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Collapse of three vortices on a sphere (*)(**)

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Summary. — The self-similar collapse of three point vortices moving on the surface of a sphere of radius R is analysed and compared with known results from the corresponding planar problem described in (AREF H., *Motion of three vortices*, *Phys. Fluids*, **22** (1979) 393-400; NOVIKOV E. A., *Dynamics and statistics of a system of vortices, Sov. Phys. JETP*, **41** (1975) 937-943; NOVIKOV E. A. and SEDOV Y., *Vortex collapse, Sov. Phys. JETP*, **50** (1979) 297-301; SYNGE J. L., *On the motion of three vortices, Can. J. Math.*, **1** (1949) 257-270). An important conserved quantity is the center of vorticity vector $\mathbf{c} = (\sum_{i=1}^{3} \Gamma_i \mathbf{x}_i)/\sum_{i=1}^{3} \Gamma_i$, which must have length R for collapse to occur. Collapse trajectories occur in pairs, called "partner states", which have two distinct collapse times $\tau^- < \tau^+$. The collapse time that is achieved for a given configuration depends on the sign of the parallelpiped volume formed by the vortex position vectors, hence depends on whether the vortices $(\Gamma_1, \Gamma_2, \Gamma_3)$ are arranged in a right-handed or left-handed sense. From a given collapsing configuration, one can obtain the partner state by reversing the signs of the Γ_i's, or, alternatively, by using a discrete symmetry associated with the initial configuration that leaves all relative distances unchanged, but reverses the sign of the parallelepiped volume. In the plane, there is only one collapse time associated with a given configuration—the partner state is one that expands self-similarly (AREF H., *Motion of three vortices*, *Phys. Fluids*, **22** (1979) 393-400). Formulas for the collapsing trajectories are derived and compared with the planar formulas. The collapse trajectories are then projected onto the stereographic plane where a new Hamiltonian system is derived governing the vortex motion. In this projected plane, the solutions are not self-similar. In the last section, the collapse process is studied using tri-linear coordinates, which reduces the system to a planar one.

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1. – Introduction

When three point vortices of differing signs move on the surface of a sphere, it is possible for them to collapse self-similarly in finite time [1]. The corresponding collapse process for planar point vortices has been well studied [2-5], yet despite the fact that the spherical problem is more geophysically relevant [6, 7], spherical collapse has only recently been studied [1]. In this paper, we describe the collapse process in detail, and contrast it with the planar collapse process for which much more is known. Aside from its inherent mathematical interest, there is recent evidence [8, 9] that three vortex collapse in the plane and on the sphere is the most frequent interaction for finite-sized vortices in dilute 2D turbulence simulations. In general terms, the motion of vortices on a sphere is important if one wants to understand the large scale mixing and dynamics associated with atmospheric and oceanographic structures that evolve over large distances and have long lifetimes. In this case, the curvature of the Earth becomes important and one can no longer use the tangent plane approximation. There are, of course, other important physical mechanisms at work in most settings of this type, including rotation, diffusion, and vertical density stratification, but we will limit ourselves here to include only the effect of the single layer spherical geometry on the collapse process.

The equations of motion for three vortices moving either in the plane or on the surface of a sphere of radius R are [1]

$$\dot{\mathbf{x}}_i = \sum_{j \neq i}^{3} \frac{\Gamma_j}{2\pi} \, \frac{\mathbf{n_j} \times (\mathbf{x}_i - \mathbf{x}_j)}{\|\mathbf{x}_i - \mathbf{x}_j\|^2} \, ,$$

where Γ_j is the vortex strength, while $\mathbf{n}_j$ is the unit normal vector to the surface at the vortex location $\mathbf{x}_j$. For the planar problem [2, 3, 5], $\mathbf{n}_j = \mathbf{e}_z$, where $\mathbf{e}_z$ is the constant unit normal to the plane and $\mathbf{x}_j \equiv (x_j, y_j)$ is the vortex position. In this case, since the normal vector is constant, one need not include it and the equations are additionally simplified by using complex notation $z_j = x_j + iy_j$, as is well known [10, 11]. On a sphere of radius R, the normal vector is given by $\mathbf{n}_j = \mathbf{x}_j / R$, with the position vector $\mathbf{x}_j \equiv (x_j, y_j, z_j)$ pointing from the center of the sphere to the vortex Γ_j. On the sphere, one also has the constraint $\|\mathbf{x}_j\| = R$. This unified way of writing either the planar or spherical system mimicks the Biot-Savart law [11].

In spherical coordinates (see fig. 1), the equations of motion become [12-14, 1]

$$\dot{\theta}_i = -\frac{1}{4\pi R^2} \sum_{j \neq i}^{3} \frac{\Gamma_j \sin(\theta_j) \sin(\phi_i - \phi_j)}{1 - \cos(\gamma_{ij})} \, ,$$

$$\sin(\theta_i)\, \dot{\phi}_i = \frac{1}{4\pi R^2} \sum_{j \neq i}^{3} \frac{\Gamma_j (\sin(\theta_i)\cos(\theta_j) - \cos(\theta_i)\sin(\theta_j)\cos(\phi_i - \phi_j))}{1 - \cos(\gamma_{ij})} \, ,$$

where $\cos(\gamma_{ij}) \equiv \cos(\theta_i)\cos(\theta_j) + \sin(\theta_i)\sin(\theta_j)\cos(\phi_i - \phi_j)$. Note that the denominator is the chord distance between two vortices $R^2(1 - \cos(\gamma_{ij})) = l_{ij}^2 \equiv \|\mathbf{x}_i - \mathbf{x}_j\|^2$. This system is in Hamiltonian form, with the Hamiltonian given by

$$\mathscr{H} = \frac{1}{4\pi R^2} \sum_{i<j} \Gamma_i \Gamma_j \ln(l_{ij}^2) \, . \tag{1.1}$$

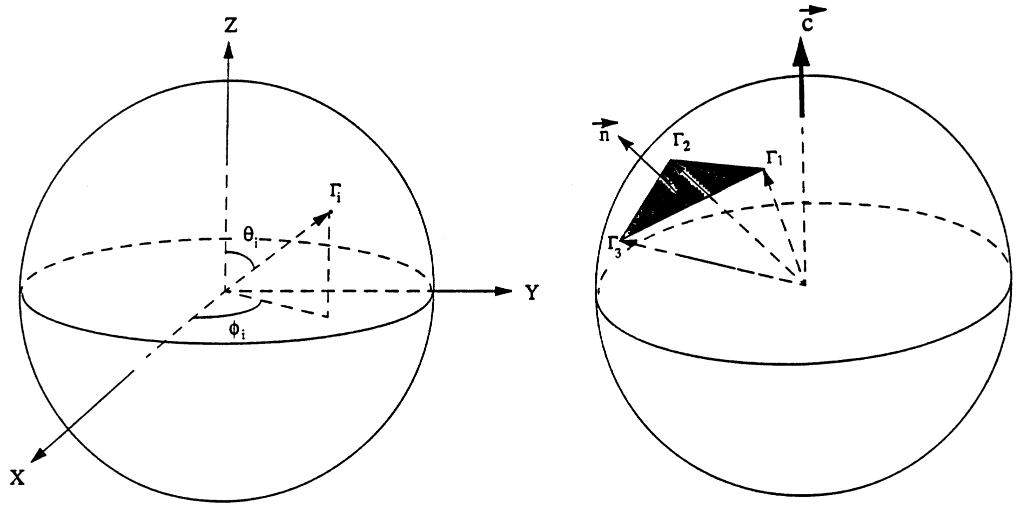

Fig. 1. – Defining features of the three vortex problem. Location of vortex Γ_i in spherical coordinates (left); vortex triangle (right).

The canonically conjugate variables are $P_i \equiv \sqrt{|\Gamma_i|}\cos(\theta_i)$ and $Q_i \equiv \sqrt{|\Gamma_i|}\phi_i$ giving rise to the standard equations

$$\dot{P}_i = \frac{\partial \mathcal{H}}{\partial Q_i},$$

$$\dot{Q}_i = -\frac{\partial \mathcal{H}}{\partial P_i}.$$

In addition to the Hamiltonian which represents the interaction energy of the vortices, one has the following quantities expressing the conservation of momentum:

$$Q = \frac{1}{R}\sum_{i=1}^{3}\Gamma_i x_i \equiv \sum_{i=1}^{3}\Gamma_i \sin(\theta_i)\cos(\phi_i),$$

$$P = \frac{1}{R}\sum_{i=1}^{3}\Gamma_i y_i \equiv \sum_{i=1}^{3}\Gamma_i \sin(\theta_i)\sin(\phi_i),$$

$$S = \frac{1}{R}\sum_{i=1}^{3}\Gamma_i z_i \equiv \sum_{i=1}^{3}\Gamma_i \cos(\theta_i).$$

The Poisson bracket algebra for the sphere was introduced in [1], where it was shown that the three-vortex problem on the sphere is integrable. It follows from the conservation of Q, P and S that the center of vorticity vector

$$\mathbf{c} = \left(\sum_{i=1}^{3}\Gamma_i \mathbf{x}_i\right)/\sigma \tag{1.2}$$

is conserved, with $\sigma = \sum_{i=1}^{3} \Gamma_i \neq 0$ being the total vorticity. Since it is always possible to orient the axes so that the vector $\mathbf{c}$ is aligned with the z-axis, we will use this convention. In addition, the vortices must collapse in finite time at the tip of the center of vorticity vector, so we have the condition

$$\|\mathbf{c}\| = R\,. \tag{1.3}$$

Our analysis is based on the equations for the chord lengths (l_{12}, l_{23}, l_{31}):

$$\frac{\mathrm{d}(l_{12}^2)}{\mathrm{d}t} = \frac{\Gamma_3 V}{\pi R}\left[\frac{1}{l_{23}^2} - \frac{1}{l_{31}^2}\right], \tag{1.4}$$

$$\frac{\mathrm{d}(l_{23}^2)}{\mathrm{d}t} = \frac{\Gamma_1 V}{\pi R}\left[\frac{1}{l_{31}^2} - \frac{1}{l_{12}^2}\right], \tag{1.5}$$

$$\frac{\mathrm{d}(l_{31}^2)}{\mathrm{d}t} = \frac{\Gamma_2 V}{\pi R}\left[\frac{1}{l_{12}^2} - \frac{1}{l_{23}^2}\right], \tag{1.6}$$

where V is the volume of the parallelepiped formed by the vectors $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$, *i.e.*

$$V = \mathbf{x}_1 \cdot (\mathbf{x}_2 \times \mathbf{x}_3)\,.$$

The vector normal to the triangle plane is given by

$$\begin{aligned}\mathbf{n} &= (\mathbf{x}_1 - \mathbf{x}_2) \times (\mathbf{x}_2 - \mathbf{x}_3)\\ &= \mathbf{x}_1 \times \mathbf{x}_2 + \mathbf{x}_2 \times \mathbf{x}_3 + \mathbf{x}_3 \times \mathbf{x}_1\,.\end{aligned}$$

Written in terms of the chord lengths, the volume formula is

$$V \equiv \pm\frac{1}{2}(16R^2A^2 - l_{12}^2 l_{23}^2 l_{31}^2)^{1/2},$$

where A is the triangle area

$$A \equiv \pm\frac{1}{4}(2l_{12}^2 l_{23}^2 + 2l_{23}^2 l_{31}^2 + 2l_{31}^2 l_{12}^2 - l_{12}^4 - l_{23}^4 - l_{31}^4)^{1/2}.$$

The $\pm$ sign depends on whether the three vortices form a right-handed $(+)$ or left-handed $(-)$ system. The volume V can also be written in terms of A, R, and a, where a is the radius of the circle in which the vortex triangle is embedded:

$$V = \pm 2AR\sqrt{1 - \frac{a^2}{R^2}}\,.$$

In the limiting case, where a/R is small, it is easy to see that the leading term is given by $V \sim 2AR$, in which case our equations agree with the planar equations studied by Aref and Synge [2, 5]. In this limit therefore, the vortex motion should correspond to the planar motion. Hence, we would expect that (asymptotically) near the collapse time, the dominant behavior should agree with the planar problem, which is known to be of power law form [2].

Two useful alternative ways of writing the volume V are

$$V = \mathbf{c} \cdot \mathbf{n} = \mathbf{x}_i \cdot \mathbf{n}$$

which gives a simple constraint:

$$(\mathbf{c} - \mathbf{x}_i) \cdot \mathbf{n} = 0 \,. \tag{1.7}$$

This condition states that the vector $(\mathbf{c} - \mathbf{x}_i)$ must lie in the plane of the vortex triangle.

Equations (1.4), (1.5), (1.6) have the two invariants:

$$C_1 = \Gamma_1 \Gamma_2 l_{12}^2 + \Gamma_2 \Gamma_3 l_{23}^2 + \Gamma_3 \Gamma_1 l_{31}^2 \equiv 0 \,, \tag{1.8}$$

$$C_2 = (l_{12}^2)^{1/\Gamma_3} \cdot (l_{23}^2)^{1/\Gamma_1} \cdot (l_{31}^2)^{1/\Gamma_2} \tag{1.9}$$

which arise from the conservation of momentum and energy. The first invariant is zero due to the fact that the chord lengths vanish at collapse. This implies that the vortex strengths cannot all have the same sign, so we use the convention $\Gamma_1 > 0$, $\Gamma_2 > 0$, $\Gamma_3 < 0$. A useful formula is given by

$$\|\mathbf{c}\|^2 = R^2 - C_1/\sigma \tag{10}$$

from which we conclude $C_1 = 0 \Leftrightarrow \|\mathbf{c}\| = R$, assuming $\sigma \neq 0$.

2. – Collapse process

We start with the ansatz that the relative distances between vortices remain constant throughout their motion:

$$l_{12}^2 = \lambda_1 l_{31}^2 , \tag{2.1}$$

$$l_{23}^2 = \lambda_2 l_{31}^2 , \tag{2.2}$$

where

$$\lambda_1 = \left(\frac{l_{12}(0)}{l_{31}(0)} \right)^2 ,$$

$$\lambda_2 = \left(\frac{l_{23}(0)}{l_{31}(0)} \right)^2 .$$

The second conserved quantity, C_2 yields

$$(l_{31}^2(t))^{\Sigma 1/\Gamma_i} = \text{const}$$

implying

$$\sum_{i=1}^{3} \frac{1}{\Gamma_i} = 0 \,.$$

We can now state the necessary and sufficient conditions for spherical collapse to occur:

1. $\|\mathbf{c}\| = R$.

2. $\sum_{i=1}^{3} \frac{1}{\Gamma_i} = 0$.

3. The vortices are not in equilibrium.

The proof is given in [1], where all equilibrium configurations have also been categorized. Their nonlinear stability properties are studied in [15]. There are two other observations one can make regarding the first two conditions. Using these conditions together, it is possible to prove that neither equilateral, nor isosceles triangles can collapse, hence $l_{12} \neq l_{23} \neq l_{31}$. Furthermore, if we make the assumption that $\Gamma_1, \Gamma_2 > 0$, $\Gamma_3 < 0$, then l_{12}, which is the chord length joining the two vortices of like sign, must have length lying in between the other two.

To understand the collapse process further, and in particular why there are two distinct collapse times (which we denote $0 < \tau^- < \tau^+ < \infty$) associated with a set of initial conditions to (1.4), (1.5), (1.6), we make use of the constraint (1.7). Figure 2 shows the relevant geometry, with the tip of the $\mathbf{c}$ vector, denoted C^*, lying at the North Pole. The vortex plane, denoted P, always intersects this point, hence the vortices lie on the circles formed by intersecting the plane with the sphere. Our first goal is to compute the angle between $\mathbf{c}$ and $\mathbf{n}$, which we denote $\alpha(t)$. Hence,

$$\mathbf{c} \cdot \mathbf{n} = \|\mathbf{c}\| \, \|\mathbf{n}\| \cos(\alpha) = V .$$

Since $\|\mathbf{n}\| = 2|A| \geqslant 0$ and $\|\mathbf{c}\| = R$, we have

$$\cos(\alpha) = V/2|A|R = \pm\sqrt{1 - a^2/R^2} . \tag{2.3}$$

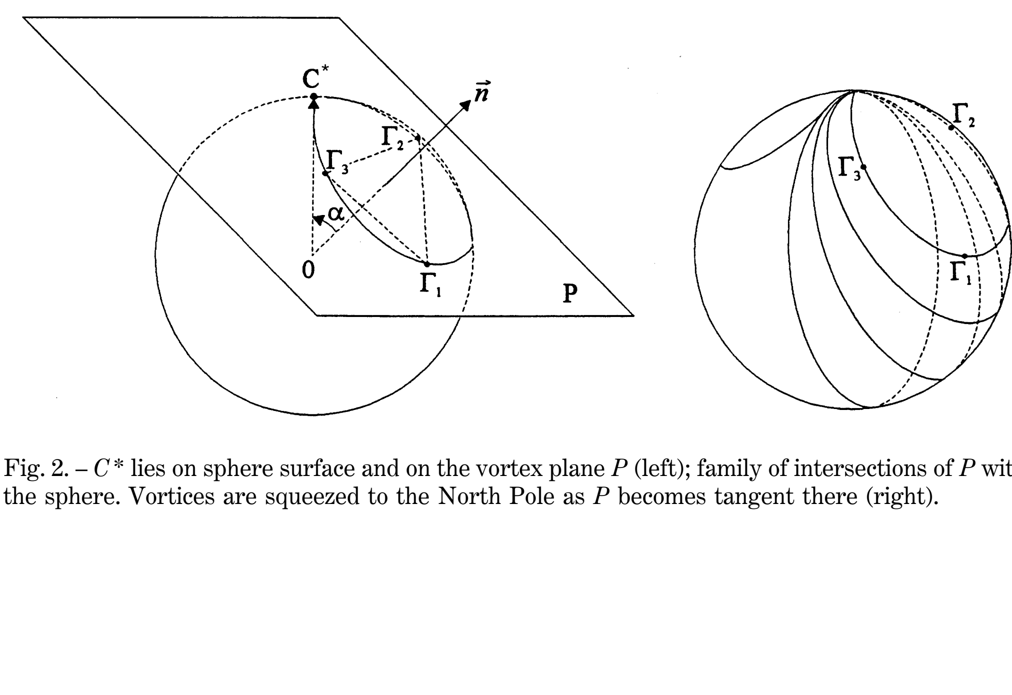

Fig. 2. – C^* lies on sphere surface and on the vortex plane P (left); family of intersections of P with the sphere. Vortices are squeezed to the North Pole as P becomes tangent there (right).

Differentiating this formula gives

$$\dot{\alpha} = \frac{1}{R^2 \sin(2\alpha)} \frac{\mathrm{d}}{\mathrm{d}t}(a^2),$$

which, after using some identities, gives

$$\dot{\alpha} = \frac{\varrho}{\sin(2\alpha)} \frac{\Gamma_2 V}{\pi R}\left(\frac{1}{l_{12}^2} - \frac{1}{l_{23}^2}\right), \tag{2.4}$$

where

$$\varrho = \frac{\lambda_1 \lambda_2}{R^2[2(\lambda_1 + \lambda_2) - (\lambda_1 - \lambda_2)^2 - 1]} > 0\,.$$

From (2.3) we can infer that

- $V \geqslant 0 \Rightarrow 0 < \alpha \leqslant (\pi/2) \Rightarrow \sin(2\alpha) \geqslant 0$,
- $V < 0 \Rightarrow (\pi/2) < \alpha < \pi \Rightarrow \sin(2\alpha) < 0$.

Thus, from (2.4) it is clear that

- $\dot{\alpha} > 0$ if $l_{12} < l_{23}$,
- $\dot{\alpha} < 0$ if $l_{12} > l_{23}$.

Suppose we have the collapsing configuration shown in fig. 3(a), which we call configuration (I). It is set up so that $\Gamma_1, \Gamma_2 > 0$, $\Gamma_3 < 0$, $l_{12} > l_{23}$ and $V(0) > 0$. Then by (2.4), we have $\dot{\alpha} < 0$ and hence $\alpha \downarrow 0$ as $t \to \tau^-$. To get the partner state associated with configuration (I), consider the same set-up, but with the signs of the Γ's reversed. Because it is the partner state associated with (I), we label this configuration ($\mathrm{I_p}$). We have $\dot{\alpha} > 0$, hence $\alpha \uparrow \pi$ as $t \to \tau^+$. The partner states are related to each other by the opposite directions in which the plane P swings in order to become tangent to the sphere at the North Pole, thereby squeezing the vortices to their ultimate collapse. Another way of achieving the partner state related to (I) is by using the discrete symmetries inherent in the problem [1]. Consider a configuration (II) obtained by reversing the signs of the x and/or y coordinates of configuration (I). All the chord lengths l_{ij} remain as in (I), as do the vortex strengths $\Gamma_1, \Gamma_2 > 0$, $\Gamma_3 < 0$. Once again, $\dot{\alpha} < 0$, hence $\alpha \downarrow 0$ as $t \to \tau^+$. Then, configuration ($\mathrm{II_p}$) is obtained by reversing the signs of the Γ's, giving $\dot{\alpha} > 0$, $\alpha \uparrow \pi$ as $t \to \tau^-$.

The collapse times are obtained analytically by using (1.4), (1.5), (1.6) along with (2.1),(2.2) to get a scalar equation for l_{31}^2:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(l_{31}^2) &= \frac{\Gamma_1}{\pi R}\left(\frac{\lambda_1 - 1}{\lambda_1 \lambda_2}\right) V/l_{31}^2 \\ &= \frac{\Gamma_2}{\pi R}\left(\frac{\lambda_2 - \lambda_1}{\lambda_1 \lambda_2}\right) V/l_{31}^2 \\ &= \frac{\Gamma_3}{\pi R}\left(\frac{1 - \lambda_2}{\lambda_1 \lambda_2}\right) V/l_{31}^2,\end{aligned}$$

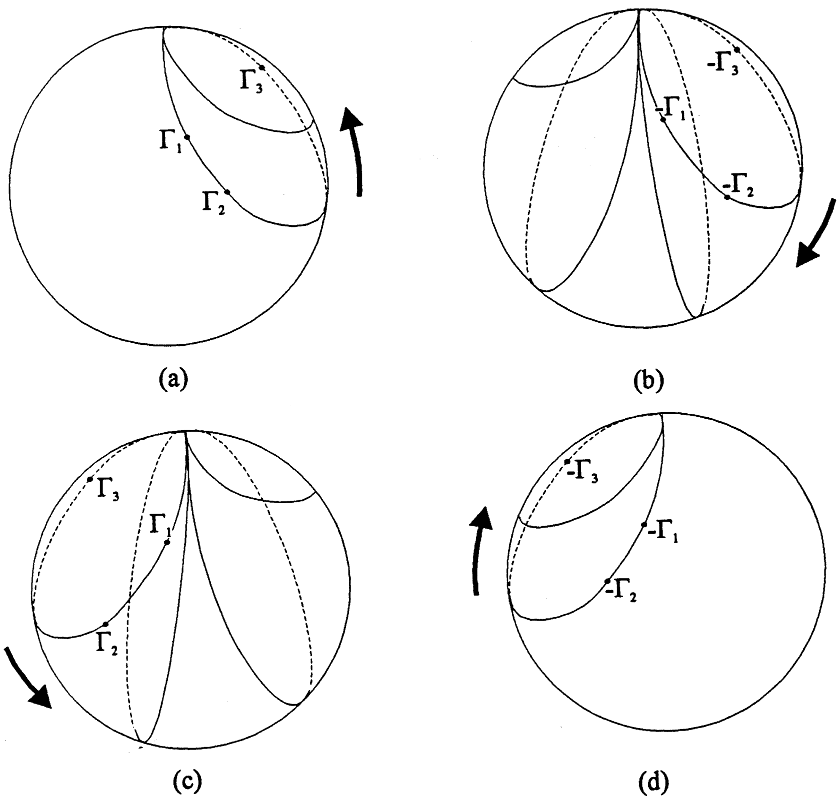

Fig. 3. – Partner states (a) I, (b) I_p , (c) II, (d) II_p. Configurations I and II_p collapse at time τ^-, while I_p and II collapse at time τ^+. In all cases the initial lengths are the same.

which gives a relationship to be satisfied among the vortex strengths and the initial conditions:

$$\Gamma_1(\lambda_1 - 1) = \Gamma_2(\lambda_2 - \lambda_1) = \Gamma_3(1 - \lambda_2).$$

These conditions can be derived from the previous conditions (1.8) and (1.9).

After some algebra (see [1] for more details), one can derive the solution

$$l_{ij}(t) = l_{ij}(0)\left(1 + \frac{t}{\tau^{\mp}}\right)^{1/2}\left(1 - \frac{t}{\tau^{\pm}}\right)^{1/2}, \tag{2.5}$$

where

$$\tau^{\pm} = \frac{4\pi R^2\sqrt{\gamma}}{\Gamma_2\,|(\lambda_1 - \lambda_2)|}(1 \pm \beta) > 0\,,$$

$$\tau^+ > \tau^-$$

with

$$\gamma = 2(\lambda_1 + \lambda_2) - (\lambda_1 - \lambda_2)^2 - 1,$$

$$\beta = \sqrt{1 - \varrho l_{13}^2(0)}\,,$$

$$\varrho = \lambda_1 \lambda_2 / R^2 \gamma\,.$$

Near collision, (2.5) has the asymptotic expansion

$$l_{ij}(t) \sim A \cdot \left(1 - \frac{t}{\tau^{\pm}}\right)^{1/2} + B \cdot \left(1 - \frac{t}{\tau^{\pm}}\right)^{3/2} + \mathrm{O}\left(\left(1 - \frac{t}{\tau^{\pm}}\right)^{5/2}\right)$$

with constants A and B given in [1]. The planar result of Aref [2] gives the exact collapse formula

$$l_{ij}(t) = l_{ij}(0) \cdot \left(1 - \frac{t}{\tau}\right)^{1/2}.$$

Another important quantity is the vortex azimuthal velocity, $\dot{\phi}_i \equiv \omega_i$, whose formula is given by

$$\omega_i = \frac{\omega_{\mathrm{p}} - (\Gamma_j + \Gamma_k)/8\pi R^2}{1 + (\Gamma_j \Gamma_k l_{jk}^2 / 4 \Gamma_i R^2 \sigma)}\,,$$

where ω_{p} is the planar angular velocity given by

$$\omega_{\mathrm{p}} = \frac{1}{4\pi} \sum_{i,j=1}^{3} \frac{\Gamma_i + \Gamma_j}{l_{ij}^2}\,.$$

See [1] for more details.

Hence, there are three differences between the spherical collapse process and the planar collapse process:

– The spherical collapse has two distinct collapse times, whereas the planar collapse has one. In the plane, the analogue of the partner state is a self-similar expanding state [2] which cannot occur on the sphere because of the extra length scale R which puts an upper bound on the maximum chord length.

– The exact formulas for chord lengths are different for the sphere and the plane, however the leading term near collapse agrees with the planar result, with higher-order corrections due to the spherical geometry.

– In the plane, the vortices all rotate with the same angular velocity as they collapse, whereas on the sphere, each has a distinct angular velocity.

3. – Stereographic projection

The change of variable

$$r_i = \tan(\theta_i / 2)$$

results in a stereographic projection of the vortex Γ_i onto the extended complex plane

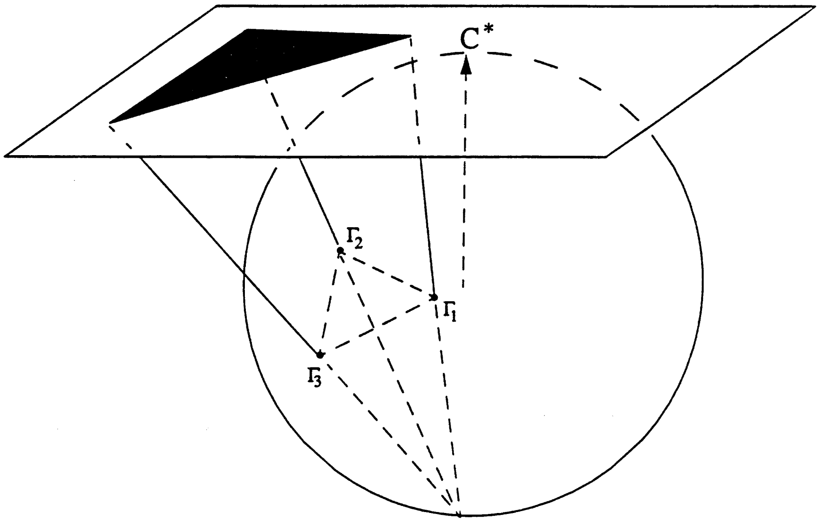

Fig. 4. – Stereographic projection of three vortices.

$\mathscr{C}$ which is tangent to the sphere at the North Pole, as shown in fig. 4. This point of tangency is at the origin of $\mathscr{C}$, while the South Pole maps to the point at infinity. An important aspect of the stereographic projection is that it is conformal [16] and vector fields on the sphere are mapped in a one-to-one fashion to vector fields on $\mathscr{C}$.

By a straightforward computation, one gets the new Hamiltonian in $\mathscr{C}$ as

$$\mathscr{H} = \frac{1}{8\pi R^2} \sum_{i<j} \Gamma_i \Gamma_j \log \left(\frac{r_i^2 + r_j^2 - 2 r_i r_j \cos(\phi_i - \phi_j)}{(1 + r_i^2)(1 + r_j^2)} \right) \tag{3.1}$$

with the new equations of motion

$$\Gamma_i \frac{\mathrm{d}}{\mathrm{d}t}(r_i^2) = -(1 + r_i^2)^2 \frac{\partial \mathscr{H}}{\partial \phi_i},$$

$$\Gamma_i \frac{\mathrm{d}\phi_i}{\mathrm{d}t} = (1 + r_i^2)^2 \frac{\partial \mathscr{H}}{\partial r_i^2},$$

where (r_i, ϕ_i) are the polar coordinates of the vortex Γ_i in the complex plane $\mathscr{C}$.

Our main goal in this section is to show that the collision process in the stereographic plane is not self-similar. To prove this, we will show that the ratios r_{12}/r_{23} and r_{12}/r_{31} are functions of time, where r_{ij} is the distance between vortices Γ_i and Γ_j in $\mathscr{C}$. It is straightforward to show that

$$r_{ij}^2 = \frac{l_{ij}^2}{4R^2[1 + (\Gamma_j \Gamma_k / 4R^2 \sigma \Gamma_i) l_{jk}^2][1 + (\Gamma_k \Gamma_i / 4R^2 \sigma \Gamma_j) l_{ki}^2]},$$

where $i \neq j \neq k$, $i, j, k = 1, \ldots, 3$. Then we can write l_{12} and l_{23} in terms of l_{31} to get

$$r_{12}^2 = \frac{\lambda_1}{4R^2} \frac{l_{31}^2}{(1+\alpha_1 l_{31}^2)(1+\alpha_2 l_{31}^2)},$$

$$r_{23}^2 = \frac{\lambda_2}{4R^2} \frac{l_{31}^2}{(1+\alpha_2 l_{31}^2)(1+\alpha_3 l_{31}^2)},$$

$$r_{31}^2 = \frac{1}{4R^2} \frac{l_{31}^2}{(1+\alpha_1 l_{31}^2)(1+\alpha_3 l_{31}^2)},$$

where

$$\alpha_1 = \frac{\Gamma_2 \Gamma_3 \lambda_2}{4R^2 \sigma \Gamma_1},$$

$$\alpha_2 = \frac{\Gamma_3 \Gamma_1}{4R^2 \sigma \Gamma_2},$$

$$\alpha_3 = \frac{\Gamma_1 \Gamma_2 \lambda_1}{4R^2 \sigma \Gamma_3}.$$

From these formulas, it is clear that

– $r_{ij} \to 0$ as $t \to \tau^{\pm}$,

– r_{12}/r_{23} and r_{12}/r_{31} are functions of time unless $\alpha_1 = \alpha_2 = \alpha_3$, which is not possible. This shows that the collapse is not self-similar.

We end this section with several remarks:

1. As shown in the previous section, the angle α between **c** and **n** is not constant, which is the reason the collapse formulas on the projected $\mathcal{C}$ plane are not self-similar.

2. The Hamiltonian system (3.1) is useful for several other purposes as well. In particular, in studying the streamline topology for the N-vortex problem on the sphere, it is advantageous to study the projected streamlines on the $\mathcal{C}$ plane. This is currently being pursued [17].

4. – Tri-linear phase plane

In the papers of Aref [2] and Synge [5], special tri-linear coordinates were introduced in order to reduce the system (1.4), (1.5), (1.6) to a planar one. For the spherical problem, these same coordinates were used in [1] and we refer the reader to these papers for more detailed discussions. For our purposes, we are interested only in using the coordinates to shed light on the collapse process, hence we introduce the following coordinates based on the chord lengths:

$$b_1 = l_{23}^2/4\Gamma_1 R^2, \qquad b_2 = l_{31}^2/4\Gamma_2 R^2, \qquad b_3 = l_{12}^2/4\Gamma_3 R^2,$$

so that we have the following identity:

$$b_1 + b_2 + b_3 = C_1/4R^2(\Gamma_1\Gamma_2\Gamma_3) = 0\,.$$

Then, one can eliminate the coordinate b_3 in favor of b_1 and b_2, and based on the conserved quantity C_1, one gets

$$\left(\frac{b_1}{b_1+b_2}\right)^{1/\Gamma_1}\cdot\left(\frac{b_2}{b_1+b_2}\right)^{1/\Gamma_2} = \text{const}\,,$$

which implies that $b_1/b_2 = \text{const}$. Because of the finite-size sphere radius, only a bounded region of the (b_1, b_2)-plane is accessible, the region defined by the condition $V^2 \geqslant 0$, which one can write as

$$2(\Gamma_1\Gamma_2 b_1 b_2 + \Gamma_2\Gamma_3 b_2 b_3 + \Gamma_3\Gamma_1 b_3 b_1) - \Gamma_1^2 b_1^2 - \Gamma_2^2 b_2^2 - \Gamma_3^2 b_3^2 \geqslant 4\Gamma_1\Gamma_2\Gamma_3 b_1 b_2 b_3\,.$$

It is useful to think of the phase plane as having two faces, the front corresponding to $V > 0$ and the back corresponding to $V < 0$, with the boundary joining the two being $V = 0$ which corresponds to great circle states. Figure 5 shows the (b_1, b_2) phase plane, with each collapsing trajectory lying on a ray going through the origin. For definiteness, we show the case $\Gamma_1 = \Gamma_2 = 1$, $\Gamma_3 = -1/2$. The partner states associated with a given collapsing configuration are shown on the diagram. The partner states shown in fig. 3(a) and (c) have identical l_{ij}'s and Γ's, therefore are located by the same point P. However, for case (a) we have $V > 0$, while for case (c) we have $V < 0$. In both

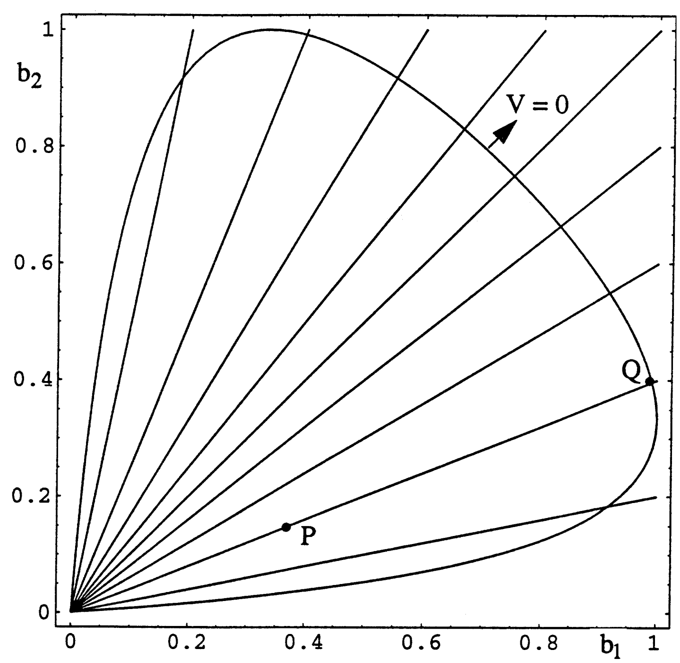

Fig. 5. – Phase plane showing the partner collapsing states. Both states start off at P, but on opposite faces. The one on the front face collapses directly towards the origin, the other evolves first to Q on the $V = 0$ curve, before collapsing to the origin on the front face.

cases, the sign of $\dot{V}$ is the same. For case (a), the trajectory evolves straight to the origin along the ray (on the front face $V > 0$), collapsing at time τ^-. State (c) evolves away from the origin (on the back face $V < 0$) until it hits the $V = 0$ curve, corresponding to a great circle configuration, then evolves to the origin on the front face $V > 0$, collapsing at the later time $\tau^+ > \tau^-$. The difference between this process and the corresponding one in the plane, described in [2], is that for the planar problem, there is nothing to bound the coordinates from above, hence the accessible region is unbounded. As a result, the trajectory analogous to state (c) continues to travel away from the origin on the same ray, representing a self-similarly expanding state. An analogous explanation can be given for the partner states shown in fig. 3(b) and (d).

REFERENCES

[1] KIDAMBI R. and NEWTON P. K., *Motion of three point vortices on a sphere, Physica D*, **116** (1998) 143-175.
[2] AREF H., *Motion of three vortices*, *Phys. Fluids*, **22** (1979) 393-400.
[3] NOVIKOV E. A., *Dynamics and statistics of a system of vortices, Sov. Phys. JETP*, **41** (1975) 937-943.
[4] NOVIKOV E. A. and SEDOV Y., *Vortex collapse, Sov. Phys. JETP*, **50** (1979) 297-301.
[5] SYNGE J. L., *On the motion of three vortices, Can. J. Math.*, **1** (1949) 257-270.
[6] DIBATTISTA M. and POLVANI L., *Barotropic vortex pairs on a rotating sphere, J. Fluid Mech.*, **358** (1998) 107-133.
[7] POLVANI L. and DRITSCHEL D. G., *Wave and vortex dynamics on the surface of a sphere, J. Fluid Mech.*, **255** (1993) 35-64.
[8] DRITSCHEL D. G., *Vortex properties of 2D turbulence, Phys. Fluids A*, **5** (1993) 984-997.
[9] DRITSCHEL D. G. and ZABUSKY N. J., *On the nature of vortex interactions and models in unforced nearly inviscid two-dimensional turbulence, Phys. Fluids A*, 8 (1996) 1252-1256.
[10] AREF H, *Integrable, chaotic and turbulent vortex motion in two-dimensional flows, Ann. Rev. Fluid Mech.*, **15** (1983) 345-389.
[11] SAFFMAN P. G., *Vortex Dynamics* (Cambridge University Press) 1992.
[12] BOGOMOLOV V. A., *Dynamics of vorticity at a sphere, Fluid Dynamics*, **6** (1977) 863-870.
[13] BOGOMOLOV V. A., *Two dimensional fluid dynamics on a sphere, Izv. Atmos. Oc. Phys.*, Vol. **15** (1979) 18-22.
[14] KIMURA Y. and OKAMOTO Y., *Vortex motion on a sphere, J. Phys. Soc. Jpn.*, **56** (1987) 4203-4206.
[15] PEKARSKY S. and MARSDEN J., *Point vortices on a sphere: stability of relative equilibria, J. Math. Phys.*, **39** (1998) 5894-5907.
[16] NEEDHAM T., *Visual Complex Analysis* (Clarendon Press) 1997.
[17] KIDAMBI R. and NEWTON P. K., *Streamline topology for integrable vortex motion on a sphere*, in preparation.