# Constructal Design of an Elliptical Cavity into a Solid Conducting Wall

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## Abstract
This work reports, according to Bejan's Constructal theory, the geometric optimization of an elliptical cavity that intrudes into a solid conducting wall. The objective is to minimize the global thermal resistance between the solid and the cavity. There is uniform heat generation on the solid wall. The cavity is isothermal and the solid conducting wall is isolated on the external perimeter. The total volume and the elliptical cavity volume are fixed while the geometry of the cavity is free to vary. The cavity shape is optimal when penetrates the conducting wall completely.

## 1. Introduction
Constructal theory bases on a deterministic principle the occurrence of geometric form in flow systems [1,2] that must be treated as malleable, i.e., as morphing structures. Nature and engineering are united in the search for optimized flow architecture. Ref. [3] describes the vascularization of smart materials with self-healing functionality. The configuration is two trees matched canopy to canopy, and has freedom to move as free as possible: channel orientations, channel sizes, and system sizes. As a general rule, the same principle that in engineering is used for optimization subject to constraints can be used to predict the natural flow architectures that surround us. The constructal optimization of multi-scale structures is treated in Refs [4] and [5].

Bejan's Constructal theory proved to be fully versatile and interdisciplinary. In Ref. [6], for example, it has been extended to hyperthermia cancer treatments. The crucial problem is to keep the temperature of the normal tissue surrounding the tumour below a certain threshold so that the temperature field has to be controlled. Ref. [7] describes the path from the older method of entropy generation minimization to constructal design: the link between energy destruction and entropy generation, as the basis of the improvement of thermodynamic performance and the generation of flow configuration.

In this paper we return to the original engineering focus of Constructal theory, which is the optimization of architecture. We consider the constructal design of an elliptical cavity intruding into a conducting rectangular solid shown in Fig.1. The external solid has uniform internal heat generation and adiabatic boundary conditions on the outer surfaces, while the cavity walls are isothermal.

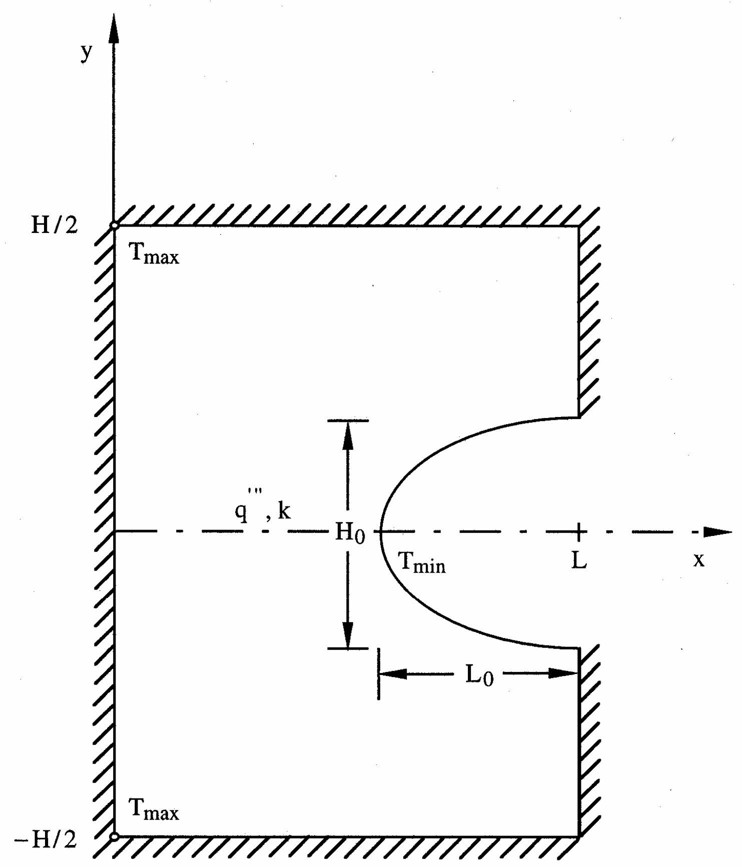


*Figure 1 – Isothermal cavity into a two-dimensional conducting body with uniform heat generation.*

We investigate the optimization of the entire system (solid and cavity) by means of the constructal method: the external shape of the cavity is free to change subject to volume constraints, and in the pursuit of maximal global performance. The indicator of global performance is the overall thermal resistance between the volume of the entire system (cavity and solid) and the surroundings. For sake of clarity, we consider two-dimensional geometries where the overall volume is a rectangular body and the cavity volume has an elliptical shape. The rectangular body and the elliptical cavity have variable geometric aspect ratios.

## 2. Numerical formulation and results

Consider the two-dimensional conducting body shown in Fig. 1. The external dimensions (H, L) vary. The third dimension, W, is perpendicular to the plane of the figure. The total volume occupied by this body is fixed,

$$V = HLW \tag{1}$$

Alternatively, we may say that the area A = HL is fixed. The dimensions of the cavity ($H_0$, $L_0$) vary. The cavity volume is fixed,

$$V_0 = \pi\, L_0\, H_0\, W/4 \qquad (2)$$

This volume constraint may be replaced by the statement that the volume fraction occupied by the cavity is fixed,

$$\phi = V_0 / V = \frac{\pi}{4}\frac{L_0 H_0}{LH} \qquad (3)$$

The solid is isotropic with the constant thermal conductivity k. It generates heat uniformly at the volumetric rate q'''[W/m$^3$]. The outer surfaces of the heat generating body are perfectly insulated. The generated heat current (q'''A) is removed by cooling the wall of the cavity. The cavity wall temperature is maintained at $T_{min}$. Temperatures rise to levels higher than $T_{min}$ in the solid. The highest temperatures (the "hot spots") are registered at points on the adiabatic perimeter, for example, in the two corners indicated with $T_{max}$ in Fig. 1.

A global constructal design constraint is the requirement that temperatures must not exceed a certain level. This makes $T_{max}$ a constraint. In the present problem statement, the design objective is represented by the maximization of the global thermal conductance $q'''A/(T_{max} - T_{min})$, or by the minimization of the global thermal resistance $(T_{max} - T_{min})/(q'''A)$. This objective is achieved through the generation of an optimal shape of the body geometry.

The numerical optimization of geometry consisted of simulating the temperature field in a large number of configurations, calculating the global thermal resistance for each configuration, and selecting the configuration with the smallest global resistance. Symmetry allowed us to perform calculations in only half of the domain, $y \geq 0$. The conduction equation for the solid region is

$$\frac{\partial^2 \tilde{T}}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{T}}{\partial \tilde{y}^2} + 1 = 0 \qquad (4)$$

where the dimensionless variables are

$$\tilde{T} = \frac{T - T_{min}}{q'''A/k} \qquad (5)$$

$$\left(\tilde{x}, \tilde{y}, \tilde{H}, \tilde{L}, \tilde{H}_e, \tilde{H}_0, \tilde{L}_0\right) = \frac{\left(x, y, H, L, H_e, H_0, L_0\right)}{A^{1/2}} \qquad (6)$$

The boundary conditions are indicated in Fig. 1. The maximal excess temperature, $\tilde{T}_{max}$ , is also the dimensionless global thermal resistance of the construct,

$$\tilde{T}_{max} = \frac{T_{max} - T_{min}}{q'''A/k} \qquad (7)$$

Equation (4) was solved using a finite elements code, based on triangular elements, developed in MATLAB environment, specifically, the pde (partial-differential-equations) toolbox [8]. The procedure to achieve mesh independence is the same used in Refs. [9, 10] and it is not shown.

The numerical work consisted of determining the temperature field in a large number of configurations of the type shown in Fig.1. Fig. 2 shows that the thermal resistance can be minimized by selecting the shape of the cavity. The thermal resistance decreases when the volume fraction ($\phi$) occupied by the cavity increases.

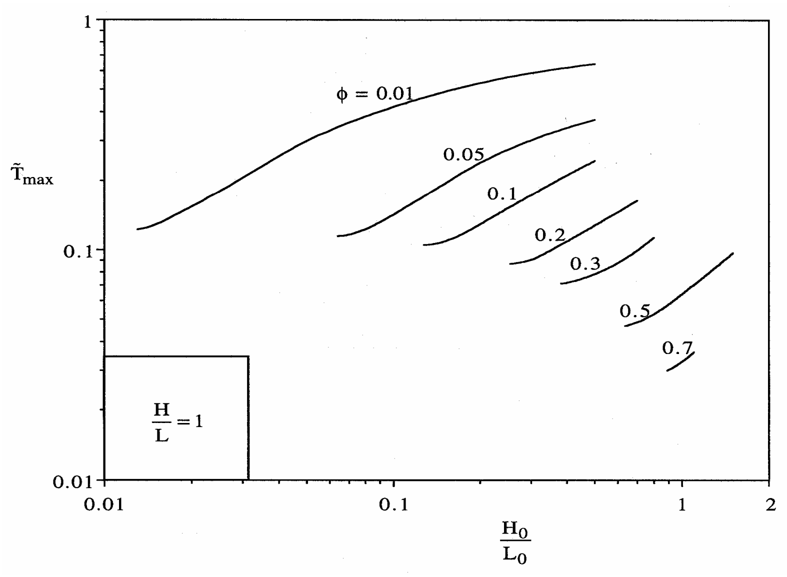


*Figure 2 – The minimization of the global thermal resistance when the external shape of the heat generating body is fixed.*

The results calculated in Fig. 2 are consolidated in Fig. 3, where the minimal (optimized) global thermal resistance and optimal internal shape of the elliptical cavity are shown as a function of the volume fraction of the cavity, $\phi$.

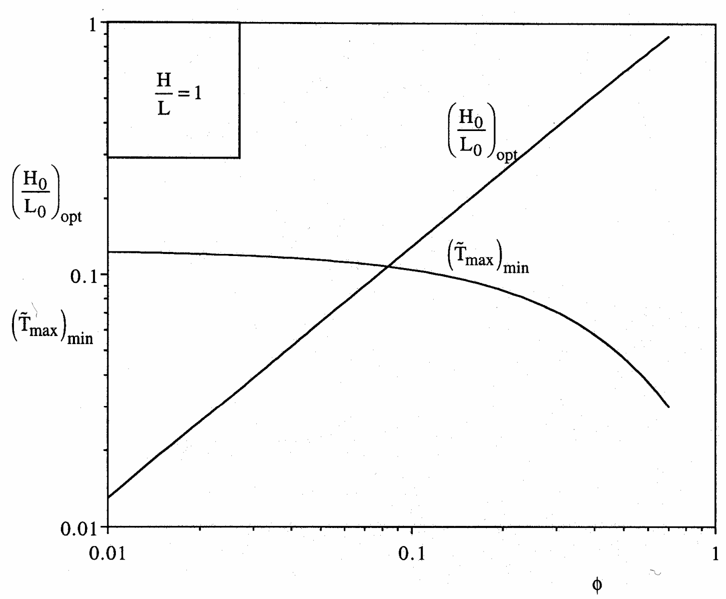


*Figure 3 – The optimized geometry and performance when the external shape is square.*

The second level of the numerical optimization scheme consisted of repeating the preceding work (Figs. 2 and 3) for several values of the second shape parameter, H/L. The most important finding (Fig. 4) is the existence of the optimal H/L ratio, i.e., the geometry of Fig. 1 can be optimized with respect to two degrees of freedom. The optimal ratio $(H/L)_{opt}$ is its smallest value possible for each value of the volume fraction $\phi$, i. e., the elliptical cavity performs better when it penetrates almost completely into the wall.

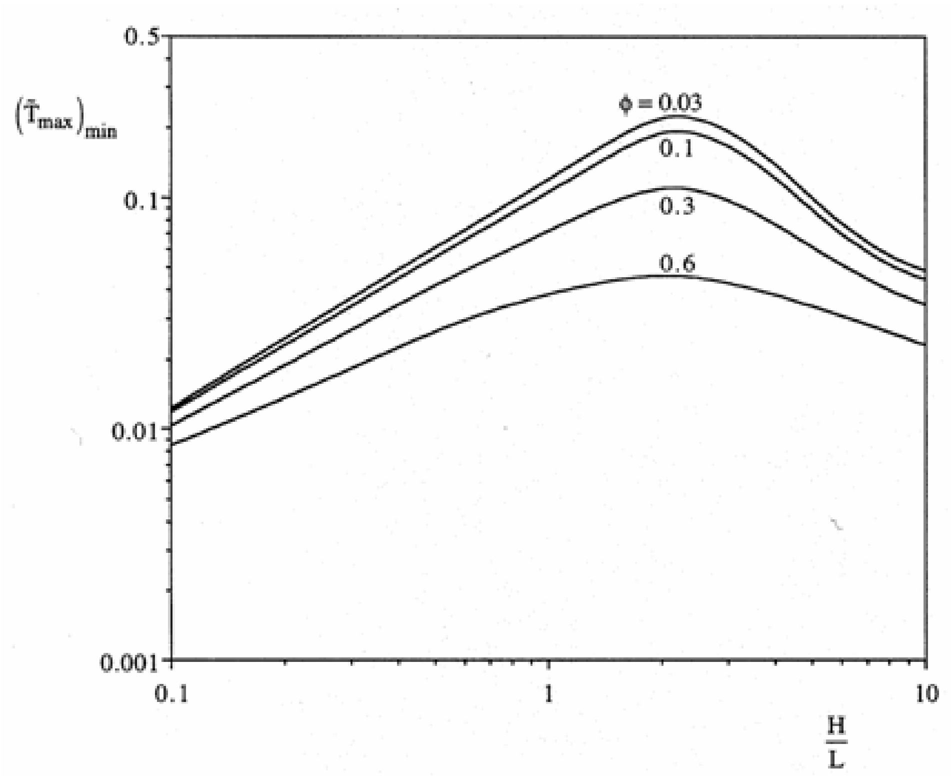


*Figure 4 – The effect of the external shape H/L on the global thermal resistance minimized as in Fig. 2.*

Figure 5 shows the optimal internal shape of the elliptical cavity corresponding to the global thermal resistances explored in Fig. 4. This figure shows that there is a unique best shape of the internal cavity for all (H/L) values, except for H/L = 2.

When H/L is equal to 2 can exist two possibilities for the optimal internal shape $(H_0/L_0)_{opt}$, depending on the value of the volume fraction. Figure 6 shows some of the best shapes for several values of the volume fraction $\phi$ and H/L = 2.

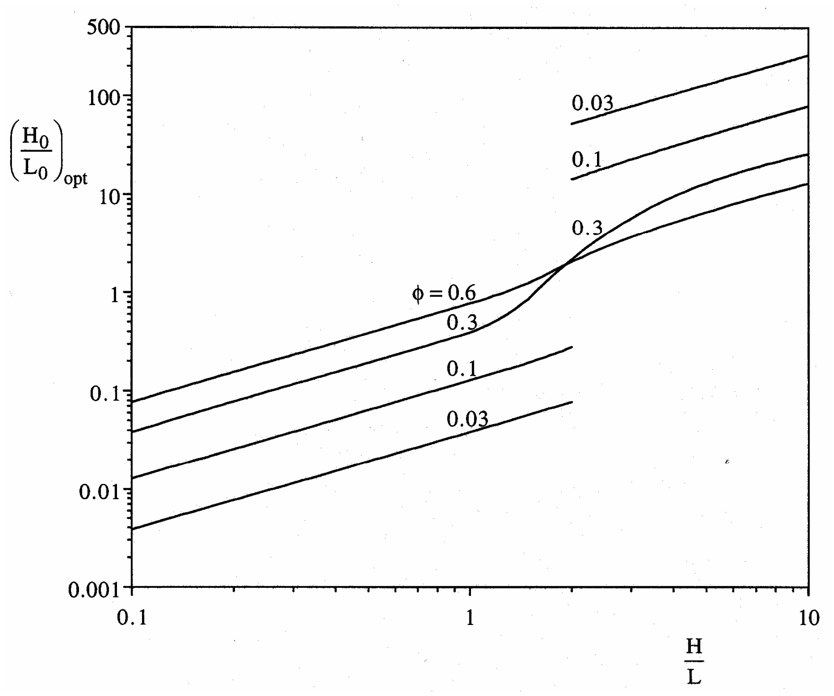


*Figure 5 – The effect of the external shape H/L on the optimal ratio $(H_0/L_0)_{opt}$ as in Fig. 2.*

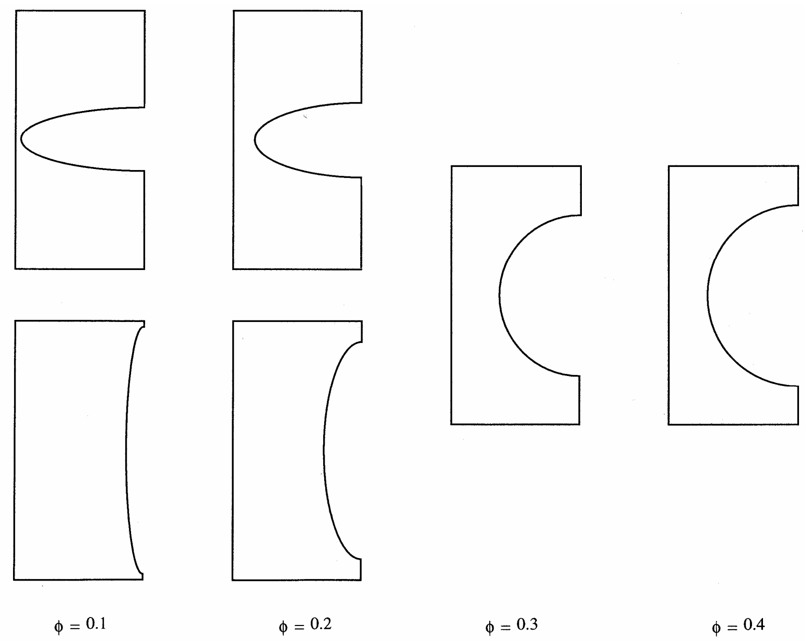


*Figure 6 – The best shapes calculated in Fig. 5 when H/L = 2.*

## 3. Concluding Remarks

We considered the original engineering focus of the constructal method, i.e. the optimization of architecture referred to a basic configuration: an isothermal elliptical cavity intruding into a rectangular conducting body. In the second step of the numerical investigation, a "double minimization" of the thermal resistance, i.e., the optimization of geometry with respect to all degrees of freedom has been contemplated. Based on observation, we found that the best cavity is the one that penetrates the solid completely and that the thermal resistance decreases when the volume fraction ($\phi$) occupied by the cavity increases. It is also interesting that there is an unique best shape of the internal cavity for all external shape (H/L) values, except for H/L = 2. When H/L is equal to 2 can exist two possibilities for the optimal internal shape of the elliptical cavity $(H_0/L_0)_{opt}$, depending on the value of the volume fraction.

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