Alpha models for 3D Eulerian mean fluid circulation (*)(**)

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Summary. — We provide a summary of the known analytical properties of the alpha models, including an outline of their derivation and the associated assumptions, their simplification for the case of constant dispersion length (alpha) and their conservation properties. We also offer interpretations of nonlinear dynamics of the viscous alpha models and indicate the differences one might expect from the dynamics of the Navier-Stokes equations.

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1. – Introduction

Holm *et al.* [1, 2] introduced the "alpha models" for the mean motion of ideal incompressible fluids as the n-dimensional generalization of the one-dimensional Camassa-Holm (CH) equation. The one-dimensional CH equation describes shallow water waves with nonlinear dispersion and admits soliton solutions called "peakons" [3]. Its n-dimensional generalization describes the slow time dynamics of fluids in which nonlinear dispersion accounts for the effects of the small-scale rapid variability upon the mean motion. The fluid transport velocity is found by inversion of a Helmholtz operator acting on the fluid circulation (or momentum) velocity. This operator contains the length scale that corresponds to the magnitude of the fluctuation covariance; the application of this operator smoothes the transport velocity relative to the circulation velocity. This length scales is denoted by α in refs. [1-3], hence the name *alpha models* for these mean fluid motion theories.

The alpha models for self-consistent mean fluid dynamics are derived by applying temporal averaging procedures to Hamilton's principle for an ideal incompressible

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fluid flow. The resulting mean fluid motion equations are obtained by using the Euler-Poincaré variational framework [1, 2]. (As explained in [1], Euler-Poincaré equations are the Lagrangian version of Lie-Poisson Hamiltonian systems.) Therefore, these equations possess conservation laws for energy and momentum, as well as a Kelvin-Noether circulation theorem that establishes how the time average (or perhaps statistical) properties of the fluctuations affect the circulation of the mean flow. These ideal fluid equations also describe geodesic motion on the volume-preserving diffeomorphism group for a metric containing the H^1 norm of the mean fluid velocity. Their geometrical properties are discussed in [4]. Their relation to Eulerian and Lagrangian mean fluid theories are discussed in [5]. In recognition of their origins, these mean fluid motion equations may be known equally well by either the name alpha models, or CH equations.

Chen *et al.* [6-8] introduced phenomenological viscosity into the CH equation and proposed the resulting viscous Camassa-Holm equation (VCHE), or Navier-Stokes alpha (NS-α) model, as a closure approximation for the Reynolds averaged equations of the incompressible Navier-Stokes fluid. They tested this approximation on turbulent channel and pipe flows with steady mean, by finding analytical solutions of the VCHE for the mean velocity and the Reynolds shear stress and comparing them with experiments [9]. They found that the steady VCHE profiles are consistent with data obtained from mean flow turbulence measurements in most of the flow region for channels and pipes at moderate-to-high Reynolds numbers. Thus, Chen *et al.* demonstrated a connection between turbulence and the VCHE for steady, or mean solutions. In fact, the *time-dependent* VCHE in a periodic box has unique classical solutions and a global attractor whose fractal dimension is finite and scales according to Kolmogorov's estimate, $N \sim (L/l_{\mathrm{d}})^3$, where $l_{\mathrm{d}} = (\nu^3/\varepsilon)^{1/4}$ is the Kolmogorov dissipation length [10]. We note that the time-dependent VCHE, or NS-α model is *not* equivalent to the Navier-Stokes equations with hyperviscosity, see [6-8, 10]. Chen *et al.* [11] used direct numerical simulations to compare the statistics and structures of the velocity and vorticity fields at moderate Reynolds numbers of the viscous alpha model with the corresponding results for the Navier-Stokes equations. The principal conclusion of this comparison is that the viscous alpha model simulations can reproduce most of the large scale features of Navier-Stokes turbulence *even when these simulations do not resolve the fine scale dynamics*, at least in the case of forced turbulence in a periodic box.

The present paper summarizes the known analytical properties of the alpha models in three-dimensional Euclidean space, including an outline of their derivation and the associated assumptions, their simplification for the case of constant alpha, and their conservation properties. We also offer interpretations of nonlinear dynamics of the alpha models and indicate the differences one might expect from the dynamics of the Navier-Stokes equations.

2. – Eulerian mean theory and alpha model equations

Holm, Marsden, and Ratiu [1, 2] used variational asymptotics to obtain evolution equations for the Eulerian mean hydrodynamic motion of ideal incompressible fluids, employing an approximation of Hamilton's principle for Euler's equation in a Euclidean space setting. The method assumes that the Euler flow may be decomposed into its mean and fluctuating components at a fixed position in space. In their approach, a first-order Taylor expansion in the fluctuation amplitude is used to approximate the velocity field with the result that the L^2 metric in Hamilton's principle giving rise to the Euler equations is replaced by an H^1 metric that produces the evolution equation of the Eulerian mean flow. We shall call this evolution equation the Euler alpha model, or the n-dimensional CH equation.

Holm *et al.* [4] give a geometrically intrinsic (*i.e.*, coordinate free) derivation of these averaged equations by the procedure of *variational asymptotics*, namely, by deriving an averaged Lagrangian and using this Lagrangian to generate the equations via Hamilton's principle. This intrinsic setting is useful because many interesting flows, *e.g.*, flows on spheres such as those in geophysics, do occur on manifolds. We follow Holm [5] in presenting the derivation in Euclidean space.

We develop the Euler-Poincaré theory of advected fluctuations from the viewpoint of Eulerian averaging. Our point of departure is a Lagrangian comprised of the fluid kinetic energy in the Eulerian description, in which volume preservation is imposed by a Lagrange multiplier P (the pressure),

$$L(\omega) = \int \mathrm{d}^3 x \left\{ \frac{D}{2} \,|\mathbf{U}(\mathbf{x}, t; \omega)|^2 + P(\mathbf{x}, t; \omega)(1 - D(\mathbf{x}, t; \omega)) \right\}, \tag{1}$$

where D is the Eulerian volume element. We assume that there are two time scales for the motion: the fast time ω and the slow time t (although the parameter ω could also denote an ensemble index). The traditional Reynolds decomposition of fluid velocity into its fast and slow components is expressed at a given position $\mathbf{x}$ in terms of the Eulerian mean fluid velocity $\mathbf{u}$ as

$$\mathbf{U}(\mathbf{x}, t; \omega) \equiv \mathbf{u}(\mathbf{x}, t) + \mathbf{u}'(\mathbf{x}, t; \omega). \tag{2}$$

Following Holm [5], we assume the Eulerian velocity fluctuation $\mathbf{u}'(\mathbf{x}, t; \omega)$ is related to an Eulerian fluid parcel displacement fluctuation—denoted as $\boldsymbol{\zeta}(\mathbf{x}, t; \omega)$—by

$$\frac{\partial \boldsymbol{\zeta}}{\partial t} + \mathbf{u} \cdot \nabla \boldsymbol{\zeta} = \boldsymbol{\zeta} \cdot \nabla \mathbf{u} + \mathbf{u}'(\mathbf{x}, t; \omega). \tag{3}$$

For purely Eulerian velocity fluctuations as in eq. (2), this relation separates into two relations: the "Taylor-like" hypothesis of [4],

$$\frac{\partial \boldsymbol{\zeta}}{\partial t} + \mathbf{u} \cdot \nabla \boldsymbol{\zeta} = 0, \tag{4}$$

and the relation [4, 5]

$$0 = \boldsymbol{\zeta} \cdot \nabla \mathbf{u} + \mathbf{u}'(\mathbf{x}, t; \omega).$$

Hence, the Reynolds velocity decomposition (2) separates the Lagrangian (1) into its mean and fluctuating pieces as

$$L(\omega) = \int \mathrm{d}^3 x \left\{ \frac{D}{2} \,|\mathbf{u}(\mathbf{x}, t) + \mathbf{u}'(\mathbf{x}, t; \omega)|^2 + P(\mathbf{x}, t)(1 - D(\mathbf{x}, t)) \right\}. \tag{6}$$

No modification is needed in the pressure constraint in this Lagrangian, because the Eulerian mean *preserves* the condition that the velocity be divergence free; hence,

$\nabla \cdot \mathbf{u} = 0$. It remains only to take the Eulerian mean of this Lagrangian, in which we assume $\langle \boldsymbol{\zeta} \rangle = 0$. The Eulerian mean averaging process at fixed position $\mathbf{x}$ is denoted $\langle \cdot \rangle$ with, *e.g.*,

$$\mathbf{u}(\mathbf{x}, t) = \langle \mathbf{U}(\mathbf{x}, t; \omega) \rangle \equiv \lim_{T \to \infty} \frac{1}{T} \int_0^T \mathbf{U}(\mathbf{x}, t; \omega)\, \mathrm{d}\omega \,. \tag{7}$$

By eq. (5), the Eulerian mean kinetic energy due to the velocity fluctuation satisfies

$$\langle |\mathbf{u}'|^2 \rangle = \langle \zeta^k \zeta^l \rangle\, \mathbf{u}_{,k} \cdot \mathbf{u}_{,l} . \tag{8}$$

Thus, we find the following Eulerian mean Lagrangian:

$$\langle L \rangle = \int \mathrm{d}^3 x \left\{ \frac{D}{2} [\, |\mathbf{u}(\mathbf{x}, t)|^2 + \langle \zeta^k \zeta^l \rangle\, \mathbf{u}_{,k} \cdot \mathbf{u}_{,l}] + P(\mathbf{x}, t)(1 - D(\mathbf{x}, t)) \right\}. \tag{9}$$

The advection relation (4) implies the same advective velocity for each component of the symmetric Eulerian mean covariance tensor $\langle \zeta^k \zeta^l \rangle$. Thus, we have

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \langle \zeta^k \zeta^l \rangle = 0 \,. \tag{10}$$

Together, this relation and the continuity equation for the volume element D,

$$\frac{\partial D}{\partial t} + \nabla \cdot D\mathbf{u} = 0 \,, \tag{11}$$

complete the auxiliary equations needed for deriving the equation of motion for the Eulerian mean velocity $\mathbf{u}$ from the averaged Lagrangian $\langle L \rangle$ in (6) by using the Euler-Poincaré theory.

The results of [1] allow one to compute the Euler-Poincaré equation for the Lagrangian $\langle L \rangle$ in (6) depending on the Eulerian mean velocity $\mathbf{u}$, and the advected quantities D and $\langle \zeta^k \zeta^l \rangle$ as

$$0 = \left(\frac{\partial}{\partial t} + u^j \frac{\partial}{\partial x^j} \right) \frac{1}{D} \frac{\delta \langle L \rangle}{\delta u^i} + \frac{1}{D} \frac{\delta \langle L \rangle}{\delta u^j} u^j_{,i} - \frac{\partial}{\partial x^i} \frac{\delta \langle L \rangle}{\delta D} + \frac{1}{D} \frac{\delta \langle L \rangle}{\delta \langle \zeta^k \zeta^l \rangle} \frac{\partial}{\partial x^i} \langle \zeta^k \zeta^l \rangle . \tag{12}$$

We compute the following variational derivatives of the averaged approximate Lagrangian $\langle L \rangle$ in eq. (6):

$$\begin{cases} \dfrac{1}{D} \dfrac{\delta \langle L \rangle}{\delta \mathbf{u}} = \mathbf{u} - \dfrac{1}{D} (\partial_k D \langle \zeta^k \zeta^l \rangle\, \partial_l)\, \mathbf{u} \equiv \mathbf{v} \,, \\[2ex] \dfrac{\delta \langle L \rangle}{\delta D} = -P + \dfrac{1}{2} |\mathbf{u}|^2 + \dfrac{1}{2} \langle \zeta^k \zeta^l \rangle (\mathbf{u}_{,k} \cdot \mathbf{u}_{,l}) \equiv -P_{\mathrm{tot}}, \\[2ex] \dfrac{\delta \langle L \rangle}{\delta P} = 1 - D \,, \qquad \dfrac{\delta \langle L \rangle}{\delta \langle \zeta^k \zeta^l \rangle} = \dfrac{D}{2} (\mathbf{u}_{,k} \cdot \mathbf{u}_{,l}) . \end{cases} \tag{13}$$

The Euler-Poincaré equation (12) for this averaged Lagrangian takes the form

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{v} + v_j \nabla u^j + \nabla P_{\mathrm{tot}} = -\frac{1}{2}(\mathbf{u}_{,k}\cdot\mathbf{u}_{,l})\,\nabla\langle\zeta^k\zeta^l\rangle, \tag{14}$$

where

$$\mathbf{v} = \mathbf{u} - \widetilde{\Delta}_D\mathbf{u} \quad \text{with } \widetilde{\Delta}_D \equiv \frac{1}{D}(\partial_k D\langle\zeta^k\zeta^l\rangle\,\partial_l) \text{ and } \nabla\cdot\mathbf{u} = 0\,. \tag{15}$$

Its definition as a variational derivative indicates that $\mathbf{v}$ is a specific momentum in a certain sense dual to the velocity $\mathbf{u}$. For more discussion of physical interpretations of $\mathbf{u}$ and $\mathbf{v}$, see [5]. The Euler-Poincaré equations (14)-(15) define the Eulerian mean motion (EMM) model. Incompressibility of the Eulerian mean velocity $\mathbf{u}$ follows from the continuity equation (11) and the constraint $\delta\langle L\rangle/\delta P = 0$. A natural set of boundary conditions is

$$\mathbf{v}\cdot\widehat{\mathbf{n}} = 0\,, \qquad \mathbf{u} = 0\,, \qquad \text{and} \qquad \widehat{\mathbf{n}}\cdot\langle\boldsymbol{\xi}\boldsymbol{\xi}\rangle = 0\,, \quad \text{on a fixed boundary}\,. \tag{16}$$

Then, provided the Helmholtz operator $1-\widetilde{\Delta}_D$ for $D=1$ may be inverted, the Eulerian mean pressure P may be obtained by solving an elliptic equation.

2·1. *Reducing the* EMM *equation to the n-dimensional* CH *equation.* – When the Eulerian mean covariance is isotropic and homogeneous, so that $\langle\zeta^k\zeta^l\rangle = \alpha^2\delta^{kl}$ (for a constant length scale α, whose magnitude is set by the initial conditions for the Eulerian mean covariance) then the EMM equation (14) *reduces* to the n-dimensional CH equation, or Euler alpha model, introduced in [1, 2], namely,

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{v} + v_j\nabla u^j + \nabla P_{\mathrm{tot}} = 0\,, \qquad \nabla\cdot\mathbf{u} = 0 = \nabla\cdot\mathbf{v}\,, \tag{17}$$

where

$$\mathbf{v} = \mathbf{u} - \alpha^2\Delta\mathbf{u}\,, \qquad P_{\mathrm{tot}} = P - \frac{1}{2}|\mathbf{u}|^2 - \frac{\alpha^2}{2}|\nabla\mathbf{u}|^2. \tag{18}$$

This n-dimensional CH equation set, or Euler alpha model, is an invariant subsystem of the EMM system (14), with definition (15) and advection law (10), because the homogeneous isotropic initial condition $\langle\zeta^k\zeta^l\rangle = \alpha^2\delta^{kl}$ is invariant under the dynamics of eq. (10). Hence, any of the formulae above remain valid if we set $\langle\zeta^k\zeta^l\rangle = \alpha^2\delta^{kl}$, with constant α.

Equations of the type (17) but with additional dissipative terms were considered previously in the theory of second-grade fluids [12] and were treated recently in the mathematical literature [13, 14]. Second-grade fluid models are derived from continuum mechanical principles of objectivity and material frame indifference, after which thermodynamic principles such as the Clausius-Duhem relation and stability of stationary equilibrium states are imposed to restrict the allowed values of the parameters in these models. In contrast, the CH equation (17) is derived here by applying asymptotic expansions, Eulerian means, and an assumption of isotropy of fluctuations in Hamilton's principle for an ideal incompressible fluid. This derivation provides the interpretation of the length scale α as the typical amplitude of the rapid fluctuations whose Eulerian mean is taken in Hamilton's principle.

The n-dimensional CH equation (17) with definitions (18) implies the conservation of energy $\frac{1}{2}\int d^3x\mathbf{u}\cdot\mathbf{v}$ and helicity $\frac{1}{2}\int d^3x\mathbf{v}\cdot\mathrm{curl}\,\mathbf{v}$. Its steady vortical flows include the analogs of the Beltrami flows $\mathrm{curl}\,\mathbf{v}=\lambda\mathbf{u}$. In the periodic case, we define $\mathbf{v_k}$ as the $\mathbf{k}$-th Fourier mode of the specific momentum $\mathbf{v}\equiv(1-\alpha^2\Delta)\,\mathbf{u}$; so that $\mathbf{v_k}\equiv(1+\alpha^2|\mathbf{k}|^2)\,\mathbf{u}_k$. Then eq. (17) becomes [1,2]

$$\frac{d}{dt}\mathbf{v_k}-i\Pi_\perp\left(\sum_{\mathbf{p}+\mathbf{n}=\mathbf{k}}\frac{\mathbf{v_p}}{1+\alpha^2|\mathbf{p}|^2}\times(\mathbf{n}\times\mathbf{v_n})\right)=0\,, \tag{19}$$

where $\Pi_\perp=\delta_{ij}-k_ik_j/|\mathbf{k}|^2$ is the Leray projection onto Fourier modes transverse to $\mathbf{k}$ (this ensures incompressiblity). Hence, the nonlinear coupling among the modes is suppressed by the denominator when $1+\alpha^2|\mathbf{p}|^2\gg|\mathbf{n}|$.

An essential feature of the n-dimensional CH equations (17)-(18) is that its specific momentum $\mathbf{v}$ is transported by a velocity $\mathbf{u}$ that is smoothed, or filtered, by application of the inverse elliptic Helmoholtz operator $(1-\alpha^2\Delta)$. The effect on length scales smaller than α is that steep gradients of the specific momentum $\mathbf{v}$ tend not to steepen much further, and that thin vortex tubes tend not to get much thinner as they are transported. Furthermore, as numerical simulations indicate [11], the effect on length scales larger than α is negligible. Hence, the n-dimensional CH equation, or Euler alpha model, preserves the assumptions under which it is derived.

2·2. *Physical interpretation of* $\mathbf{v}$ *as the Lagrangian mean velocity.* – The Stokes mean drift velocity is defined by [15]

$$\langle\mathbf{U}\rangle^{\mathrm{S}}\equiv\langle\boldsymbol{\zeta}\cdot\nabla\mathbf{u}'\rangle. \tag{20}$$

Hence, eq. (5) implies

$$\langle\mathbf{U}\rangle^{\mathrm{S}}=-\langle\boldsymbol{\zeta}\cdot\nabla\boldsymbol{\zeta}\cdot\nabla\rangle\,\mathbf{u}=-\widetilde{\Delta}\mathbf{u}+o(|\boldsymbol{\zeta}|^2)\,, \tag{21}$$

where

$$\widetilde{\Delta}\equiv(\partial_k\langle\zeta^k\zeta^l\rangle\,\partial_l)=\widetilde{\Delta}_D\,|_{D=1}\,, \tag{22}$$

and we argue that $\nabla\cdot\boldsymbol{\zeta}=o(|\boldsymbol{\zeta}|^2)$. Thus, we find that $\mathbf{v}$ satisifies, to order $o(|\boldsymbol{\zeta}|^2)$,

$$\mathbf{v}\equiv\mathbf{u}-\widetilde{\Delta}\mathbf{u}=\mathbf{u}+\langle\mathbf{U}\rangle^{\mathrm{S}}=\langle\mathbf{U}\rangle^{L}. \tag{23}$$

Therefore to this order, $\mathbf{v}$ in the EMM theory is the Lagrangian mean velocity.

2·3. *Kelvin circulation theorem for* EMM *and* CH *equations.* – Since they are Euler-Poincaré, both the Eulerian mean motion (EMM) equation (14) and its invariant reduced form the CH equations (17)-(18) for $\langle\zeta^k\zeta^l\rangle=\alpha^2\delta^{kl}$ possess the corresponding Kelvin-Noether circulation theorem,

$$\frac{d}{dt}\oint_{\gamma(\mathbf{u})}\mathbf{v}\cdot d\mathbf{x}=-\frac{1}{2}\int\!\!\int_{S(\mathbf{u})}\nabla(\mathbf{u}_{,k}\cdot\mathbf{u}_{,l})\times\nabla\langle\zeta^k\zeta^l\rangle\cdot d\mathbf{S}\,, \tag{24}$$

for any closed curve $\gamma(\mathbf{u})$ that moves with the Eulerian mean fluid velocity $\mathbf{u}$ and surface $S(\mathbf{u})$ with boundary $\gamma(\mathbf{u})$. Thus in this Kelvin-Noether circulation theorem, the presence of spatial gradients in the Eulerian mean fluctuation covariance $\langle \zeta^k \zeta^l \rangle$ creates circulation of the Lagrangian mean velocity $\mathbf{v} = \mathbf{u} - \widetilde{\Delta}\mathbf{u}$.

2·4. *Vortex stretching equation for the Eulerian mean model.* – In three dimensions, the EMM equation (14) may be expressed in its equivalent "curl" form, as

$$(25) \qquad \frac{\partial}{\partial t}\mathbf{v} - \mathbf{u} \times (\boldsymbol{\nabla} \times \mathbf{v}) + \boldsymbol{\nabla}(P_{\mathrm{tot}} + \mathbf{u}\cdot\mathbf{v}) = -\frac{1}{2}(\mathbf{u}_{,k}\cdot\mathbf{u}_{,l})\, \boldsymbol{\nabla}\langle \zeta^k \zeta^l \rangle, \qquad \boldsymbol{\nabla}\cdot\mathbf{u} = 0\,.$$

The curl of this equation in turn yields an equation for transport and creation for the Lagrangian mean vorticy, $\mathbf{q} \equiv \operatorname{curl}\mathbf{v}$,

$$(26) \qquad \frac{\partial \mathbf{q}}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{q} = \mathbf{q}\cdot\boldsymbol{\nabla}\mathbf{u} - \frac{1}{2}\boldsymbol{\nabla}(\mathbf{u}_{,k}\cdot\mathbf{u}_{,l}) \times \boldsymbol{\nabla}\langle \zeta^k \zeta^l \rangle, \qquad \text{where } \mathbf{q} \equiv \operatorname{curl}\mathbf{v}\,,$$

and we have used incompressibility of $\mathbf{u}$. Thus $\mathbf{u}$ is the transport velocity for the generalized vorticity $\mathbf{q}$, and the expected vortex stretching term $\mathbf{q}\cdot\boldsymbol{\nabla}\mathbf{u}$ is accompanied by an additional vortex creation term proportional to the Eulerian mean covariance gradient. Of course, this additional term is also responsible for the creation of circulation of $\mathbf{v}$ in the Kelvin-Noether circulation theorem (24) and vanishes when the Eulerian mean covariance is homogeneous in space, thereby recovering the corresponding result for the three-dimensional CH equation [1, 2].

2·5. *Energetics of the Eulerian mean model.* – Noether's theorem guarantees the conservation of energy for the Euler-Poincaré equations (14), since the Eulerian mean Lagrangian $\langle L \rangle$ in eq. (6) has no explicity dependence on time. This constant energy is given by

$$(27) \qquad E_{\mathrm{t}} = \frac{1}{2}\int \mathrm{d}^3x(|\mathbf{u}|^2 + \langle \zeta^k \zeta^l \rangle\, \mathbf{u}_{,k}\cdot\mathbf{u}_{,l}) = \frac{1}{2}\int \mathrm{d}^3x\,\mathbf{u}\cdot\mathbf{v}\,.$$

Thus, the total kinetic energy is the integrated product of the Eulerian mean and Lagrangian mean velocities. In this kinetic energy, the Eulerian mean covariance of the fluctuations couples to the gradients of the Eulerian mean velocity. So there is a cost in kinetic energy for the system either to increase these gradients, or to increase the Eulerian mean covariance.

2·6. *Momentum conservation-stress tensor formulation.* – Noether's theorem also guarantees conservation of momentum for the Euler-Poincaré equation (14), since the Eulerian mean Lagrangian $\langle L \rangle$ in eq. (6) has no explicit spatial dependence. In momentum conservation form, eq. (14) becomes

$$(28) \qquad \frac{\partial v_i}{\partial t} = -\frac{\partial}{\partial x^j}(v_i u^j + P\delta^j_i - \mathbf{u}_{,k}\cdot\mathbf{u}_{,i}\langle \zeta^k \zeta^j \rangle)\,.$$

Natural boundary conditions are given in eq. (16).

2·7. *A second-momentum turbulence closure model for* EMM. – When dissipation and forcing are added to the EMM equation of motion (14) by using the phenomenological viscosity $\nu\widetilde{\Delta}\mathbf{v}$ and forcing $\mathbf{F}$, one finds a *second-moment Eulerian* *mean turbulence model* given by

$$\left(\frac{\partial}{\partial t}+\mathbf{u}\cdot\boldsymbol{\nabla}\right)\mathbf{v}+v_j\boldsymbol{\nabla}u^j+\boldsymbol{\nabla}P_{\text{tot}}+\frac{1}{2}(\mathbf{u}_{,k}\cdot\mathbf{u}_{,l})\,\boldsymbol{\nabla}\langle\zeta^k\zeta^l\rangle=\nu\,\widetilde{\Delta}\mathbf{v}+\mathbf{F}\,,\qquad\text{where }\ \boldsymbol{\nabla}\cdot\mathbf{v}=0\,, \tag{29}$$

with viscous boundary conditions $\mathbf{v}=0$, $\mathbf{u}=0$ at a fixed boundary. Note that the Eulerian mean fluctuation covariance $\langle\zeta^k\zeta^j\rangle$ appears in the dissipation operator $\widetilde{\Delta}$. In the absence of the forcing $\mathbf{F}$, this viscous EMM turbulence model dissipates the energy E in eq. (27) according to

$$\frac{\mathrm{d}E}{\mathrm{d}t}=-\,\nu\int\mathrm{d}^3x[\operatorname{tr}(\boldsymbol{\nabla}\mathbf{u}^{\mathrm{T}}\cdot\langle\boldsymbol{\zeta}\boldsymbol{\zeta}\rangle\cdot\boldsymbol{\nabla}\mathbf{u})+\widetilde{\Delta}\mathbf{u}\cdot\widetilde{\Delta}\mathbf{u}]\,. \tag{30}$$

This negative definite energy dissipation law is a consequence of adding viscosity with $\widetilde{\Delta}$, instead of using the ordinary Laplacian operator. In the isotropic homogeneous case of this model, where $\langle\zeta^k\zeta^l\rangle=\alpha^2\delta^{kl}$ (for a constant length scale α) we find the viscous Camassa-Holm equation (VCHE), or the Navier-Stokes alpha model,

$$\left(\frac{\partial}{\partial t}+\mathbf{u}\cdot\boldsymbol{\nabla}\right)\mathbf{v}+v_j\boldsymbol{\nabla}u^j+\boldsymbol{\nabla}P_{\text{tot}}=\nu\alpha^2\Delta\mathbf{v}+\mathbf{F}\,,\qquad\text{where}\qquad\boldsymbol{\nabla}\cdot\mathbf{u}=0\,, \tag{31}$$

where Δ is the usual Laplacian operator for this case and $\mathbf{v}$ and P_{tot} are defined in (18).

In Fourier space, the viscous alpha model equation (25) with isotropic viscosity can be written for the Lagrangian mean velocity $\mathbf{v}$ as follows, cf. eq. (19),

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}_{\mathbf{k}}-i\Pi_{\perp}\left(\sum_{\mathbf{p}+\mathbf{n}=\mathbf{k}}\frac{\mathbf{v}_{\mathbf{p}}}{1+\alpha^2|\mathbf{p}|^2}\times(\mathbf{n}\times\mathbf{v}_{\mathbf{n}})\right)=-\,\nu|\mathbf{k}|^2\mathbf{v}_{\mathbf{k}}+\mathbf{F}_{\mathbf{k}},\quad\text{where}\quad\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}}=0\,. \tag{32}$$

The Eulerian mean velocity satisfies $\mathbf{u}=(1-\alpha^2\Delta)^{-1}\mathbf{v}$ and $\mathbf{F}_{\mathbf{k}}$ is the forcing term for the $\mathbf{k}$-th velocity component. Since $\mathbf{u}_k=\mathbf{v}_{\mathbf{k}}/(1+\alpha^2|\mathbf{k}|^2)$, the quantity $1/\alpha$ acts as a cutoff wave number for the *nonlinearity* in the alpha model.

2·8. *Constitutive interpretation of the* VCHE, *or* NS-α *model*. – Chen *et al.* [6-8] gave a continuum mechanical interpretation to the VCHE closure model, by rewriting the VCHE (31) in the equivalent *constitutive form*,

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}=\operatorname{div}\mathbf{T}\,,\qquad\mathbf{T}=-\,p\mathbf{I}+2\nu(1-\alpha^2\Delta)\,\mathbf{D}+2\alpha^2\dot{\mathbf{D}}\,, \tag{33}$$

with $\boldsymbol{\nabla}\cdot\mathbf{v}=0$, $\mathbf{D}=(1/2)(\boldsymbol{\nabla}\mathbf{u}+\boldsymbol{\nabla}\mathbf{u}^{\mathrm{T}})$, $\boldsymbol{\Omega}=(1/2)(\boldsymbol{\nabla}\mathbf{u}-\boldsymbol{\nabla}\mathbf{u}^{\mathrm{T}})$, and co-rotational (Jaumann) derivative given by $\dot{\mathbf{D}}=\mathrm{d}\mathbf{D}/\mathrm{d}t+\mathbf{D}\boldsymbol{\Omega}-\boldsymbol{\Omega}\mathbf{D}$, with $\mathrm{d}/\mathrm{d}t=\partial/\partial t+\mathbf{u}\cdot\boldsymbol{\nabla}$. In this form, one recognizes the constitutive form of VCHE as a variant of the rate-dependent incompressible homogeneous fluid of second grade [16, 17], whose viscous dissipation, however, is *modified* by the Helmholtz operator $(1-\alpha^2\Delta)$. There is a tradition at least since Rivlin [18] of modeling turbulence by using continuum mechanics principles such as objectivity and material frame indifference (see also [19]). For example, this sort of approach is taken in deriving Reynolds stress algebraic equation models [20]. Rate-dependent closure models of mean turbulence such as the VCHE have also been obtained by a two-scale DIA approach [21] and by renormalization group methods [22].

2·9. *Comparison of* VCHE *with* LES *and* RANS *models*. – Reynolds-averaged Navier-Stokes (RANS) models of turbulence are part of the classic theoretical development of the subject [23-25]. The related Large Eddy Simulation (LES) turbulence modeling approach [26-28], provides an operational definition of the intuitive idea of Eulerian resolved scales of motion in turbulent flow. In this approach a filtering function $\mathscr{F}(\mathbf{r})$ is introduced and the Eulerian velocity field $\mathbf{U}_{\mathrm{E}}$ is filtered in an integral sense, as

$$\bar{\mathbf{u}}(\mathbf{r}) \equiv \int_{R^3} \mathrm{d}^3 r' \, \mathscr{F}(\mathbf{r}-\mathbf{r}') \, \mathbf{U}_{\mathrm{E}}(\mathbf{r}') \,. \tag{34}$$

This convolution of $\mathbf{U}_{\mathrm{E}}$ with $\mathscr{F}$ defines the large scale, resolved, or filtered velocity, $\bar{\mathbf{u}}$. The corresponding small scale, or subgrid scale velocity, $\mathbf{u}'$, is then defined as the difference,

$$\mathbf{u}'(\mathbf{r}) \equiv \mathbf{U}_{\mathrm{E}}(\mathbf{r}) - \bar{\mathbf{u}}(\mathbf{r}) \,. \tag{35}$$

When this filtering operation is applied to the Navier-Stokes system, the following dynamical equation is obtained for the filtered velocity, $\bar{\mathbf{u}}$, cf. eq. (33),

$$\frac{\partial}{\partial t} \bar{\mathbf{u}} + \bar{\mathbf{u}} \cdot \nabla \bar{\mathbf{u}} = -\operatorname{div} \overline{\mathbf{T}} - \nabla \bar{p} + \nu \Delta \bar{\mathbf{u}} \,, \qquad \nabla \cdot \bar{\mathbf{u}} = 0 \,, \tag{36}$$

in which $\bar{p}$ is the filtered pressure field (required to maintain $\nabla \cdot \bar{\mathbf{u}} = 0$) and the tensor difference

$$\overline{\mathbf{T}} = (\overline{\mathbf{U}_{\mathrm{E}} \mathbf{U}_{\mathrm{E}}}) - \bar{\mathbf{u}} \bar{\mathbf{u}} \,, \tag{37}$$

represents the subgrid scale stress due to the turbulent eddies. This subgrid scale stress tensor appears in the same form as the Reynolds stress tensor obtained from Reynolds averaging the Navier-Stokes equation.

The results of Chen *et al.* [6-8], may be given either an LES, or RANS interpretation simply by comparing the constitutive form of the VCHE, or NS-α model in (33) term by term with eq. (36). Additional LES interpretations, discussions and numerical results for forced-turbulence simulations of the VCHE, or the Navier-Stokes alpha model, are presented in [11].

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REFERENCES

[1] Holm D. D., Marsden J. E. and Ratiu T. S., *Adv. Math.*, **137** (1998) 1.
[2] Holm D. D., Marsden J. E. and Ratiu T. S., *Phys. Rev. Lett.*, **80** (1998) 4173.
[3] Camassa R. and Holm D. D., *Phys. Rev. Lett.*, **71** (1993) 1661.

[4] Holm D. D., Kouranbaeva S., Marsden J. E., Ratiu T. and Shkoller S., *Fields Inst. Comm.*, Arnold Vol. **2** (American Mathematical Society, Rhode Island) 1998, to be published.
[5] Holm D. D., *Physica D*, **133** (1999) 215-269.
[6] Chen S., Foias C., Holm D. D., Olson E., Titi E. S. and Wynne S., *Phys. Rev. Lett.*, **81** (1998) 5338.
[7] Chen S., Foias C., Holm D. D., Olson E., Titi E. S. and Wynne S., *Phys. Fluids*, **11** (1999) 2343-2353.
[8] Chen S., Foias C., Holm D. D., Olson E., Titi E. S. and Wynne S., *Physica D*, **133** (1999) 49-65.
[9] Zagarola M. V., PhD thesis, Princeton University (1996).
[10] Foias C., Holm D. D. and Titi E. S., *The 3D viscous Camassa-Holm equations: their relation to the Navier-Stokes equations and turbulence theory*, in preparation.
[11] Chen S., Holm D. D., Margolin L. G. and Zhang R., *Physica D*, **133** (1999) 66-83.
[12] Dunn J. E. and Fosdick R. L., *Arch. Rat. Mech. Anal.*, **56** (1974) 191.
[13] Cioranescu D. and Girault V., *C. R. Acad Sci. Paris, Ser.* 1, **322** (1996) 1163.
[14] Cioranescu D. and Girault V., *Int. J. Non-Lin. Mech.*, **32** (1997) 317.
[15] Andrews D. G. and McIntyre M. E., *J. Fluid Mech.*, **89** (1978) 609.
[16] Dunn J. E. and Fosdick R. L., *Arch. Rat. Mech. Anal.*, **56** (1974) 191.
[17] Dunn J. E. and Rajagopal K. R., *Int. J. Eng. Sci.*, **33** (1995) 689.
[18] Rivlin R. S., *Q. Appl. Math.*, **15** (1957) 212.
[19] Chorin A. J., *Phys. Rev. Lett.*, **60** (1988) 1947.
[20] Shih T. H., Zhu J. and Lumley J. L., *Comput. Methods Appl. Mech. Eng.*, **125** (1995) 287.
[21] Yoshizawa A., *Phys. Fluids*, **27** (1984) 1377.
[22] Rubinstein R. and Barton J. M., *Phys. Fluids A*, **2** (1990) 1472.
[23] Hinze J. O., *Turbulence*, 2nd edition (Mc-Graw-Hill, New York) 1975.
[24] Townsend A. A., *The Structure of Turbulent Flows* (Cambridge University Press) 1967.
[25] Lumley J. L. and Tennekes H., *A First Course in Turbulence* (MIT Press) 1972.
[26] Reynolds W. C., *Fundamentals of turbulence for turbulence modeling and simulation*, in *Lecture Notes for Von Karman Institute*, *AGARD Lecture Note Series* (NATO, New York) 1987, pp. 1-66.
[27] Piomelli U., *Applications of large eddy simulations in engineering: an overview*, in *Large-eddy Simulation of Complex Engineering and Geophysical Flows*, edited by B. Galperin and S. A. Orszag (Cambridge University Press) 1993.
[28] Lesieur M. and O. Métais, *Annu. Rev. Fluid Mech.*, **28** (1996) 45.