Remarks on stability of the rotating shallow-water vortices in the frontal dynamics regime (*)(**)

M. BEN JELLOUL and V. ZEITLIN

Laboratoire de Météorologie Dynamique, B.P. 99, Université P. et M. Curie
4 pl. Jussieu, 75252 Paris Cedex 05, France

(ricevuto il 18 Novembre 1998; revisionato il 24 Maggio 1999; approvato il 7 Giugno 1999)

Summary. — Stability properties of large-scale strongly nonlinear isolated vortices in the rotating shallow water on the *f*-plane are analysed. Working first in the framework of the balanced frontal dynamics equations we demonstrate that vortices of arbitrary sign with monotonous profiles of the free-surface elevation are formally stable and establish criteria for nonlinear stability. We then discuss stability in the framework of the full rotating shallow-water equations and obtain a conditional stability criterion.

PACS 92.10 – Physics of the oceans.
PACS 47.20 – Hydrodynamics stability.
PACS 92.10.Ei – Coriolis effects.
PACS 01.30.Cc – Conference proceedings.

1. – Introduction

It is well known that a hierarchy of the so-called balanced models for the slow (with respect to the rapid gravity waves) vortical component of the flow may be deduced from the rotating shallow-water equations (RSW in what follows) by filtering the high-frequency fine-scale motions. The derivation of the balanced equations is straightforward in the case when the vortical motion in question has a single characteristic spatial scale (as, *e.g.*, for an isolated vortex). The balanced equations follow then from the direct asymptotic expansions in Rossby number and Burger number ([1-4]. The obvious advantage of the balanced models is that they allow to reduce a set of the RSW equations to a single equation for the pressure (the free-surface elevation) variable. The most known balanced equation is, of course, the standard quasi-geostrophic (QG) vorticity equation. However, by increasing the degree

(*) Paper presented at the International Workshop on "Vortex Dynamics in Geophysical Flows", Castro Marina (LE), Italy, 22-26 June 1998.
(**) The authors of this paper have agreed to not receive the proofs for correction.

of nonlinearity of the vortex motions in question, one arrives at the intermediate geostrophic (IG), nonlinear geostrophic (NLQG) and, finally, frontal dynamics (FD) equations, respectively. In the present paper our main focus will be on the vortices in the FD regime, *i.e.* those corresponding to small Rossby numbers and small Burger numbers (cf. [3, 4]). In terms of the characteristic nonlinearities and scales this means that the vortices in question have surface elevations comparable with the unperturbed depth of the fluid and that their size is much larger than the Rossby radius. In what follows we shall limit our study to the f-plane dynamics where the so-called scalar nonlinearity leading to the steepening of the vortex profile in the zonal direction and vortex breakdown (cf. [6]) are absent. The analysis of vortices in such regime was first undertaken in [3,5]. Recently, vortices in this regime were observed experimentally [6] in the rotating parabolic shallow-water layer (note that scalar nonlinearity disappears in the parabolic geometry, [7, 4]). In this experiment anticyclonic vortices were easier to generate and had much longer lifetimes than cyclonic ones, a possible reason for this phenomenon stemming from a difference in stability properties of frontal cyclones and anticyclones claimed in [5]. However, a trivial explanation related to the experimental difficulties with producing an order one deep in a thin fluid layer could not be excluded neither (cf. [6]). The cyclone-anticyclone asymmetry was the main motivation of the present study of nonlinear stability of the FD vortices. We show below that, in what concerns the FD equations, both cyclones and anticyclones may be stable. We also make some comments on vortex stability in the framework of the full RSW equations. No definite conclusions may be drawn there at the present stage.
The paper is organised as follows. In sect. **2** we display the Hamiltonian structure of the underlying equations and remind the basics of the formal and nonlinear stability analyses. In sect. **3** we study formal and nonlinear stability of the FD vortices and in sect. **4** we comment on the stability properties of the RSW solutions.

2. – Hamiltonian structure of the RSW and FD equations and stabilty analysis algorithm

2·1. RSW *equations*. – The standard system of the rotating shallow-water (RSW) equations in the f-plane approximation is written as follows:

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} + f*\mathbf{u} + g\nabla h = 0\,, \\ \dfrac{\partial \mathbf{u}}{\partial t} + \nabla\cdot(\mathbf{u}h) = 0\,. \end{cases} \tag{1}$$

Here $\mathbf{u} = (u, v)$ is the horizontal velocity, h is the free surface elevation, $*\mathbf{u} = (-v, u)$ and f is the (constant) Coriolis parameter. $\mathbf{r}$ will denote the radius-vector on the plane expressed in Cartesian (x, y) or polar (r, θ) coordinates. The nondimensionalisation with the help of the characteristic turnover velocity U and the characteristic vortex scale L in the horizontal direction and with the unperturbed fluid depth H in the vertical direction naturally introduces the Rossby and Burger numbers

$$\mathrm{Ro} = \varepsilon = \frac{U}{fL}\,, \tag{2}$$

$$\mathrm{Bu} = s = \frac{R_{\mathrm{d}}^2}{L^2} = \frac{gH}{f^2 L^2}\,. \tag{3}$$

The system of equations (1) possesses a Hamiltonian structure (see, *e.g.*, [8]) which is provided by the nondimensional Hamiltonian

$$H_{\mathrm{SW}}[\mathbf{u}, h] = \frac{1}{2}\int\!\!\int \mathrm{d}x\,\mathrm{d}y\left[\frac{\varepsilon h^2}{2s} + \frac{\varepsilon h \mathbf{u}^2}{2}\right], \tag{4}$$

where

$$\mathbf{u} = \frac{\mathbf{m} - h\mathbf{R}}{\varepsilon h} \tag{5}$$

and $\boldsymbol{\nabla} \wedge \mathbf{R} = f\mathbf{k}$ is rescaled to have $f = 1$, and by the Poisson bracket

$$\{F, G\}_{\mathrm{SW}} = \int\!\!\int \mathrm{d}x\,\mathrm{d}y\left\{\frac{\delta F}{\delta m_i}\left[(\partial_j m_i + m_j\partial_i)\frac{\delta G}{\delta m_j} + h\partial_i\frac{\delta G}{\delta h}\right] + \frac{\delta F}{\delta h}\partial_j h\frac{\delta G}{\delta m_j}\right\}. \tag{6}$$

We should remind, without dwelling on mathematical subtleties, that in general the dynamical evolution of a set of variables $\omega(x, y; t)$ is Hamiltonian if it may be written in the form

$$\dot{\omega} = \{\omega, H[\omega]\}\,, \tag{7}$$

where $H[\omega]$—a Hamiltonian—is a functional of ω and for any pair of functionals $\{A[\omega], B[\omega]\}$ denotes a Poisson bracket obeying a cyclic Jacobi identity.

An infinite series of integrals of motion—so-called Casimir functionals—arise from the continuous symmetry under relabelling of the Lagrangian particles

$$C_{\mathrm{SW}}[\mathbf{u}, h] = \int\!\!\int \mathrm{d}x\,\mathrm{d}y\, h\Phi(q)\,. \tag{8}$$

They are functionals of the dynamical variables $\mathbf{u}$, h and their Poisson bracket (6) with any function of dynamical variables is identically zero. Here the nondimensional potential vorticity q is defined by

$$q = \frac{1 + \varepsilon\omega}{h}\,, \qquad \omega = (\boldsymbol{\nabla} \wedge \mathbf{u})\cdot\mathbf{k}\,, \tag{9}$$

where $\mathbf{k}$ here and before denotes a unit vector normal to the plane. The Hamiltonian (4) is invariant with respect to spatial rotations. In the case where the boundary conditions are also invariant, an additional conserved quantity—the angular momentum—appears:

$$I_{\mathrm{SW}}[\mathbf{u}, h] = \int\!\!\int \mathrm{d}x\,\mathrm{d}y(\mathbf{m} \wedge \mathbf{r})\cdot\mathbf{k}\,. \tag{10}$$

2·2. FD *equations*. – The FD equations arise from the direct asymptotic expansions of eqs. (1) under the assumption $s \simeq \varepsilon$ and by introducing multiple time scales [1, 3, 4].

In slow time $\tau = \varepsilon^{-2} t$ the equation of motion for the free-surface elevation is

$$\frac{\partial h}{\partial \tau} = \mathrm{J}\left(h,\, h\nabla^2 h + \frac{(\nabla h)^2}{2}\right); \tag{11}$$

here $\mathrm{J}(A, B) = \partial_x A \partial_y B - \partial_x B \partial_y A$ denotes the Jacobian of any pair of functions A and B.

The Hamiltonian structure is given by the following Hamiltonian and Lie-Poisson bracket:

$$H_{\mathrm{FD}}[h] \equiv \frac{1}{2}\int\!\!\int \mathrm{d}x\,\mathrm{d}y\, h(\nabla h)^2, \tag{12}$$

$$\{F, G\}_{\mathrm{FD}} = \int\!\!\int \mathrm{d}x\,\mathrm{d}y\, h\, \mathrm{J}\left(\frac{\delta F}{\delta h}, \frac{\delta G}{\delta h}\right). \tag{13}$$

The Casimir functionals are

$$C_{\mathrm{FD}}[h] \equiv \int\!\!\int \mathrm{d}x\,\mathrm{d}y\, \Phi(h), \tag{14}$$

where Φ is an arbitrary real-valued function. Assuming axisymmetric boundary conditions, one may check the conservation of the angular momentum

$$I_{\mathrm{FD}}[h] \equiv \frac{1}{2}\int\!\!\int \mathrm{d}x\,\mathrm{d}y\, \mathbf{r}^2 h\,. \tag{15}$$

2·3. *Stability of stationary solutions.* – Let us now remind the definitions of formal and nonlinear stability of stationary solutions for arbitrary Hamiltonian system (7).

Formal stability

Suppose an equilibrium solution $\omega_{\mathrm{e}}(t) = \omega_{\mathrm{e}}(0)$ is found such as $\{\omega_{\mathrm{e}}, H[\omega_{\mathrm{e}}]\} = 0$. Then this solution is formally stable if there exists a conserved quantity (an invariant of motion) such that its first variation calculated on ω_{e} is zero and the second variation is sign-definite.

Nonlinear stability

Nonlinear stability of an equilibrium solution is just the Lyapunov stability, *i.e.* ω_{e} is nonlinearly stable if there exists a norm $\|\cdot\|$ such that

$$\forall \varepsilon > 0, \qquad \exists \delta > 0 \colon \|\omega(0) - \omega_{\mathrm{e}}\| < \delta \Rightarrow \|\omega(t) - \omega_{\mathrm{e}}\| < \varepsilon, \qquad \forall t > 0\,.$$

Usually, it is the second variation of the Hamiltonian (or the Hamiltonian "augmented" by additional invariants introduced with their proper Lagrange multipliers) which, if sign-definite, provides a norm for nonlinear stability proofs. It does provide a norm for *linear* stability, however additional convexity estimates are needed to get the full nonlinear stability (cf. [9]).

Note that in Hamiltonian systems possessing Casimir invariants, the dynamical evolution in the phase space takes place on the surface of fixed Casimirs. So the stability analysis may be done either on the fixed Casimirs' surface by considering the "isovortical" variations, *i.e.* those variations of the dynamical variables that do not change the Casimir invariants (cf. [10]), or in the full phase space by considering arbitrary variations of the dynamical variables applied to the augmented Hamiltonian (cf. [9]). In the latter case the formal stability analysis algorithm consists in finding the Lagrange multipliers by vanishing the first variation of the augmented Hamiltonian and then analysing the second variation.

3. – Stability properties of the FD vortices

Any axisymmetric profile $h_\mathrm{e}(r)$ is, obviously, a stationary solution of (11). In the present paper we are studying the stability of the *isolated* vortices. An isolated vortex is the one with zero circulation at spatial infinity, where the velocity in FD regime is determined from h via the geostrophic balance equations. The simplest isolated vorticity configuration consists of a vortex core surrounded by a ring of opposite-sign vorticity. In what follows only smooth profiles of vorticity will be considered. Note that, according to our nondimensionalisation $h_\mathrm{e}(r)$ tends to unity when r goes to infinity. Monotonous $h_\mathrm{e}(r)$ correspond to "non-sheared" vortices having azimuthal velocity of a definite sign.

3·1. *Formal stability analysis.* – We apply now the standard formal stability analysis algorithm by constructing an augmented Hamiltonian $H^\mathrm{C}_\mathrm{FD} = H_\mathrm{FD} + C_\mathrm{FD}$ and calculating its first variation

$$DH^\mathrm{C}_\mathrm{FD}[h_\mathrm{e}] = \int\!\!\int \mathrm{d}x\,\mathrm{d}y\left(-h_\mathrm{e}\nabla^2 h_\mathrm{e} - \frac{(\nabla h_\mathrm{e})^2}{2} + \Phi'(h_\mathrm{e})\right)\delta h\,. \tag{16}$$

Vanishing this expression gives

$$\Phi'(h_\mathrm{e}) = h_\mathrm{e}\nabla^2 h_\mathrm{e} + \frac{(\nabla h_\mathrm{e})^2}{2}\,. \tag{17}$$

The second variation is then

$$D^2 H^\mathrm{C}_\mathrm{FD}[h_\mathrm{e}] = \int\!\!\int \mathrm{d}x\,\mathrm{d}y[(\Phi''(h_\mathrm{e}) - \nabla^2 h_\mathrm{e})(\delta h)^2 + h_\mathrm{e}(\nabla\delta h)^2]\,. \tag{18}$$

If the vortex height profile is monotonic, then $\Phi''(h_\mathrm{e})$ as a function of r may be obtained from (17)

$$\Phi''(h_\mathrm{e}) = \nabla^2 h_\mathrm{e} + \frac{h_\mathrm{e}(\nabla^2 h_\mathrm{e})' + ((\nabla h_\mathrm{e})^2/2)'}{h'_\mathrm{e}}\,, \tag{19}$$

where primes in the r.h.s. denote differentiation with respect to r. Although h'_e vanishes at zero and at infinity, this expression has no singularities there according to our assumptions on the structure of the vortex profile. After some algebra one gets

$$h_e h_e'^2 \left[\nabla \left(\frac{\delta h}{h_e'} \right) \right]^2 = \tag{20}$$

$$= h_e (\nabla \delta h)^2 + \left[\frac{h_e (\nabla^2 h_e)' + ((\nabla h_e)^2 /2)'}{h_e'} \right] (\delta h)^2 + [\nabla^2 h_e' - (\nabla^2 h_e)'] \frac{h_e}{h_e'} (\delta h)^2 .$$

Note that for the radial part of the Laplacian of $h_e(r)$

$$\nabla^2 h_e(r) = h''_e(r) + \frac{h_e'(r)}{r} \tag{21}$$

the bracket in the last term in (20) does not vanish giving $-h_e'/r^2$. Hence, the second variation of the Hamiltonian for Fourier-expandible perturbations

$$\delta h(r, \theta, t) = \sum_{n \in \mathbb{N}} \widehat{\delta h}_n (r, t)\, e^{in\theta} + \text{c.c.} \tag{22}$$

is

$$D^2 H_{\mathrm{FD}}^{\mathrm{C}} [h_e] = \int\!\!\int \mathrm{d}x\, \mathrm{d}y \left\{ h_e h_e'^2 \left[\nabla \left(\frac{\delta h}{h_e'} \right) \right]^2 - \frac{h_e}{r^2} (\delta h)^2 \right\} = \tag{23}$$

$$= \sum_{n \in \mathbb{N}} \int 2\pi r\, \mathrm{d}r \left[h_e h_e'^2 \left| \left(\frac{\widehat{\delta h}_n}{h_e'} \right)' \right|^2 + \frac{(n^2 - 1)\, h_e}{r^2} \,|\, \widehat{\delta h}_n \,|^2 \right].$$

The axisymmetric $n = 0$ perturbation mode may be considered as part of the initial steady profile. Therefore, if we restrict the vortex profile perturbations to those with zero angular mean, then $n = 0$ mode is excluded, (23) is positive-definite and we get formal stability of any monotonous vortex profile. Note that both monotonous cyclones and anticyclones are formally stable with respect to zero angular mean modal perturbations according to this result.

3·2. *Nonlinear stability analysis.* – As usual in the stability analysis (cf. [9]) we form the following combination from the variation of $H_{\mathrm{FD}}^{\mathrm{C}}$ with respect to finite perturbations Δh of h_e and $DH_{\mathrm{FD}}^{\mathrm{C}}[h_e]$:

$$\operatorname{var}(H_{\mathrm{FD}}^{\mathrm{C}}) := [H_{\mathrm{FD}}^{\mathrm{C}}[h_e + \Delta h] - H_{\mathrm{FD}}^{\mathrm{C}}[h_e] - \mathrm{D}H_{\mathrm{FD}}^{\mathrm{C}}[h_e] \cdot \Delta h = \tag{24}$$

$$= \operatorname{var}(\Phi) + \int\!\!\int \mathrm{d}x\, \mathrm{d}y (-\nabla^2 h_e (\Delta h)^2 + (h_e + \Delta h)(\nabla(\Delta h))^2) ,$$

where

$$\operatorname{var}(\Phi) := \int\!\!\int \mathrm{d}x\, \mathrm{d}y (\Phi(h_e + \Delta h) - \Phi(h_e) - \Phi'(h_e)\, \Delta h) . \tag{25}$$

In order to establish upper and lower bounds for $\operatorname{var}(H_{\mathrm{FD}}^{\mathrm{C}})$ we, first, have to bound $\operatorname{var}(\Phi)$. For this we need to bound the second derivative of $\Phi(h)$ in the domain $\mathscr{D}$ of the flow. For axisymmetric vortex profiles the domain $\mathscr{D}$ is either a circular disc or the full plane; in the latter case we will consider perturbations with a compact support $\mathscr{D}$. We, thus, require

$$-\infty < a \leqslant \Phi''(h_\mathrm{e}) \leqslant A < +\infty\,. \tag{26}$$

Note that $\Phi''(h)$ is still given by (19) and the fact that it is bounded means that $h_\mathrm{e}(r)$ is necessarily monotonous. Then

$$a\|\Delta h\|^2 \leqslant \mathrm{var}\,(\Phi) \leqslant A\|\Delta h\|^2 \tag{27}$$

and we get the following bounds for $\mathrm{var}\,(H^\mathrm{C}_\mathrm{FD})$:

$$\Big[a + \inf_{\mathscr{D}}(-\nabla^2 h_\mathrm{e}) + C_{\mathscr{D}}^{-1}\inf_{\mathscr{D}}(h_\mathrm{e}+\Delta h)\Big]\|\Delta h\|^2 \leqslant \mathrm{var}\,(H^\mathrm{C}_\mathrm{FD}) \tag{28}$$

and

$$\mathrm{var}\,(H^\mathrm{C}_\mathrm{FD}) \leqslant \Big[C_{\mathscr{D}}\Big(A + \sup_{\mathscr{D}}(-\nabla^2 h_\mathrm{e})\Big) + \sup_{\mathscr{D}}(h_\mathrm{e}+\Delta h)\Big]\|\nabla\Delta h\|^2, \tag{29}$$

where $\|\cdot\|$ denotes the L_2 norm and $C_{\mathscr{D}}$ is a constant depending on $\mathscr{D}$. We use here the Poincaré inequality

$$\iint_{\mathscr{D}} \mathrm{d}x\,\mathrm{d}y\,|f|^2 \leqslant C_{\mathscr{D}}\iint_{\mathscr{D}} \mathrm{d}x\,\mathrm{d}y\,|\nabla f|^2 \tag{30}$$

which is valid for compact domains, domains bounded in one direction or finite-measure sets on the plane. Hence, if we limit ourselves to the finite-variation perturbations whose maximum and minimal values are bounded (*i.e.* perturbations which do not change the peak value of h_e are physically plausible; note that in any case $\inf_{\mathscr{D}}(h_\mathrm{e}+\Delta h) > 0$) and if the following inequality holds:

$$a + \inf_{\mathscr{D}}(-\nabla^2 h_\mathrm{e}) + C_{\mathscr{D}}^{-1}\inf_{\mathscr{D}}(h_\mathrm{e}+\Delta h) > 0\,, \tag{31}$$

then there exist positive constants c and C such that

$$c\|\Delta h\|^2 \leqslant \mathrm{var}\,(H^\mathrm{C}_\mathrm{FD}) \leqslant C\|\nabla\Delta h\|^2. \tag{32}$$

This inequality holds for all $t > 0$ as H^C_FD is an integral of motion. Hence,

$$c\|\Delta h(t)\| \leqslant C\|\nabla\Delta h(0)\|\,. \tag{33}$$

This relation provides a criterion for Lyapunov stability (note that this is a slight generalisation of the nonlinear stability definition given above as two different norms enter) of the FD vortices. The condition (31) is not symmetric with respect to the change of vortex sign but eddies of both signs may verify it. A detailed analysis of $\Phi(h_\mathrm{e}(r))$ and, hence, $h_\mathrm{e}(r)$, cf. (19), satisfying this criterion will be presented elsewhere.

4. – Stability of localised vortices in the full RSW equations

In the preceding section we have shown that the FD vortices with monotonous profile of the free surface elevation are formally stable. The FD equations are, however, highly "processed" with respect to the parent RSW equations and the question arises what is the counterpart of this stability property in the full RSW equations.

Let us define the augmented Hamiltonian (cf. [11]) by

$$H^{\mathrm{C}}_{\mathrm{SW}} = H_{\mathrm{SW}} + C_{\mathrm{SW}} + \Omega I_{\mathrm{SW}}, \tag{34}$$

where Ω is an arbitrary real constant. Note that this is a most general form of H^{C} while we managed to establish formal stability in FD without including angular momentum. The equilibrium solution $\mathbf{u}_{\mathrm{e}}$, h_{e} satisfies the following equations:

$$\mathbf{u}_{\mathrm{e}}\cdot\boldsymbol{\nabla}\left(\frac{sh_{\mathrm{e}}}{\varepsilon} + \frac{\varepsilon\mathbf{u}_{\mathrm{e}}^2}{2}\right) = 0\,, \tag{35}$$

$$\mathbf{u}_{\mathrm{e}}\cdot\boldsymbol{\nabla} q_{\mathrm{e}} = 0\,. \tag{36}$$

Hence, there exists a real-valued function K verifying

$$\frac{sh_{\mathrm{e}}}{\varepsilon} + \frac{\varepsilon\mathbf{u}_{\mathrm{e}}^2}{2} = K(q_{\mathrm{e}})\,. \tag{37}$$

The first variation of the augmented Hamiltonian,

$$\begin{aligned} &\mathrm{D}H^{\mathrm{C}}_{\mathrm{SW}}[\mathbf{u}_{\mathrm{e}}, h_{\mathrm{e}}]\cdot(\delta\mathbf{u}, \delta h) = \\ &= \int\!\!\int \mathrm{d}x\,\mathrm{d}y\left\{\left[\frac{sh_{\mathrm{e}}}{\varepsilon} + \frac{\varepsilon\mathbf{u}_{\mathrm{e}}^2}{2} + \Phi(q_{\mathrm{e}}) - q_{\mathrm{e}}\Phi'(q_{\mathrm{e}}) - \Omega(\varepsilon r\mathbf{u}_{\mathrm{e}}\cdot\mathbf{e}_\theta + r^2/2)\right]\delta h + \right.\\ &\qquad\left. + [\varepsilon h\mathbf{u} - \varepsilon\mathbf{k}\wedge\boldsymbol{\nabla}\Phi'(q_{\mathrm{e}}) - \Omega h_{\mathrm{e}} r\mathbf{e}_\theta]\cdot\delta\mathbf{u}\right\} \end{aligned} \tag{38}$$

is zero, provided

$$K(q_{\mathrm{e}}) + \Phi(q_{\mathrm{e}}) - q_{\mathrm{e}}\Phi'(q_{\mathrm{e}}) - \Omega\varepsilon r\mathbf{u}_{\mathrm{e}}\cdot\mathbf{e}_\theta = 0\,, \tag{39}$$

where $\mathbf{e}_\theta$, $\mathbf{e}_r$ are the unit vectors in polar coordinates. The second variation is

$$\begin{aligned} &\mathrm{D}^2 H^{\mathrm{C}}_{\mathrm{SW}}[\mathbf{u}_{\mathrm{e}}, h_{\mathrm{e}}]\cdot(\delta\mathbf{u}, \delta h) = \\ &= \int\!\!\int \mathrm{d}x\,\mathrm{d}y\left[\frac{\varepsilon(\delta(h\tilde{\mathbf{u}}))^2}{h_{\mathrm{e}}} + \left(\frac{s}{\varepsilon} - \frac{\varepsilon|u_{\mathrm{e}} - \Omega r|^2}{h_{\mathrm{e}}}\right)|\delta h|^2 + h_{\mathrm{e}}\Phi''(q_{\mathrm{e}})(\delta q)^2\right], \end{aligned} \tag{40}$$

where $\tilde{\mathbf{u}} = \mathbf{u} - \Omega r\mathbf{e}_\theta$ and

$$\Phi''(q_{\mathrm{e}}) = \frac{h_{\mathrm{e}}(r)[u_{\mathrm{e}}(r) - \Omega r]}{q_{\mathrm{e}}'(r)}\,, \tag{41}$$

$u_{\mathrm{e}}(r)$ denoting the azimuthal velocity component.

This derivation follows literally the one given in [9] for an arbitrary barotropic fluid in a rotating framework, RSW being a particular example of such system. Obviously, it is sufficient that the second and the third term in the integrand of (40) be positive for the positive-definitedness of the second variation of the augmented Hamiltonian and, hence, formal stability. These conditions give exactly the ones obtained in [9, 11] (in a nondimensional form). However, as is easy to see, this stability criterion is empty for isolated vortices. Indeed for rapidly decaying $u_e(r)$, $h_e(r)$ the second term in the integrand of (40) can be everywhere positive only for $\Omega = 0$. Now, the quantity $u_e(r)/q_e'(r)$ cannot be positive-definite in the whole intergration domain. Indeed, the localised equilibrium solution verifies

$$u_e = \frac{-r + \sqrt{r^2 + 4sh_e'(r)}}{2\varepsilon}, \tag{42}$$

so u_e and h_e' always have the same sign. For the vortex to be isolated its core should be surrounded by a ring of the opposite vorticity (we are mostly interested in non-sheared azimuthal velocity configurations; the argument below may be, however, extended to sheared eddies, too). Let us consider the case of a cyclonic core (the proof is similar in the anticyclonic case). Then u_e is everywhere positive being zero at the origin and decaying rapidly at infinity. The vorticity $\omega_e = u_e' + u_e/r$ is then positive in the core region and the potential vorticity value is $q_e(r) = [(1 + \varepsilon\omega_e(r))/h_e(r)]$ (remember that $h_e(r)$ is everywhere positive and growing in this case). Now, in the ring region vorticity is negative and $h_e(r)$ is larger than in the core. Hence, the potential vorticity is smaller in the ring than in the core and its derivative should be negative at least somewhere in between.

One can, neveretheless, establish necessary conditions for the sign-definiteness of (40). Note that there exists a class of perturbations rendering $D^2 H_{SW}^C[h_e, \mathbf{u}_e]\cdot(\delta\mathbf{u}, \delta h)$ positive. It is clear that potential vorticity does not depend on the potential part of the velocity. By decomposing velocity perturbation into vortical and potential parts $\delta\mathbf{u} = \mathbf{k}\wedge\boldsymbol{\nabla}\delta\psi + \boldsymbol{\nabla}\delta\xi$ and taking perturbations with $\delta\psi = \delta h = 0$ one gets

$$D^2 H_{SW}^C[\mathbf{u}_e, h_e]\cdot(\boldsymbol{\nabla}\delta\xi, 0) = \iint dx\, dy(\varepsilon h_e(\boldsymbol{\nabla}\delta\xi)^2) \geqslant 0\,. \tag{43}$$

Hence, $D^2 H_{SW}^C h_e, \mathbf{u}_e(\delta\mathbf{u}, \delta h)$ cannot be negative-definite.

If now $\Phi''(q_e)$ is not positive-definite $D^2 H_{SW}^C h_e, \mathbf{u}_e(\delta\mathbf{u}, \delta h)$ may be rendered negative by the following choice of perturbations. Take

$$\delta\mathbf{u}\cdot\mathbf{e}_r = \delta u = \sum_{n\in N} \widehat{\delta u_n}(r)\, e^{in\theta} + \text{c.c.}, \tag{44}$$

$$\delta\mathbf{u}\cdot\mathbf{e}_\theta = 0\,, \tag{45}$$

$$\delta h = 0\,. \tag{46}$$

Then the second variation for a single harmonic $\widehat{\delta u_n}(r)$ becomes

$$D^2 H_{SW}^C[\mathbf{u}_e, h_e]\cdot(\delta\mathbf{u}, 0) = \int 2\pi r\, dr\left[\varepsilon h_e\,|\,\widehat{\delta u_n}\,|^2 + \frac{\Phi''(q_e)\, n^2\,|\,\widehat{\delta u_n}\,|^2}{h_e r^2}\right]. \tag{47}$$

If $\widehat{\delta u_n}(r)$ is peaked in the region where $\Phi''(q_e)$ is negative, one can choose n as large as necessary to make $D^2 H_{SW}^C[\mathbf{u}_e, h_e]\cdot(\delta\mathbf{u}, 0)$ negative.

Therefore, $\Phi''(q_e)$ should be necessarily positive-definite for sign-definiteness of $D^2 H_{SW}^C[\mathbf{u}_e, h_e]\cdot(\delta\mathbf{u}, \delta h)$. As was just demonstrated, $u_e(r)/q_e'(r)$ cannot. Hence, this latter quantity should be negative-definite along with $[u_e(r) - \Omega r]/u_e(r)$. Yet, these conditions are not sufficient to guarantee stability unless the free-surface perturbation δh are confined to the region where $(s/\varepsilon - \varepsilon|u_e - \Omega r|^2/h_e)$ is non-negative. This gives a conditional stability condition. Note, however, that the monotonicity of potential vorticity is essential and that potential vorticity degenerates to the inverse surface elevation in the FD approximation which makes a link with the above-established FD stability result.

5. – Discussion

Thus, we have demonstrated formal stability of vortices with monotonous elevation profile within the framework of the FD equations and established criteria for nonlinear stability. The fact that the elevation represents, in fact, potential vorticity in this regime corroborates the earlier results on stability of vortices with monotonous vorticity profiles in 2D Euler and QG equations (cf. [12, 13]). We have seen that the monotonicity of the potential vorticity is important in the framework of the full RSW equations, as well. However, we were unable to establish unconditional formal RSW stability criteria for isolated vortices. Of course, the sign-definiteness of the second variation of the augmented Hamiltonian is a *sufficient* formal stability condition and the fact that we cannot establish it does not, obviously, mean that RSW vortices are unstable. However, the fact that stability conditions valid in a balanced model have no clear counterpart in the full equations is somewhat disturbing. The energy-Casimir approach is the only regular method to get stability proofs and, hence, in the absence of these latter the only way to proceed is numerics. In this connection the analysis of potentially dangerous perturbations rendering the second variation of the augmented RSW Hamiltonian negative which was done in the last section may give some useful hints.

We should mention that our analysis of formal and nonlinear stability of the FD vortices is similar to that undertaken in [14] and then in [15] for linear fronts (*i.e.* equilibrium solutions of the form $h_{\mathrm{e}} = h_{\mathrm{e}}(y)$). Note, that on the level of the formal stability the different geometry leads to the appearance of the second term in (23). On the level of nonlinear stability the authors of [14] and [15] have limited themselves to the energy bounds. However, the energy is a cubic function of h in FD and, hence, is not a norm because it does not satisfy the triangle inequality.

REFERENCES

[1] WILLIAMS G. P. and YAMAGATA T., *Geostrophic regimes, intermediate solitary vortices and Jovian eddies, J. Atmos. Sci.*, **41** (1984) 453-478.

[2] ROMANOVA N. N. and TSEITLIN V., *On quasigeostrophic motions in barotropic and baroclinic fluids, Atm. Ocean Phys.-Izvestija*, **20** (1984) 85-91.

[3] CUSHMAN-ROISIN B., *Frontal geostrophic dynamics, J. Phys. Oceanogr.*, **16** (1986) 132-143.

[4] STEGNER A. and ZEITLIN V., *What can asymptotic expansions tell us about large-scale quasi-geostophic anticyclonic vortices?, Nonlin. Proc. Geophys.*, **2** (1985) 186-193.

[5] CUSHMAN-ROISIN B. and TANG B., *Geostrophic turbulence and emergence of eddies beyond the radius of deformation, J. Phys. Oceanogr.*, **20** (1990) 97-113.

[6] STEGNER A. and ZEITLIN V., *From quasigeostrophic to strongly nonlinear monopolar vortices in a paraboloidal shallow-water-layer experiment, J. Fluid Mech.*, **356** (1998) 1-24.

[7] NYCANDER J., *The difference between monopole vortices in planetary flows and laboratory experiments, J. Fluid Mech.*, **254** (1993) 561-577.

[8] ALLEN J. S. and HOLM D. D., *Extended-geostrophic Hamiltonian models for rotating shallow water motion, Physica D*, **98** (1996) 229-248.
[9] HOLM D. D. *et al.*, *Nonlinear stability of fluid and plasma equilibria, Phys. Rep.,* **123** (1985) 1-116.
[10] ARNOLD V. I., *Mathematical Methods of Classical Mechanics* (Springer, Berlin) 1978.
[11] RIPA P., *On the stability of elliptical vortex solutions of the shallow-water equations, J. Fluid Mech.*, **183** (1987) 343-363.
[12] CARNEVALE G. F. and SHEPHERD T. G., *On the Interpretation of Andrews' Theorem, Geophys. Astrophys. Fluid Dyn.*, **51** (1990) 1-17.
[13] DRITSCHEL D. G., *Nonlinear stability bounds for inviscid, two-dimensional, parallel or circular flows with monotonous vorticity and the analogous three-dimensional quasi-geostrophic flows, J. Fluid Mech.*, **191** (1988) 575-581.
[14] SWATERS G. E., *On the baroclinic dynamics, Hamiltonian formulation and general stability characteristics of density-driven surface currents and fronts over a sloping continental shelf, Philos. Trans. R. Soc. London, Ser. A*, **345** (1993) 295-325.
[15] DUAN J. and WIGGINS S., *Nonlinear stability of one-layer geostrophic fronts, Physica D*, **98** (1996) 335-342.