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A new theoretical model for the spectra of turbulent kinetic energy which includes a proposal for the divergence of spectral energy

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Summary. — The paper solves the spectral equation of turbulent kinetic energy for a steady-state and stable horizontally homogeneous flow. Differently from latter approach it explicitly considers a model for the divergence of spectral energy due to pressure fluctuations. Using Green's function it is postulated that this divergence is a linear combination of the other terms of the spectral equation. Physically acceptable results indicate that it can be dependent only on the second and third derivatives of the spectrum function. The model's results show good agreement with previous studies as well as clear dependence with stability parameters according to the observations done by Kaimal *et al.*

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1. – Introduction

A fairly complete understanding of the spectral and cospectral behavior of turbulence in atmospheric boundary layer has emerged from observational studies conducted by various investigators over the last four decades. Only a comparatively small number of theoretical papers are concerned with the spectral dynamics, however.

The spectral characteristics of atmospheric turbulence are a balance of mechanical and buoyant production, turbulent transport, pressure effects and dissipation. In the case of a stationary and homogeneous flow this equation is written in the following symbolic form (a detailed explanation of the procedure for its derivation is given in sect. **2**):

$$M(k;\, z)\, \Omega(z) + \frac{g}{T_0} H(k;\, z) + T_e(k;\, z) + T_{we}(k;\, z) - 2\nu k^2 E(k;\, z) = 0\,. \tag{1}$$

The first and second terms in eq. (1) represent the energy-feeding of turbulent flow by the mean flow due to mechanical and buoyant effects. $M(k;z)$ is the spectral momentum flux and $H(k;z)$ is the spectral temperature flux. The third term embodies the spectral transfer of kinetic energy. The fourth term includes the divergence of the vertical transport of kinetic energy by pressure fluctuations. The last term describes the molecular dissipation of velocity fluctuations.

In order to solve this equation, a closure hypothesis relating the different terms with the spectral turbulent kinetic energy is necessary. The closure problem as applied to the atmospheric spectral distribution of energy was first stated in the pioneering works of Von Kármán (1937), Onsager (1945), Heisenberg (1948) and Batchelor and Towsend (1949). Closure models rely heavily on physical interpretation and good intuition.

Different spectral models were constructed based on the balance equation (Straka *et al.* (1978), Moraes and Goedert (1988), Moeng and Wyngaard (1989), Moraes *et al.* (1992), Schumann (1994)). The usual closure hypothesis is to relate the unknown spectra to the spectra of turbulent kinetic energy.

In previous work (Moraes and Goedert (1988), Moraes *et al.* (1992)), Onsager's hypothesis (1945) is used to define the spectral transfer in atmospheric inhomogeneous and anisotropic turbulence and extended to close the terms that embody the mutual interaction of mean and turbulent flow. The results of these works show functions in a very good agreement with experimental measurements. Thus, it can be concluded that the spectral density equation which arises from the considerations made by them is a good expression to describe the energy spectrum. However, none of the models consider the influence of the spectral divergence term ($T_{we}(k;z)$) in the spectral equation.

It will be shown in sect. **2** that T_{we} is the term that involves the pressure-velocity covariances from Reynolds stress equations. Unfortunately, neither theory, laboratory experiment, nor field observations have successfully revealed the behavior of these covariances (Mc Bean and Elliott (1975), Moeng and Wyngaard (1986)). Physically, this is the term that describes the distribution of energy among the three movement components. It is assumed in this work, as a first hypothesis, that T_{we} plays an important role in the spectral equation, even though Moraes and Goedert (1988) and Moraes *et al.* (1992) results might suggest that it is not true. A second hypothesis that is considered is that the results of the mentioned works are a good expression for the spectral density function. It might seem that these are contradictory hypothesis, but the idea presented here is that the terms in the spectral equation behave in a different way than these works propose. This behavior must be such that once T_{we} is considered, the final result does not change. In other words, it is proposed here that if those previous works resulted in good expressions for the spectral density function, and T_{we} was not considered, it happened because the spectral divergence was naturally modeled when the parameterizations of the others terms were done. This process makes possible the derivation of a mathematical expression for T_{we}. Such an expression does not currently exist, as a consequence of difficulty in measuring pressure fluctuations.

In sect. **2**, the spectral equation is derived, in sect. **3**, it is shown that the idea that T_{we} can be modeled by parameterizing other terms has a mathematical support, through the use of Green's function definition. In sect. **4** the spectral model is presented, including the parameterizations for all terms of the spectral equation and, specially, for T_{we}. In sect. **5** the differential equation for the energy spectrum and its solution by the power series method are presented. Finally, in sect. **6**, the results of this model are presented and qualitatively compared to field measurements.

2. – The spectral energy equation

The spectral characteristics of atmospheric turbulence are generally interrelated to the mean flow quantities, the vertical gradients of mean velocity and temperature. In the case of a steady-state, horizontally homogeneous flow, the interaction among the spectrum of turbulent kinetic energy $E(k; z)$ as a function of wave number k and height z above ground, the vertical mean flow gradient $\mathrm{d}U/\mathrm{d}z$ of a mean flow U in the x-direction, and the vertical temperature gradient $\mathrm{d}T/\mathrm{d}z$ may be expressed by the dynamic equation of the energy spectrum. The starting point is Boussinesq's approximation as employed by Rotta (1972), who derived the following equation for the velocity covariances in a thermally stratified turbulent medium:

$$[(U_l)_B - (U_l)_A]\,\frac{\partial Q_{i,j}}{\partial \xi_l} + \left(\frac{\partial U_i}{\partial x_l}\right)_A Q_{l,i} + \left(\frac{\partial U_j}{\partial x_l}\right)_B Q_{i,l} + \frac{\partial}{\partial \xi_l}(S_{i,lj} - S_{il,j}) = \tag{2}$$

$$= -\frac{1}{\varrho}\left(\frac{\partial K_{i,B}}{\partial \xi_j} - \frac{\partial K_{A,j}}{\partial \xi_i}\right) + \frac{g}{T_0}(T_{A,j}\delta_{i,3} + T_{i,B}\delta_{j,3}) + 2\nu^2\,\frac{\partial^2 Q_{i,i}}{\partial \xi_l \partial \xi_l}\,.$$

In eq. (2) i, j, l stand for the three main directions of an orthogonal coordinate system, $(U_i)_A$ is the component of the mean velocity at position A, $(x_l)_A$ is the l component of the position vector of point A, ξ_j is the separation between points A and B along the direction j, $\delta_{i,j}$ is the Kronecker delta symbol, ϱ is the mass density and ν is the kinematic viscosity of the air. Einstein's summation convention is adopted in eq. (2) and in what follows. In the foregoing equation K and T are first-order tensors defined by

$$K_{i,B} = \overline{(u_i)_A p_B}\,, \tag{3}$$

$$T_{i,B} = \overline{(u_i)_A \theta_B}\,, \tag{4}$$

where $(u_i)_A$ is the turbulent velocity component in the x-direction measured at position A and p_B and θ_B are turbulent components of pressure and temperature, respectively, at position B. The second-rank tensor Q and third-rank tensor S are defined by

$$Q_{i,j} = \overline{(u_i)_A (u_j)_B} \tag{5}$$

and

$$S_{i,jl} = \overline{(u_i)_A (u_j)_B (u_l)_B}\,. \tag{6}$$

A comma among indices in the above equations separates values measured at position A from values measured at position B.

Local time dependence and Coriolis' force effects are not included in eq. (2). This simplification is justified by a time scale criterion: the time scale for those contributions is much greater than the turbulence time scale. The correlation tensors K, T, Q and S are dependent on ξ, the separation between the points where the different quantities are considered. These quantities also depend on z, the height above ground, a fact that reflects the inhomogeneity of turbulence in the vertical direction.

A situation with no vertical mean velocity and where the horizontal components depend only on z is chosen to proceed with the analysis (*i.e.* horizontally homogeneous flow with no subsidence). The contraction of indices i, j in eq. (2) yields

$$\text{(7)}\quad \left[(U_A - U_B)\frac{\partial}{\partial \xi_x} + (V_A - V_B)\frac{\partial}{\partial \xi_y}\right] Q_{i,i}(\xi;z) +$$

$$+\left(\frac{\partial U}{\partial z}\right)_A Q_{3,1}(\xi;z) + \left(\frac{\partial V}{\partial z}\right)_A Q_{3,2}(\xi;z) + \left(\frac{\partial U}{\partial z}\right)_B Q_{1,3}(\xi;z) + \left(\frac{\partial V}{\partial z}\right)_B Q_{2,3}(\xi;z) +$$

$$+\frac{\partial}{\partial \xi_l}[S_{i,lj}(\xi;z) - S_{il,j}(\xi;z) = -\frac{1}{\varrho}\left[\frac{\partial K_{i,B}(\xi;z)}{\partial \xi_i} - \frac{\partial K_{A,i}(\xi;z)}{\partial \xi_i}\right] +$$

$$+\frac{g}{T_0}[T_{A,3}(\xi;z) + T_{3,B}(\xi;z)] + 2\nu\frac{\partial^2 Q_{i,i}(\xi;z)}{\partial \xi_l \partial \xi_l}.$$

Finally an equation for the spectral balance of energy is obtained from the Fourier transform in ξ-space of eq. (7) integrated over a surface of constant wave number k, that is, over a sphere in Fourier space. The resulting equation has the following form:

$$\text{(1)}\quad M(k;z)\,\Omega(z) + \frac{g}{T_0}H(k;z) + T_e(k;z) + T_{we}(k;z) - 2\nu k^2 E(k;z) = 0\,,$$

where

$$\text{(8)}\quad M(k;z) = \frac{1}{\Omega(z)}\oint_{k_n k_n} \mathrm{F}\left\{[\mathrm{i}K_x(U_A - U_B) + \mathrm{i}k_y(V_A - V_B]\,Q_{i,i} - \right.$$

$$\left. -\left(\frac{\partial U}{\partial z}\right)_A Q_{3,i} - \left(\frac{\partial V}{\partial z}\right)_A Q_{3,2} - \left(\frac{\partial U}{\partial z}\right)_B Q_{i,3} - \left(\frac{\partial V}{\partial z}\right)_B Q_{2,3}\right\}\mathrm{d}\sigma\,,$$

$$\text{(9)}\quad H(k;z) = \oint_{k_n k_n} \mathrm{F}(T_{A,3} + T_{3,B})\,\mathrm{d}\sigma\,,$$

$$\text{(10)}\quad T_e(k;z) = \oint_{k_n k_n} \mathrm{i}k_x\,\mathrm{F}(S_{i,lj} - S_{il,j})\,\mathrm{d}\sigma\,,$$

$$\text{(11)}\quad T_{we}(k;z) = \frac{1}{\varrho}\oint_{k_n k_n} \mathrm{i}k_l\,\mathrm{F}(K_{l,B} - K_{A,l})\,\mathrm{d}\sigma$$

and

$$\text{(12)}\quad E(k;z) = \oint_{k_n k_n} \mathrm{F}(Q_{i,i})\,\mathrm{d}\sigma\,.$$

In eqs. (8)-(12) $d\sigma$ is the differential of the spherical surface in k-space, $i = \sqrt{-1}$, $\Omega(z)$ is the wind shear and F represents the Fourier transform of the indicated argument.

The terms in this equation are parameterized in terms of the second- or first-order moments in order to close the equation. One of these parameterizations involves pressure covariances, such as the pressure-velocity terms in the Reynolds stress equations.

3. – The basis for the new model

Various methods for solving the dynamical equation for the energy spectrum have been suggested. They are based on intuitive physical pictures of the mechanism for the transfer of energy between the different scales of the turbulent flow.

The start point of the present work is Moraes and Goedert's (1998) and Moraes *et al.*'s (1992) models. These models do not consider the spectral divergence of the turbulent kinetic energy T_{we} but both models provide good results. We hypothesize that parameterization of other terms in the turbulent kinetic energy equation (M, H and T_e) implicitly includes the parameterization of T_{we}. To illustrate it, consider the equation for the turbulent pressure p in an incompressible fluid, *i.e.* the Poisson equation, which is derived by taking the divergence of the equation for the fluctuating velocity filed:

$$\nabla^2 p = \frac{\varrho_0 g}{\theta_0}\frac{\partial\theta}{\partial z} + \varrho_0 \frac{\partial^2 u_i u_j}{\partial x_i \partial x_j} + \varrho_0 \frac{\partial^2 \overline{u_i u_j}}{\partial x_i \partial x_j} - 2\varrho_0 \frac{\partial U_j}{\partial x_i}\frac{\partial u_i}{\partial x_j}, \tag{13}$$

where ϱ_0, g, θ_0, u_i and U represent the mean air density, gravitational acceleration, temperature near the ground, turbulent velocity and mean velocity, respectively. In this equation, Einstein's summation convention is used. Since the solution of (13) involves integration over the entire flow domain, the pressure fluctuation at a point depends on the flow field at all points in the flow domain. Equation (13) also indicates that there are three distinct contributions to pressure fluctuations: turbulent-turbulent interaction, mean shear, and buoyancy effects.

All variables in eq. (13) are considered at the same point r. Now, multiplying the above equation by the turbulent velocity at the point r' and averaging the resulting equation, we obtain

$$\frac{1}{\varrho_0}\nabla^2 \overline{p(r)\, u_l(r')} = \frac{g}{\theta_0}\frac{\partial \overline{\theta(r)\, u_l(r')}}{\partial z} + \frac{\partial^2 \overline{u_i(r)\, u_j(r)\, u_l(r')}}{\partial x_i \partial x_j} - 2\frac{\partial U_j(r)}{\partial x_i}\frac{\partial \overline{u_i(r)\, u_l(r')}}{\partial x_j}. \tag{14}$$

This equation can be rewritten as

$$\frac{1}{\varrho_0}\nabla^2 P_{l,\,r} = \frac{g}{\theta_0}\frac{\partial T_{l,\,r}}{\partial z} - \frac{\partial^2 S_{ij,\,l}}{\partial x_i \partial x_j} - 2\frac{\partial U_j}{\partial x_i}\frac{\partial Q_{i,\,l}}{\partial x_j}. \tag{15}$$

The solution of this equation is given in terms of Green's function defined as

$$\nabla^2 G(r, r') = -4\pi\delta(r - r'), \tag{16}$$

where δ is Dirac's function. The solution of (16) is

$$P_{l,r} = \frac{1}{4\pi} \int \varrho_0 \left(\frac{g}{\theta_0} \frac{\partial T_{l,r}}{\partial z} - \frac{\partial^2 S_{ij,l}}{\partial x_i \partial x_j} - 2 \frac{\partial U_j}{\partial x_i} \frac{\partial Q_{i,l}}{\partial x_j} \right) G(r, r') \, d^3 x \,. \tag{17}$$

From a mathematical point of view, this solution establishes a functional relationship amongst the variables, given by

$$P_{l,r} = P_{l,r}(Q_{i,l}, S_{ij,l}, T_{l,r}) \,. \tag{18}$$

The above relationship is applicable only in real space. Now, it is postulated that an equivalent relationship in Fourier space, *i.e.*

$$T_{we} = T_{we}(M(k), T_e(k), H(k)) \tag{19}$$

also exists. This functional form shows that the parameterization of $M(k)$, $T_e(k)$ and $H(k)$ includes the parameterization of T_{we}.

A possible form for T_{we} is

$$T_{we}(k) = A_{ij} \Phi_j(k) \,, \tag{20}$$

where $\Phi_1 = M(k)$, $\Phi_2 = T_e(k)$, $\Phi_3 = H(k)$ and

$$A_{ij} = \sum_{i=0}^{\infty} a_{ij} \frac{\partial^i}{\partial k^i} \tag{21}$$

with a_{ij} being constants of proportionality.

4. – The spectral model

In order to solve eq. (1), it is necessary to formulate a closure hypothesis relating $M(k; z)$, $T_e(k; z)$, $H(k; z)$ and $T_{we}(k; z)$ to $E(k; z)$.

For $M(k; z)$ and $H(k; z)$ the same approach followed in Moraes *et al.* (1992) is adopted. The mechanical production is a function of the wind shear $\partial U / \partial z$, the wave number k, the mean dissipation rate ε, and the spectrum function E. Similarity analysis leads to

$$M(k; z) = c_1 \frac{\partial U}{\partial z} \varepsilon^{-1/3} k^{-2/3} E(k; z) \,, \tag{22}$$

where c_1 is a proportionality constant.

From the same reasoning, the thermal destruction/production term can be written as a function of the vertical temperature gradient, the dissipation rate, the wave number, and the spectrum function and is given by

$$H(k; z) = c_2 \frac{\partial \Theta}{\partial z} \varepsilon^{-1/3} k^{-2/3} E(k; z) \,, \tag{23}$$

where c_2 is a proportionality constant.

The spectral transfer term $T_e(k;z)$ is defined as proportional to the spectral divergence of the spectral flux Ψ, or

$$T_e(k;z) = -\frac{\partial \Psi}{\partial k}. \tag{24}$$

The spectral flux Ψ is a function of the wave number, the dissipation rate and the spectrum function. Combining it with eq. (24) leads to

$$T_e(k;z) = \alpha^{-1}\varepsilon^{1/3}\left(\frac{5}{3}k^{2/3}E + k^{5/3}\frac{\partial E}{\partial k}\right), \tag{25}$$

where α is the Kolmogorov constant.

Monin-Obukhov similarity theory defines the following universal gradients:

$$\Phi_\varepsilon = \frac{\kappa z \varepsilon}{u_*^3}, \tag{26}$$

$$\Phi_m = \frac{\kappa z}{u_*}\frac{\partial U}{\partial z}, \tag{27}$$

$$\Phi_h = \frac{\kappa z}{\theta_*}\frac{\partial \Theta}{\partial z}, \tag{28}$$

where κ is the von Kármán constant and u_* and θ_* are, respectively, velocity and temperature scales for the surface boundary layer.

Substituting eqs. (26)-(28) allows eqs. (22), (23) and (25) to be rewritten as

$$M\frac{\partial U}{\partial z} = \frac{c_1}{\kappa^{5/3}}u_*\Phi_m^2\Phi_\varepsilon^{-1/3}(kz)^{-5/3}(kE), \tag{29}$$

$$H\frac{g}{T_0} = \frac{c_2}{\kappa^{5/3}}u_*\left(\frac{z}{L}\right)\Phi_h\Phi_\varepsilon^{-1/3}(kz)^{-5/3}(kE), \tag{30}$$

$$T_e = \frac{u_*\Phi_\varepsilon^{1/3}}{\alpha\kappa^{1/3}}(kz)^{-1/3}\left[\frac{2}{3}(kE) + (kz)\frac{\partial(kE)}{\partial(kz)}\right]. \tag{31}$$

In the above equations, all terms are expressed as functions of kE and $\partial(kE)/\partial(kz)$, multiplying different powers of kz. It allows eq. (20) to be rewritten as

$$T_{we} = \sum_{i=0}^{\infty} A_i\, f(kz)\frac{\partial^i(kE)}{\partial(kz)^i}, \tag{32}$$

where $f(kz)$ is a generic function of kz.

5. – The spectral equation and its solution

We now substitute the closure hypothesis concerning $M(k;z)$, $H(k;z)$, $T_e(k;z)$ and $T_{we}(k;z)$ into eq. (1) and obtain the equation for $E(k;z)$:

$$\sum_{i=0}^{\infty} A_i\, f(x) \frac{\mathrm{d}^i y}{\mathrm{d}x^i} + (Bx^{2/3}) \frac{\mathrm{d}y}{\mathrm{d}x} + (Cx^{-5/3} + Dx^{-1/3} + Ex)\, y = 0\,, \tag{33}$$

where

$$y = \frac{kE}{u_*^2}\,, \tag{34}$$

$$x = kz\,, \tag{35}$$

$$B = \frac{\Phi_\varepsilon^{1/3}}{\alpha \kappa^{1/3}}\,, \tag{36}$$

$$C = \frac{\Phi_\varepsilon^{-1/3}}{\kappa^{5/3}} \left[\left(\frac{4}{3} c_5 + 1 \right) \left(c_1 \Phi_m^2 + c_2 \Phi_h \frac{z}{L} \right) \right], \tag{37}$$

$$D = \frac{2}{3} \frac{c_4 \Phi_\varepsilon^{1/3}}{\alpha \kappa^{1/3}}\,, \tag{38}$$

$$E = -2 \frac{\nu\kappa}{u_*} = -\frac{2}{\mathrm{Re}}\,. \tag{39}$$

The solution of eq. (33) is assumed to be a power series of the form

$$y = x^{-2/3} \sum_{j=0}^{\infty} a_j x^{-4j/3}. \tag{40}$$

This expression is equivalent to the solution obtained in Moraes and Goedert (1988) and Moraes *et al.* (1992).

5.1. *A first approximation for* T_{we}. – Before substituting the general form of the solution in the differential equation, it is necessary to find an exact closure for T_{we}. This expression must agree with eq. (32). One first guess consists in taking this term as dependent only on the derivatives of the spectrum function with respect to the wave number, with constant coefficients. In this case, $f(kz) = 1$ in eq. (32).

When it is substituted into (33), the resultant series is given by

$$\sum_{i=0}^{\infty} \left[A_i \sum_{j=0}^{\infty} g(j^i)\, a_j x^{-\frac{4j+3i}{3}} \right] - \frac{B}{3} + C \sum_{j=0}^{\infty} a_j x^{-\frac{4j+5}{3}} + D \sum_{j=0}^{\infty} a_j x^{-\frac{4j+1}{3}} + E \sum_{j=0}^{\infty} a_j x^{-\frac{4j-3}{3}} = 0\,,$$

where $g(j^i)$ is a generic function of j. In this expansion, the coefficient of each power of x must vanish separately. The general j term is of the form

$$\sum_{i=0}^{\infty} A_i g(j^i)\, a_j x^{-\frac{4j+3i}{3}} - \frac{B}{3}(4j+2)\, a_j x^{-\frac{4j+1}{3}} + C a_j x^{-\frac{4j+5}{3}} + D a_j x^{-\frac{4j+1}{3}} + E a_j x^{-\frac{4j+5}{3}}.$$

From the term corresponding to 1/3 power (*i.e.* $j = 0$),

$$Ea_0 x^{1/3} = 0 \, . \tag{41}$$

In the PBL, $E \approx 1/\mathrm{Re} \approx 10^{-7}$ and it is possible to consider, as a good approximation, $a_0 \neq 0$ by assuming $E = 0$. This consideration implies that the model does not consider the dissipation and, therefore, does not provide a good description of $E(k; z)$ for large wave numbers. However, T_{we} is not expected to be important for these wave numbers since, according to Kolmogorov's second hypothesis, only the spectral transference (T_e) and the dissipation govern the spectral dynamics of these small eddies.

The term corresponding to zero power gives

$$A_0 a_0 x^0 = 0 \, . \tag{42}$$

In order to have $a_0 \neq 0$, $A_0 = 0$ and the first term in eq. (32) is null, *i.e.* T_{we} is dependent only on the derivatives of the spectrum function.

The term corresponding to $x^{-1/3}$ leads to

$$-\frac{2}{3} B + D = 0 \, . \tag{43}$$

In x^{-1}:

$$Ea_1 + A_1 g(j)\, a_0 = 0 \, . \tag{44}$$

Note that, once $E \approx 0$, it does not imply that $a_1 = 0$. On the other hand, as $a_0 \neq 0$, then $A_1 = 0$ With this condition, T_{we} cannot depend on the first derivative of the turbulent kinetic energy.

From the $-5/3$ power it is found that

$$[Ca_0 + (-B + D)\, a_1 + Ea_2]\, x^{-5/3} = 0 \tag{45}$$

and consequently

$$a_1 = -\frac{C}{D - B} a_0 . \tag{46}$$

From the -2 power:

$$A_2 g(j^2)\, a_0 x^{-2} = 0 \, . \tag{47}$$

Since $a_0 \neq 0$, then $A_2 = 0$ and the term dependent on the second derivative is also equal to zero.

From the -3 power:

$$\left[-g(j^3)\, A_3 a_0 + Ca_1 + \left(-\frac{4}{3} B + D \right) a_2 + Ea_3 \right] x^{-3} = 0 \tag{48}$$

and the following relation arises:

$$a_2 = -\frac{[Ca_1 - (80/27)\, A_3 a_0]}{(-(4/3)\, B + D)} \, . \tag{49}$$

From the above equation it is possible to see that the term dependent on the third derivative is different from zero. From the same analysis for the other powers, it is observed that the coefficients $A_0, A_1, A_2, A_4, A_5, A_6, A_8, A_9, A_{10}, \ldots$ must be zero and the coefficients $A_3, A_7, A_{11} \ldots, A_{4n-1}, \ldots,$ can be different from zero. From (32) the general expression for T_{we} is given as

$$T_{we} = A_3 g(j^3) \frac{\partial^3 (kE)}{\partial (kz)^3} + A_7 g(j^7) \frac{\partial^7 (kE)}{\partial (kz)^7} + \ldots + A_{4n-1} g(j^{4n-1}) \frac{\partial^{4n-1} (kE)}{\partial (kz)^{4n-1}} + \ldots . \tag{50}$$

The recurrence relation between successive coefficients in the series of eq. (40) has the form

$$a_j = K_0 a_{j-1} + K_1 g(j^3) A_3 a_{j-2} + K_2 g(j^7) A_7 a_{j-3} + \ldots + K_n g(j^{4n-1}) A_{4n-1} a_{j-n-1}, \tag{51}$$

where the K's coefficients are functions of B, C, D and E. A condition for the series to convergence is that a_j has to decrease with increasing j. Consequently, $A_n \ll A_{n-1}$. It is convenient to choose only A_3 different from zero because the other A's are multiplying functions with high powers of j and this could lead to a divergent series. Hence, a simplified model for T_{we} becomes

$$T_{we}(kz) = A_3 \frac{\partial^3 (kE)}{\partial (kz)^3} . \tag{52}$$

From the above model, it is possible to derive the recurrence relation between the a_j's from eq. (51). It is a straightforward matter to show that

$$a_j = - \frac{\left[C a_{j-1} - \dfrac{(4j-6)(4j-3)(4j)}{27} a_{j-2} \right]}{\left(- \dfrac{4j+2}{3} B + D \right)} . \tag{53}$$

Equations (40) and (53) are the solution to the turbulent kinetic energy equation.

5·2. *An improved approximation for* T_{we}. – In the previous subsection, a model for T_{we} with the generic function $f(kz)$ in eq. (32) equal to one was hypothesized. Subsequent analysis showed that once this hypothesis is assumed, the only possibility for T_{we} is to be proportional to the third derivative of the spectrum function. Also, for small values of (kz), each term of the series tends to infinity. As (kE) must be finite, it implies that the a_j coefficients must approach zero faster than $x^{-4j/3}$. A consequence is that A_3 must be very small $(O(10^{-4}))$ for this condition to be satisfied (Acevedo, 1995). With this order of magnitude for A_3, the T_{we} term in eq. (33) results in a very small spectral divergence term relative to the other terms. To avoid this problem, it is desirable to have a recurrence relation in which the constant which multiplies T_{we} is no longer in the numerator, but in the denominator of the expression. In this case, large values of this constant will help to make the a_j values approach zero faster than $x^{-4j/3}$.

This kind of solution can be achieved using adequate values for $f(kz)$ in eq. (32). Furthermore at least two derivatives of (kE) with respect to (kz) must exist in (50). The way to decide on the optimum choice is to consider an acceptable approximation for the spectral divergence term. In Mc Bean and Elliot (1975) measurements lead to a spectral divergence due to pressure-velocity correlations that has a positive peak at small wave number, followed by a negative and another positive peak as the wave number increases. The derivatives of the spectrum function that have this shape are the second and the third. So, the following expression for T_{we} is postulated:

$$T_{we} = A_2 (kz)^{5/3} \frac{\partial^2 (kE)}{\partial (kz)^2} + A_3 (kz)^{8/3} \frac{\partial^3 (kE)}{\partial (kz)^3}\,, \tag{54}$$

where $f(kz)$ were chosen so that A_2 would be in the denominator of the recurrence relation. Inserting (54) into (1):

$$\sum_{j=0}^{\infty} \left[-\frac{A_3}{27}(4j+2)(4j+5)(4j+8) + \frac{A_2}{9}(4j+2)(4j+5) - \frac{B}{3}(4j+2) + D \right] a_j x^{-\frac{4j+1}{3}} + \\ + \sum_{j=0}^{\infty} C a_j x^{-\frac{4j+5}{3}} + \sum_{j=0}^{\infty} E a_j x^{-\frac{4j-3}{3}} = 0\,. \tag{55}$$

Equation (41) is again achieved when analyzing the exponent $x^{1/3}$. So, again, dissipation will not be considered by assuming $E = 0$. The analysis for $x^{-1/3}$ leads to

$$E a_1 - \frac{80}{27} A_3 a_0 + \frac{10}{9} A_2 a_0 - \frac{2}{3} B a_0 + D a_0 = 0\,. \tag{56}$$

As $E = 0$ and $D = 2B/3$, the above equation leads to the following relation between the coefficients of the derivatives that take part in the T_{we} expression:

$$A_3 = \frac{3}{8} A_2. \tag{57}$$

From the above equation, it is clear why it is necessary to choose at least two derivatives in the postulated expression for T_{we}. If only one derivative had been considered, it would be necessary to have the coefficient of this derivative equal to zero.

The analysis of the other exponents in x leads to the following recurrence relation between the a_j coefficients:

$$a_j = -\frac{(-3C)\, a_{j-1}}{j \left[\dfrac{(4j+2)(4j+5)\, A_2}{6} + 4B \right]}\,. \tag{58}$$

This result combined with eq. (40) constitutes one model for the spectrum of turbulence in a stratified atmospheric surface layer. Comparing the preceding model, this one has the advantage of converging for any value of A_2.

6. – Results and discussion

Equation (40) is the solution of the spectrum function. Two recurrence relations to the a_j coefficients were achieved, given by eqs. (55) and (60). Here, only the results from the second hypothesis are shown, once the first solution, as already discussed, imposes a very small coefficient multiplying T_{we}, making its contribution negligible.

Before showing the results it is necessary to find the values of the coefficients A_2, c_1 and c_2 for different values of the stability parameters. It is also necessary to assume some specific forms for the nondimensional similarity functions. Different experimental works were made in order to provide expressions to the universal functions in terms of the stability parameter z/L. In this work Businger *et al.* (1971) expressions are used:

$$
\begin{cases} \Phi_m = 1 + 4.7z/L \,, \\ \Phi_h = 1 + 3.7z/L \,, \\ \Phi_\varepsilon = 1.25(1 + 4.7z/L) \,. \end{cases} \tag{59}
$$

In the case of neutral stability ($z/L = 0$), there is no thermal production or destruction and thus $c_2 = 0$. The values of A_2 and c_1 must be adjusted so that the peak of the spectrum occurs at similar wave numbers as those found in the literature. As was discussed, the solution converges independently of the values of A_2 chosen, provided it is negative. The approach used here is to choose different values of A_2, with adequate value of c_1 that makes the peak occur in the desired wave number range. Such analysis provides the following results: $c_1 = -1.63$ for $A_2 = -0.1$; $c_1 = -1.44$ for $A_2 = -0.05$; $c_1 = -1.35$ for $A_2 = -0.025$ and $c_1 = -1.21$ for $A_2 = -0.01$. The first value tested for A_2 was -0.1, arbitrarily chosen. The other values come from decreasing A_2 until a desirable result is achieved, as discussed next. Figure 1 shows the spectrum function for these values of the coefficients.

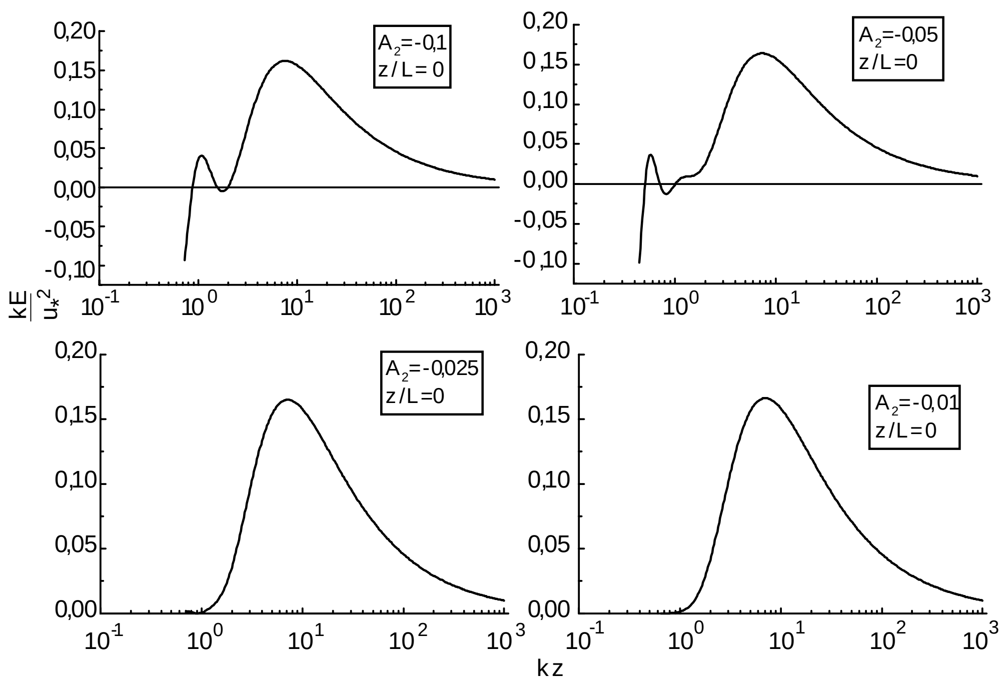

Fig. 1. – Spectrum function for different values of the A_2 coefficient and neutral stability.

In fig. 1, it is clear that the solution is not good for the cases when $A_2 = -0.1$ and $A_2 = -0.05$, once in the low wave numbers of the spectrum, these solutions show another maximum, not desirable. Physically, this feature can be considered as the result of an overestimation of T_{we}, in a way that makes the other terms behave differently than usual to keep the balance in the spectral equation. On the other hand, it does not happen when $A_2 = -0.025$ and $A_2 = -0.01$, showing that in this case T_{we} is not so large that it changes the spectral shape. The value which will be assumed here for this constant is $A_2 = -0.025$. It is better than $A_2 = -0.01$ because, for the latter, T_{we} is too small.

For other values of the stability parameter, the values of the constants continue to be $c_1 = -1.35$ and $A_2 = -0.025$. In this case, c_2 will be different from zero and will be -5.52 for $z/L = 0.2$ and -4.26 for $z/L = 0.5$. These numbers arise from the same considerations of making the peak in the computed spectrum agree with results found in the literature. The fact that c_2 is a function of z/L is natural as long as this is the constant that multiplies the thermal production/destruction term. Figure 2 shows the computed spectrum dependence on the stability parameter.

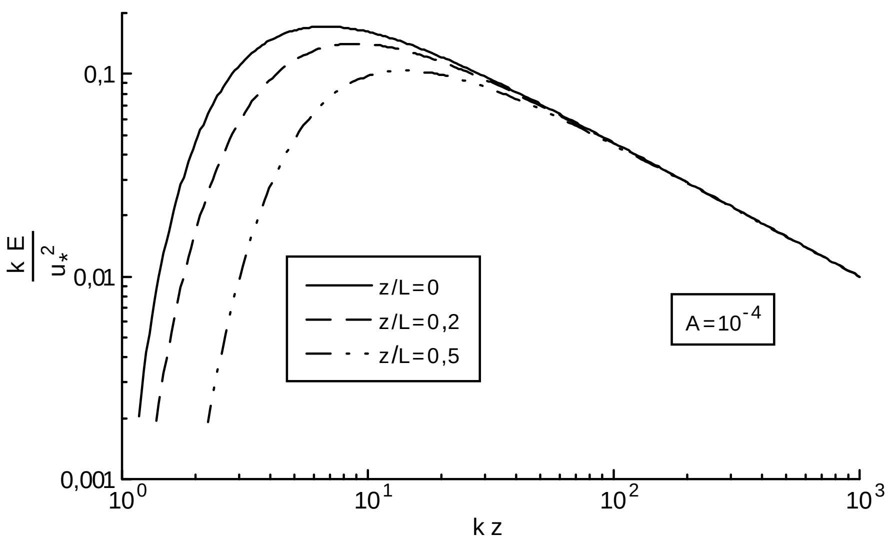

Fig. 2. – Spectrum function for different values of the stability parameter.

This figure shows the variation of the spectrum with the stability parameter. The measurements by Kaimal *et al.* (1972) suggest that this feature is to be expected in the atmospheric surface layer. It is important to stress that Kaimals' results are unidimensional while the present work provides a tridimensional solution.

The dependence of the terms of the spectral equation on the stability parameter is shown in fig. 3. It is clear that the increase in H as a destruction term coincides with an increasing in the stability parameter. It occurs along with a decrease of T_e, which is the other destruction term at low wave numbers. It is not possible to see the positive part of T_e which would be expected at high wave numbers. It is a direct consequence of the fact that this model does not consider the dissipation, which is negative for high wave numbers. This way, to keep the balance in the spectral equation, the positive part

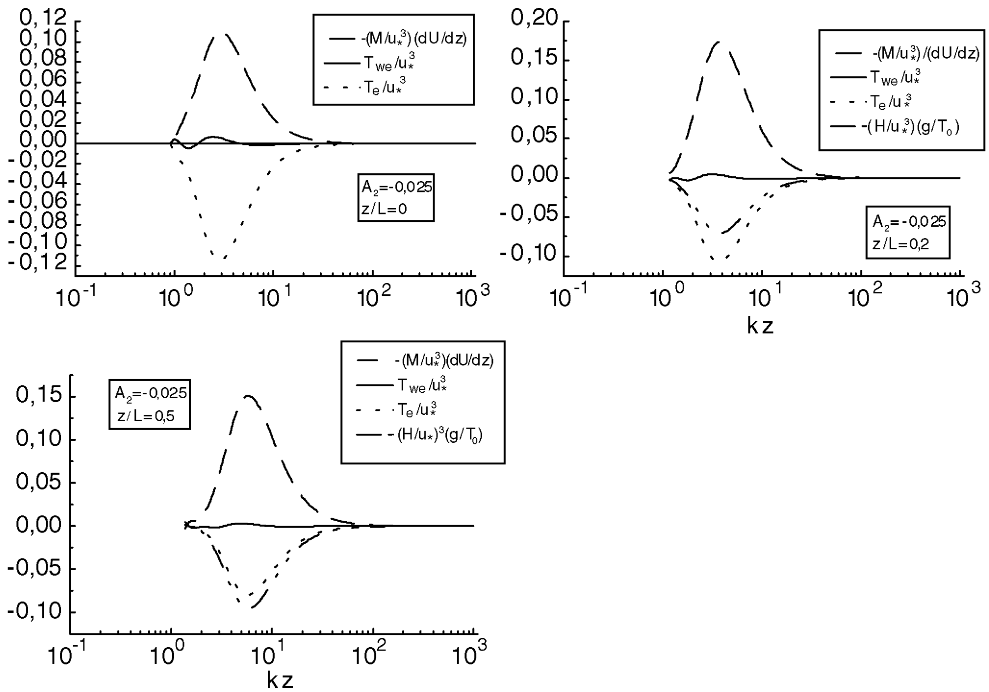

Fig. 3. – Schematic view of the terms of eq. (1) for different values of the stability parameter.

of T_e will also not appear. Regarding T_{we}, this figure shows that it is about an order of magnitude lower than the other terms.

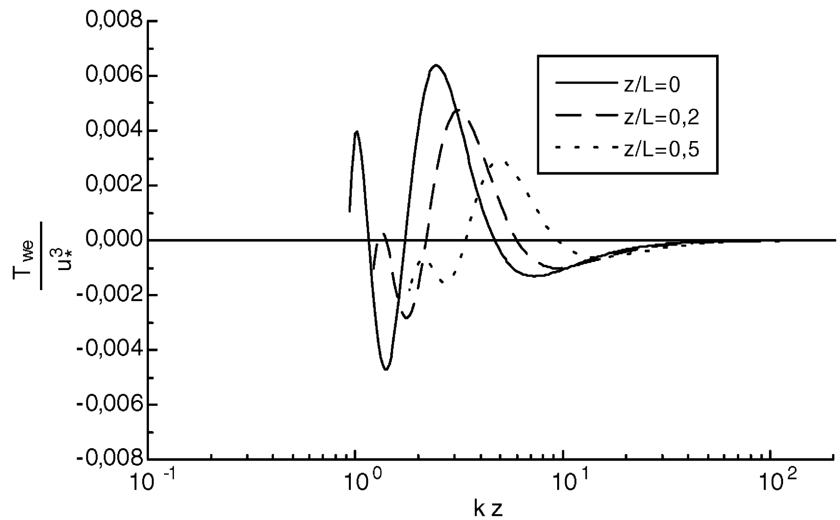

Fig. 4. – Spectral divergence due to pressure-velocity correlations and its variation with respect to the stability parameter.

Figure 4 shows a closer look in T_{we} and its dependence on the stability parameter. It can be seen that this term has a negative part at high wave numbers, becoming positive and negative again as wave numbers decrease. This term is responsible for a distribution of energy from lower and larger wave numbers' intervals to an intermediate range. Also there is a shift and decrease in the magnitude of the peaks. Another positive peak for low wave numbers appears, specially for the lower stability conditions. This, however, is a questionable feature. For very small wave numbers, the solution (eq. (40)) is highly dependent on the approximation in the series and can lead to undesirable results.

7. – Conclusions

In this work a model for the turbulent spectra in a stable surface layer which includes an expression for the spectral divergence due to pressure-velocity correlations was presented. This is the first expression proposed in the literature. Two solutions were proposed and described respectively by eqs. (40) with (53) and (40) with (58).

The final result agrees with previous models which do not explicitly consider the T_{we} term. It is a direct consequence of the fact that the starting point of this work is to assume that those results are a good description for the energy distribution in terms of the wave numbers, where T_{we} was already described in the parameterization of the other terms. Using Green's function definition, it was shown that this idea is mathematically founded.

It was postulated that T_{we} is a linear combination of different derivatives of the spectrum function with respect to the wave number. Further analysis, based on the spectral differential equation, allowed us to conclude that only some of these derivatives are important. An expression for this term, such that it depends on the second and third derivatives only, is proposed.

Finally, with the assumption made T_{we} is shown to be one order of magnitude smaller than the other terms in the spectral equation and that it has a shape consistent with Mc Bean and Elliott's (1975) measurements.

* * *

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