big), \\ K = Ab/d\,, \qquad L = a/b\,, \qquad M = c/d\,. \end{cases}$$

It is to be noted that

$$\text{(17)}\qquad T(r,\,\pi) = T(r,\,-\pi)\,, \qquad \Theta(r,\,\pi) = -\,\Theta(r,\,-\pi)\,.$$

The LT^{-1} of the exponential factor of $B(s)$, appearing in (13), is computed integrating along the closed path of fig. 1, inside which there are no poles of the exponential because this has no poles in the negative complex plane of s. The integral is therefore nil because the residuals are nil and we can write

$$\text{(18)}\qquad (1/2i\pi)\lim_{u\to\infty}\Bigg[\int_{b-iu}^{b+iu}\exp[st - x[As(a+bs^z)/(c+ds^z)]^{1/2}]\,\mathrm{d}s + \\ + \int_D^E \exp[st - x[As(a+bs^z)/(c+ds^z)]^{1/2}]\,\mathrm{d}s + \\ + \int_F^H \exp[st - x[As(a+bs^z)/(c+ds^z)]^{1/2}]\,\mathrm{d}s\Bigg] = 0\,.$$

Noting that in the integration on DE: $\theta = \pi$ and on FH: $\theta = -\pi$, then

(19) $$\theta = \pm\pi\,, \qquad s = -r\,, \qquad ds = -dr\,, \qquad s^{z/2} = r^{z/2}(\cos(\pi z/2) \pm i\sin(\pi z/2))$$

and finally

(20) $$p(x,t) = p(0) -$$

$$-(1/2i\pi)\int_0^\infty \{\exp[-rt - r^{z/2}xK^{1/2}[(L^2 + r^{2z} + 2Lr^z\cos z\pi)/(M^2 + r^{2z} + 2Mr^z\cos\pi z)]^{1/4}\cdot$$

$$\cdot\cos(0.5(\tan^{-1}(-r^z\sin z\pi/(L + r^z\cos z\pi)) - \tan^{-1}(-r^z\sin z\pi/(M + r^z\cos z\pi)) - \pi))]\}\cdot$$

$$\cdot\{\cos[r^{z/2}xK^{1/2}[(L^2 + r^{2z} + 2Lr^z\cos z\pi)/(M^2 + r^{2z} + 2Mr^z\cos\pi z)]^{1/4}\cdot$$

$$\cdot\sin(0.5(\tan^{-1}(-r^z\sin z\pi/(L + r^z\cos z\pi)) - \tan^{-1}(-r^z\sin z\pi/(M + r^z\cos z\pi)) - \pi))] +$$

$$+ i\sin[r^{z/2}xK^{1/2}[(L^2 + r^{2z} + 2Lr^z\cos z\pi)/(M^2 + r^{2z} + 2Mr^z\cos\pi z)]^{1/4}\cdot$$

$$\cdot\sin(0.5(\tan^{-1}(-r^z\sin z\pi/(L + r^z\cos z\pi)) - \tan^{-1}(-r^z\sin z\pi/(M + r^z\cos z\pi)) - \pi))]\}\,dr -$$

$$-(1/2i\pi)\int_\infty^0 \{\exp[-rt - r^{z/2}xK^{1/2}[(L^2 + r^{2z} + 2Lr^z\cos z\pi)/(M^2 + r^{2z} + 2Mr^z\cos\pi z)]^{1/4}\cdot$$

$$\cdot\cos(0.5(\tan^{-1}(r^z\sin z\pi/(L + r^z\cos z\pi)) - \tan^{-1}(r^z\sin z\pi/(M + r^z\cos z\pi)) + \pi))]\}\cdot$$

$$\cdot\{\cos[r^{z/2}xK^{1/2}[(L^2 + r^{2z} + 2Lr^z\cos z\pi)/(M^2 + r^{2z} + 2Mr^z\cos\pi z)]^{1/4}\cdot$$

$$\cdot\sin(0.5(\tan^{-1}(r^z\sin z\pi/(L + r^z\cos z\pi)) - \tan^{-1}(r^z\sin z\pi/(M + r^z\cos z\pi)) + \pi))] +$$

$$+ i\sin[r^{z/2}xK^{1/2}[(L^2 + r^{2z} + 2Lr^z\cos z\pi)/(M^2 + r^{2z} + 2Mr^z\cos\pi z)]^{1/4}\cdot$$

$$\cdot\sin(0.5(\tan^{-1}(r^z\sin z\pi/(L + r^z\cos z\pi)) - \tan^{-1}(r^z\sin z\pi/(M + r^z\cos z\pi)) + \pi))]\}\,dr\,.$$

Using (17) we can obtain from (20) the general solution in the following simpler form:

(21) $$p(x,t) = p(0) + (\mathrm{LT}^{-1}(B(s)))\cdot$$

$$\cdot(1/p)\int_0^\infty \{\exp[-rt - r^{z/2}xK^{1/2}[(L^2 + r^{2z} + 2Lr^z\cos zp)/(M^2 + r^{2z} + 2Mr^z\cos pz)]^{1/4}\cdot$$

$$\cdot\cos(0.5(\tan^{-1}(r^z\sin zp/(L + r^z\cos zp)) - \tan^{-1}(r^z\sin zp/(M + r^z\cos zp))))]\}\cdot$$

$$\cdot\sin\{r^{z/2}xK^{1/2}[(L^2 + r^{2z} + 2Lr^z\cos zp)/(M^2 + r^{2z} + 2Mr^z\cos pz)]^{1/4}\cdot$$

$$\cdot\sin(0.5(\tan^{-1}(r^z\sin zp/(L + r^z\cos zp)) - \tan^{-1}(r^z\sin zp/(M + r^z\cos zp))))\}\,dr\,.$$

Formula (21) can be numerically integrated for all values of x and t. It is to be noted that, since x and t are here assumed positive or nil, the integral in (21) is convergent and nil for $x = 0$.

When $\mathrm{LT}^{-1}B(s) = \delta(t)$, then formula (21) is the Green function of the problem and one may solve the classic problems of diffusion.

From (11) assuming $f(t) = \mathrm{LT}^{-1} B(s)$ it is also easily seen with LT extreme-values theorem that $p(x, 0) = 0 f(0) + p(0)$ for any x, which implies that the pressure, at the initial time, is the initial one. For $t = \infty$ it is $p(x, \infty) = 0 f(\infty) + p(0) = p(0)$.

The result that the solution is $p(0)$ at the initial time is in agreement with our definition of fractional order derivative which requires that the function is finite at the initial time.

The inspection of formula (21) shows that for any value of t, however small, and of x, however large, the value of the integral is always larger than zero, which implies that the signal travels with infinite velocity. However, when the boundary signal is sinusoidal the wave travels with a finite phase velocity and an amplitude depending on frequency.

4. – Filtering properties

As we mentioned, the memory formalism (4) implies filtering of the pressure gradient with response $F(\omega)$ obtained by substituting $s = i\omega$ in (5) with the assigned initial conditions (9):

$$|F(\omega)| = (c + ds^{z_1})/(a + bs^{z_2}) =$$

$$= |(c^2 + d^2\omega^{2z_1} + 2cd\omega^{z_1}\cos pz_1)/(a^2 + b^2\omega^{2z_2} + 2ab\omega^{z_2}\cos pz_2)|^{1/2},$$

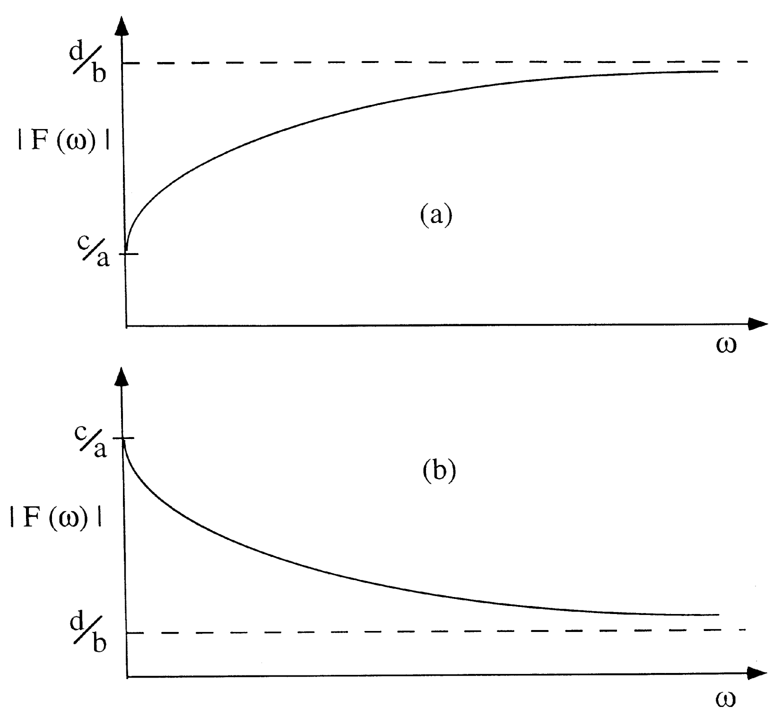

Fig. 2. – Response curves of the filter $F(\omega)$ represented by the memory formalism introduced by the model defined in (4). In (a) is the case when $d/b > c/a$, in (b) is the case when $d/b < c/a$.

$$\arg F(\omega) =$$

$$= a\tan^{-1}\{d\omega^{z_1}\sin pz_1/(c+d\omega^{z_1}\cos\pi z_1)\} - a\tan^{-1}\{b\omega^{z_2}\sin pz_2/(a+b\omega^{z_2}\cos pz_2)\}\,.$$

When $z_1 = z_2$, as in the case of the Green function determined here, depending on $c/a > d/b$ or $c/a < d/b$ we have a decreasing or an increasing response curve, respectively, as shown in fig. 2 where it is noted that unless $a = 0$ or $b = 0$ there are no singularities in the response curves.

A physically interesting case is when $z_1 = z_2$ and $d = 0$ in which the filter response is finite at zero frequency and nil at infinite frequency, which may be of use in some cases of geoelectric prospecting. The relaxation time is $(b/a)^{1/z}$ and the response to an infinite frequency input is d/a, $a = 0$ would therefore imply a relaxation time of infinite duration.

Ii is of interest in solid Earth geophysics, the case when $z_1 = z_2$ and $a = 0$ (Körnig and Müller, 1989), in which there is infinite response at zero frequency input and a finite response at infinite frequency input.

In case one of the parameters b or d is nil then the procedure to obtain formulae (20) and (21) is still valid and the formulae can be readily computed arriving at simpler results.

5. – Conclusions

The memory formalisms introduced by various authors (Wyss, 1986; Schneider and Wyss, 1989; Mainardi, 1993; and Caputo, 1976, 1996a) imply singularities in the consequent filtering applied to the pressure gradient; the formalism introduced in this note implies a filtering of the pressure gradient without singularities and is therefore physically acceptable.

The form of the Green function in the case when no singularities are present in the response curve of the filter is more complicated than that of the case when singularities are present; this is the price for physical consistency.

After the preceding discussion we can also extend to the diffusion the formal mathematical analogy between anelastic and dielectric media assuming that the flux is the dual of the deformation (induction) and the pressure gradient is the dual of the stress (applied electric field) as shown by Caputo (1996b).

The new memory formalism is of interest in the diffusion phenomena of fluids in porous media. In particular the physically consistent solution obtained in the present note may also contribute to give a satisfactory explanation to the velocity of migration of earthquake's foci and to the variable velocity of migration, in the crust of the Earth, of the precursory phenomena of strong earthquakes (Caputo, 1992), which are possibly due to the diffusion of subterranean waters.

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