**On some connections between condensed matter and string theory. Mathematical connections with some sectors of Number Theory**

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**Abstract**

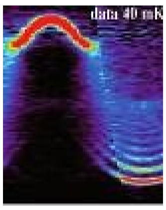
In this paper we have developed the recent research: *"Quantum criticality in an Ising chain: experimental evidence for emergent $E_8$ symmetry"* of R. Coldea et al. After that we have described, in the **Section 1**, some mathematical results of this work and some equations concerning the Ising Field Theory, we have showed, in the **Section 2**, some equations concerning the AdS/CFT and condensed matter physics to describe the superconductivity, thence the connections between matter condensed and string theory. Furthermore, in these sections, we have showed also the mathematical connections with some sectors of Number Theory, principally with the Ramanujan's modular equations and the aurea ratio (or golden ratio). Recently are appears many papers concerning some connections between condensed matter and string theory, using the AdS/CFT correspondence. We think that the research of Coldea et. al can be described by these kinds of researches.

## 1. Quantum resonances reveal "hidden" symmetry near quantum criticality [1]

Quantum phase transitions take place between distinct phases of matter at zero temperature. Experimental exploration of such phase transitions offers a unique opportunity to observe quantum states of matter with exceptional structure and properties.
One of the most theoretically studied paradigms for a quantum phase transition is the one-dimensional **Ising chain** in a transverse magnetic field. This system has been realized in experiments on the magnetic material CoNb2O6 placed in strong magnetic fields and cooled close to absolute zero. Cobalt atoms in the material act like tiny bar magnets arranged in long chains and exemplify ferromagnetism in one dimension. When a magnetic field is applied at right angles to the aligned spin directions the spins can "quantum tunnel" between the allowed "up" and "down" orientations and at a precise value of the field those quantum fluctuations "melt" the spontaneous ferromagnetic order and a quantum critical state is reached.
Near the critical point the spin excitations were theoretically predicted nearly two decades ago to have a set of resonances with an exceptional mathematical structure due to a "hidden" symmetry described by the group $E_8$, called by some "the most beautiful object in mathematics", which has not been seen experimentally before. Those quantum resonances in the spin chain are to be understood as the harmonic modes of vibration of the string of spins, where the string tension comes from the interactions between spins. These experiments indeed observed such a set of quantum resonances

and the ratio between the frequencies of the two lowest resonances approached the "**golden ratio**" $\left(1+\sqrt{5}\right)/2$ =1.618... near the critical point, as predicted by the $E_8$ model. These results emphasize that the exploration of quantum phase transitions can open up new avenues to experimentally realize otherwise inaccessible correlated quantum states of matter with complex symmetries and dynamics. In other words, the chain of spins behaves like a "magnetic guitar string" where the tension comes from interactions between spins. Near the critical field the two lowest frequencies were observed to approach the **golden ratio**, one of the key signatures of the predicted $E_8$ structure governing the magnetic spin patterns.

Recent experiments on quantum magnets suggest that quantum critical resonances may expose the underlying symmetries most clearly. Remarkably, the simplest of systems, the **Ising chain**, promises a very complex symmetry, described mathematically by the $E_8$ Lie group. Lie groups describe continuous symmetries and are important in many areas of physics. They range in complexity from the U(1) group, which appears in the low-energy description of superfluidity, superconductivity, and Bose-Einstein condensation, to $E_8$, the highest-order symmetry group discovered in mathematics, which has not yet been experimentally realized in physics.
A symmetry described by the $E_8$ Lie group with a spectrum of eight particles was long predicted to appear near the critical point of an **Ising chain**. It is possible to realize this system experimentally by using strong transverse magnetic fields to tune the quasi-one-dimensional Ising ferromagnet $CoNb_2O_6$ (cobalt niobate) through its critical point. Spin excitations are observed to change character form pairs of kinks in the ordered phase to spin-flips in the paramagnetic phase. Just below the critical field, the spin dynamics shows a fine structure with two sharp modes at low energies, in a ratio that approaches the **golden mean** predicted for the first two meson particles of the $E_8$ spectrum. These results demonstrate the power of symmetry to describe complex quantum behaviours.
In other words, the neutron data taken just below the critical field are consistent with the highly nontrivial prediction of two prominent peaks at low energies, which can be identify with the first two particles $m_1$ and $m_2$ of the off-critical **Ising model**. The ratio of the energies of those peaks varies with increasing field and approaches closely the **golden ratio** $m_2/m_1 = (1+\sqrt{5})/2 = 1.618...$ predicted for the $E_8$ masses. We identify the field where the closest agreement with the $E_8$ mass ratio is observed as the field $B_C^{1D}$ where the 1D chains would have been critical in the absence of interchain couplings. Indeed, it is in this regime that the special quantum critical symmetry theory would be expected to apply.




*What is $E_8$?*

There actually are 4 different but related things called $E_8$. $E_8$ is first of all the largest exceptional root (base) system, which is a set of vectors in an 8-dimensional real vector space satisfying certain properties. Root systems were classified by Wilhelm Killing in the 1890s. He found 4 infinite classes of Lie algebras, labelled $A_n$, $B_n$, $C_n$, and $D_n$, where n=1,2,3.... He also found 5 more *exceptional* ones: $G_2$, $F_4$, $E_6$, $E_7$, and $E_8$.

The $E_8$ root system consists of all vectors (called roots) $(a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8)$ where all $a_i$ are integers or all $a_i$ are integers plus 1/2, the sum is an even integer, and sum of the squares is 2. An example with all integers is (-1,0,1,0,0,0,0,0) (there are 112 of these) and an example with half-integers is (1/2,1/2,-1/2,-1/2,-1/2,1/2,1/2,-1/2) (there are 128 of these). $E_8$ has 240 roots. The 8 refers to the fact that there are 8 coordinates.

Secondly $E_8$ refers to the *root lattice* obtained by taking all sums (with integral coefficients) of the vectors in the root system. It consists of all vectors above with all $a_i$ integers, or all $a_i$ integers plus 1/2, and whose sum is even. The integers of squared length 2 are precisely the roots. This lattice, sometimes called the "8-dimensional diamond lattice", has a number of remarkable properties. It gives most efficient sphere-packing in 8 dimensions, and is also the unique *even, unimodular* lattice in 8 dimensions. This latter property makes it important in string theory.

Next $E_8$ is a semisimple Lie algebra. A Lie algebra is a vector space, equipped with an operation called the Lie bracket. A simple example is the set of all 2 by 2 matrices. This is a 4-dimensional vector space. The Lie bracket operation is [X,Y]=XY-YX.

$E_8$ is a 248-dimensional Lie algebra. Start with the 8 coordinates above, and add a coordinate for each of the 240 roots of the $E_8$ root system. This vector space has an operation on it, called the *Lie bracket*: if X,Y are in the Lie algebra so is the Lie bracket [X,Y]. This is like multiplication, except that it is not commutative. Unlike the example of 2 x 2 matrices, it is very hard to write down the formula for the Lie bracket on $E_8$.

This is a complex Lie algebra, i.e. the coordinates are complex numbers. Associated to this Lie algebra is a (complex) Lie group, also called $E_8$. This complex group has (complex) dimension 248. The $E_8$ Lie algebra and group were studied by Elie Cartan in 1894.

Finally $E_8$ is one of three real forms of the the complex Lie group $E_8$. Each of these three real forms has *real* dimension 248. The group which we are referring to in this web site is the split real form of $E_8$.

*Geometric description of the split real form of $E_8$*

We consider 16x16 real matrices X satisfying two conditions. First of all X is a rotation matrix, i.e. its rows and columns are orthonormal. Secondly assume $X^2=-I$. The set of all such matrices $V_0$ is a geometric object (a "real algebraic variety"), and it is 56-dimensional. There is a natural way to add a single circle to this to make a 57-dimensional variety V. (V=Spin(16)/SU(8), and is circle bundle over $V_0$, to anyone keeping score). Finally $E_8$ is a group of symmetries of V.
We note that 57 is a Lie number $L(7) = 7^2 + 7 + 1 = 49 + 7 + 1 = 57$, also of form $2T+1 = 2*28 + 1$ where 28 is a triangular number (but also perfect number) In this research, we have symmetry and aurea section (and thence Fibonacci's series). Furthermore, 57 dimensions = 55+2, with 55 that is a Fibonacci's number.

**Now we describe various equations concerning the Ising Field theory and some equations concerning the AdS/CFT and condensed matter physics to describe the superconductivity, thence the connections between matter condensed and string theory.**
**Indeed, recently are appears many papers concerning some connections between condensed matter and string theory, using the AdS/CFT correspondence. We think that the research of Coldea et. al can be described by these kinds of researches**.

1.1 **On some equations concerning the Ising Field Theory [2]**

The two-quark component is completely characterized by the following "fermionic wave function" $\Psi(p_1,p_2)$:

$$\left|\Psi^{(2)}\right\rangle = \frac{1}{2}\int_{-\infty}^{\infty}\frac{dp_1}{2\pi}\frac{dp_2}{2\pi}\Psi(p_1,p_2)\left|p_1,p_2\right\rangle. \quad (1.1)$$

By this definition, the wave function is antisymmetric, $\Psi(p_1,p_2) = -\Psi(p_2,p_1)$. If the multi-quark components in the equation

$$\left|\Psi\right\rangle = \left|\Psi^{(2)}\right\rangle + \left|\Psi^{(4)}\right\rangle + \left|\Psi^{(6)}\right\rangle + \ldots \quad (1.2)$$

are neglected, the eigenvalue problem $(\mathbf{H}-\mathrm{E})\left|\Psi_P\right\rangle = 0$ reduces to the integral equation

$$[\varepsilon(p_1)+\varepsilon(p_2)-\Delta E]\Psi(p_1,p_2) = f_0\int_{-\infty}^{\infty}2\pi\,\delta(p_1+p_2-q_1-q_2)\mathcal{G}(p_1,p_2|q_1,q_2)\Psi(q_1,q_2)\frac{dq_1}{2\pi}\frac{dq_1}{2\pi} \quad (1.3)$$

where at this point $\varepsilon(p)$ stands for the free quark energy $\varepsilon(p) = \omega(p) = \sqrt{m^2+p^2}$. In eq. (1.3) $\Delta E = E - E_v$ is the energy above the ground state, and

$$f_0 = 2\bar{\sigma}h = m^2\lambda \quad (1.4)$$

is the "string tension". In fact we will refer to the parameter $f_0$, eq. (1.4), as the "bare" string tension, as opposed to "dressed", or "effective" string tension $f$ which will replace $f_0$ when radiative corrections (originating from the multi-quark components in (1.2)) are taken into account. The kernel $\mathcal{G}$ in the right-hand side is the following matrix element:

$$\mathcal{G}(p_1,p_2|q_1,q_2) = \frac{(1/4)}{\sqrt{\omega(p_1)\omega(p_2)\omega(q_1)\omega(q_2)}}\left[\frac{\omega(p_1)+\omega(q_1)}{p_1-q_1}\frac{\omega(p_2)+\omega(q_2)}{p_2-q_2}+\right.$$
$$\left.-\frac{\omega(p_1)+\omega(q_2)}{p_1-q_2}\frac{\omega(p_2)+\omega(q_1)}{p_2-q_1}+\frac{p_1-p_2}{\omega(p_1)+\omega(p_2)}\frac{q_1-q_2}{\omega(q_1)+\omega(q_2)}\right]. \quad (1.5)$$

The kernel is singular, and the right-hand side of (1.3) involves the principal value of the singular integral. For the fermionic state with the momentum $P$ one takes the wave function of the form

$$\Psi(p_1,p_2) = (2\pi)\delta(p_1+p_2-P)\Psi_P(p_1-P/2) \quad (1.6)$$

with antisymmetric $\Psi_P(p) = -\Psi_P(-p)$. Then the equation (1.3) takes the form

$$[\varepsilon(P/2-p)+\varepsilon(P/2+p)-\Delta E]\Psi_P(p) = f_0\int_{-\infty}^{\infty}G_P(p|q)\Psi_P(q)\frac{dq}{2\pi}, \quad (1.7)$$

where the kernel $G_P(p|q)$ is the function (1.5) evaluated at $p_1 + p_2 = q_1 + q_2 = P$,

$$G_P(p|q) = \mathcal{G}\left(\frac{P}{2} + p, \frac{P}{2} - p \middle| \frac{P}{2} + q, \frac{P}{2} - q\right). \quad (1.8)$$

The fermionic wave-function $\Psi_P(p)$ is assumed to be normalizable,

$$\|\Psi_P\|^2 \equiv \frac{1}{2\Delta E}\int_{-\infty}^{\infty} |\Psi_P(p)|^2 \frac{dp}{2\pi} < \infty \ , \quad (1.9)$$

and the equation (1.7) is understood as the eigenvalue problem for the fermionic energy $\Delta E$. Observe that the kernel $G_P(p|q)$ in (1.7) has the second-order poles at $q = \pm p$, with residues 1. More precisely, one can check that

$$G_P(p|q) = \frac{1}{(p-q)^2} - \frac{1}{(p+q)^2} + G_P^{(reg)}(p|q), \quad (1.10)$$

where the last term is regular when both $p$ and $q$ take real values. Therefore the principal value integral in the right-hand side of (1.7) has the same effect as the linear potential in the following Hamiltonian of two particles interacting via the confining force:

$$H = \omega(p_1) + \omega(p_2) + 2\bar{\sigma}h|x_1 - x_2|, \quad (1.11)$$

provided the separation between the quarks is large enough, $m|x_1 - x_2| >> 1$.

Consider the eigenvalue equation (1.7) and take the limit $P \to +\infty$. In this limit, and under appropriate normalization, the fermionic wave function $\Psi_P(p)$ remains finite at $2|p|/P < 1$, but vanishes as $1/P^2$ outside this domain. For $2|p|/P < 1$ both sides of the equation (1.7) decay as $1/P$ at large $P$. Balancing the coefficients in front of $1/P$ asymptotic in both sides of (1.7) yields a nontrivial eigenvalue equation for the parameter $M^2$ in the following equation:

$$\Delta E = \sqrt{M^2 + P^2} \ . \quad (1.12)$$

Its most convenient form is obtained by changing to the rapidity variables

$$2p = P\tanh\theta \ , \quad 2q = P\tanh\theta' \quad (1.13)$$

before sending $P$ to infinity, and introducing the notation $\Psi(\theta)$ for the limit of $\Psi_P(p)$ at $P \to \infty$. It is

$$\left[m^2 - \frac{M^2}{4\cosh^2\theta}\right]\Psi(\theta) = f_0 \int_{-\infty}^{\infty} G(\theta|\theta')\Psi(\theta')\frac{d\theta'}{2\pi}, \quad (1.14)$$

where

$$G(\theta|\theta') = 2\frac{\cosh(\theta - \theta')}{\sinh^2(\theta - \theta')} + \frac{1}{4}\frac{\sinh\theta}{\cosh^2\theta}\frac{\sinh\theta'}{\cosh^2\theta'}. \quad (1.15)$$

Thence, we can rewrite the eq. (1.14) as follows

$$\left[m^2 - \frac{M^2}{4\cosh^2\theta}\right]\Psi(\theta) = f_0\int_{-\infty}^{\infty} 2\frac{\cosh(\theta-\theta')}{\sinh^2(\theta-\theta')} + \frac{1}{4}\frac{\sinh\theta}{\cosh^2\theta}\frac{\sinh\theta'}{\cosh^2\theta'}\Psi(\theta')\frac{d\theta'}{2\pi}. \quad (1.15b)$$

We will refer to (1.14), or (1.15b), as the Bethe-Salpeter (BS) equation, which yields infinite sequence of fermionic masses.

Let us make few general remarks on the properties of the BS equation (1.14). First, the wave function $\Psi(\theta)$ should be regarded as a vector in the Hilbert space with the metric

$$\|\Psi\|^2 = \int_{-\infty}^{\infty}\frac{|\Psi(\theta)|^2}{4\cosh^2\theta}\frac{d\theta}{2\pi} \quad (1.16)$$

which is just the metric (1.9) rewritten in terms of the rapidity variable $\theta$, in the limit $P=\infty$. Second, the equation is understood as the eigenvalue problem for the parameter $M^2$,

$$\hat{H}\Psi = M^2\Psi , \quad (1.17)$$

where the operator $\hat{H}$, defined as

$$\hat{H}\Psi(\theta) = 4\cosh^2\theta\left[m^2\Psi(\theta) - f_0\int_{-\infty}^{\infty} G(\theta|\theta')\Psi(\theta')\frac{d\theta}{2\pi}\right], \quad (1.18)$$

is Hermitian with respect to the metric (1.16). Its eigenvalues are real and positive. We will use the notations $\widetilde{M}_n^2$ with $n = 1,2,3,...$ for the successive eigenvalues of (1.17) and $\Psi_n(\theta)$ for the corresponding eigenfunctions. The quantities $\widetilde{M}_n$ provide certain approximations for the actual fermionic masses, and we reserve the notation $M_n$ for those actual masses.

We found it convenient to use yet another approach which is based on the Fourier-transformed version of the equation (1.14). We introduce the rapidity Fourier-transform

$$\Psi(\theta) = \int_{-\infty}^{\infty}\psi(\nu)e^{-i\nu\theta}d\nu , \qquad \psi(\nu) = \int_{-\infty}^{\infty}\Psi(\theta)e^{i\nu\theta}\frac{d\nu}{2\pi}. \quad (1.19)$$

In terms of $\psi(\nu)$ the norm (1.16) becomes

$$\|\Psi\|^2 = \frac{1}{4}\int_{-\infty}^{\infty}\psi^*(\nu)K(\nu-\nu')\psi(\nu')d\nu\, d\nu' , \quad (1.20)$$

where

$$K(\nu) = \int_{-\infty}^{\infty}\frac{e^{i\nu\theta}}{\cosh^2\theta}\frac{d\theta}{2\pi} = \frac{\nu}{2\sinh\frac{\pi\nu}{2}} , \quad (1.21)$$

and then the equation (1.14) transforms to

$$8\left(m^2 + f_0\nu \tanh\frac{\pi\nu}{2}\right)\psi(\nu) - \frac{f_0}{2}\frac{\nu}{\cosh\frac{\pi\nu}{2}}\int_{-\infty}^{\infty}\frac{\nu'}{\cosh\frac{\pi\nu'}{2}}\psi(\nu')d\nu' = M^2\int_{-\infty}^{\infty}\left[K(\nu-\nu') - K(\nu+\nu')\right]\psi(\nu')d\nu' \quad (1.22)$$

We note that this equation can be related with the following expression concerning the modes corresponding to the physical vibrations of the superstrings:

$$8 = \frac{1}{3}\frac{4\left[anti\log\frac{\int_0^{\infty}\frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}.$$

Thence, we obtain:

$$\frac{1}{3}\frac{4\left[anti\log\frac{\int_0^{\infty}\frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}\times$$

$$\times\left(m^2 + f_0\nu \tanh\frac{\pi\nu}{2}\right)\psi(\nu) - \frac{f_0}{2}\frac{\nu}{\cosh\frac{\pi\nu}{2}}\int_{-\infty}^{\infty}\frac{\nu'}{\cosh\frac{\pi\nu'}{2}}\psi(\nu')d\nu' = M^2\int_{-\infty}^{\infty}\left[K(\nu-\nu') - K(\nu+\nu')\right]\psi(\nu')d\nu' \quad (1.22b)$$

Now the kernel is regular at any real arguments, and the equation (1.22) admits numerical solution through straightforward discretization of the variable $\nu$. Before turning to the numeric, let us say few words about the general properties of the BS equation written in this form. It is not difficult to show that generic solution $\psi(\nu)$ has poles at the values of $\nu$ which solve the equation

$$1 + \lambda\nu\tanh\frac{\pi\nu}{2} = 0, \quad (1.23)$$

where again $\lambda = f_0/m^2$. At real positive $\lambda$ all solutions are purely imaginary, of the form $\pm i\kappa$, $\kappa$ being a positive root of the equation

$$\lambda\kappa\tan\frac{\pi\kappa}{2} = 1. \quad (1.24)$$

Let $\kappa_0$ be the lowest of such roots; at real positive $\lambda$ it lays in the interval $[0,1]$. The associated pole of $\psi(\nu)$ controls the large-$\theta$ behaviour of the rapidity-space wave function,

$$\Psi(\theta) \to r\,\mathrm{sign}(\theta) e^{-\kappa_0|\theta|} \quad \text{as} \quad |\theta| \to \infty, \quad (1.25)$$

where $r$ is a constant. Note that at large $\lambda$ this pole approaches the real axis, since

$$\kappa_0 \to \sqrt{\frac{2}{\pi\lambda}} \quad \text{as} \quad \lambda \to \infty. \quad (1.26)$$

At $\lambda = \infty$ every eigenfunction $\psi_n(\nu)$ has a simple pole at $\nu = 0$; correspondingly, the associated rapidity wave function $\Psi_n(\theta)$ tends to a constant $r_n$ at large $|\theta|$. One consequence of this phenomenon is the nature of the expansion around the point $\lambda = \infty$. This point appears to be a square-root branching point of the eigenvalues $\widetilde{M}_n^2$ taken as the functions of $\lambda$. Therefore the large-$\lambda$ expansions of $\widetilde{M}_n^2$ are of the form

$$\frac{\widetilde{M}_n^2}{4m^2} = Y_n^{(0)}\lambda + Y_n^{(1)}\sqrt{\lambda} + Y_n^{(2)} + \ldots + Y_n^{(k)}\lambda^{1-\frac{k}{2}} + \ldots, \quad (1.27)$$

where $Y_n^{(k)}$ are constants, and the series converge in finite domains. It is straightforward to derive the following expression for the coefficients at the terms $\approx \sqrt{\lambda}$,

$$Y_n^{(1)} = \frac{1}{2\sqrt{2\pi}}\left[\frac{|r_n|^2}{\|\Psi_n\|^2}\right]_{\lambda=\infty}, \quad (1.28)$$

where $r_n$ are the constants in the asymptotics (1.25) of the wave functions $\Psi_n$. We have various lowest eigenvalues of the BS equations (1.17) at different values of $\lambda$ obtained by numerical solution of the eq. (1.22). For $n = 9$ and $\lambda = 0.01$ the ratio $\frac{\widetilde{M}_n^2}{4m^2}$ is equal to 1.612724086 that is very near to the value of the aurea ratio i.e. $\frac{\sqrt{5}+1}{2} = 1.618033987...$

The numerical solution of (1.22) is obtained by discretization of the variable $\nu$. On the other side, at large $\lambda$, the expansion (1.27) apply. It is interesting to observe that as $n$ grows the leading coefficients $Y_n^{(0)}$ quickly approach the following simple asymptotic form

$$Y_n^{(0)} \to \pi(n - 3/8) \quad \text{as} \quad n \to \infty. \quad (1.28b)$$

In the same limit $Y_n^{(1)}$ tend to a constant value 1.2533…, while $Y_n^{(2)}$ increases logarithmically as $\frac{1}{2}\log(n - 3/8) + \text{Const}$, with Const $\approx$ 1.209.

We note that have also that:

$$1{,}612724086 - 0{,}36067977 = 1{,}252044 \cong 1{,}253;$$

and

$$0{,}36067977 = \left[(\Phi)^{-7/7} + (\Phi)^{-28/7} + (\Phi)^{-49/7} + (\Phi)^{-63/7}\right]\cdot \frac{4}{9};$$

with $\Phi = \frac{\sqrt{5}+1}{2} = 1{,}618033987...$

Furthermore, we note that, with regard the eq. (1.28):

$$\frac{1}{2\sqrt{2\pi}} = 0{,}19947 \cong (0{,}19453071 + 0{,}20601133)/2;$$

Where $0{,}19453071 = \left[(\Phi)^{-14/7} + (\Phi)^{-42/7}\right]\cdot \frac{4}{9}$; and $0{,}20601133 = (\Phi)^{-7/7}\cdot \frac{1}{3}$;

with $\Phi = \frac{\sqrt{5}+1}{2} = 1{,}618033987...$

### 1.2 On some equations concerning the Ising Field Theory in a magnetic field [3]

Now we describe some equations concerning the analytic properties of the scaling function $\Phi(\eta)$ associated with the 2D Ising model free energy in a magnetic field.
The values of the scaling function $\Phi(\eta)$ can be computed through the following dispersion relation

$$G_{high}(\xi) = -\xi^2 \int_{\xi_0}^{\infty} \frac{2\Im m G_{imh}(t)}{t(t^2+\xi^2)}\frac{dt}{\pi}, \quad (1.29)$$

once the function $D_{imh}(y) \equiv \Im m\Phi_{imh}(y)$ is known. This dispersion relation applies to the function $\Phi(\eta)$ with $\eta$ in the high-T wedge (HTW), where it can be written as

$$\Phi(\eta) = -\frac{15\eta^2}{4\pi}\int_0^{Y_0} \frac{D_{imh}(-y)y^{\frac{3}{4}}dy}{y^{\frac{15}{4}} + (-\eta)^{\frac{15}{4}}}. \quad (1.30)$$

Some extrapolation is needed in order to perform the (numerical) integration in (1.30). The best results were obtained with the following approximation

$$D_{imh}(y) = D_{imh}^{<}(y) \quad \text{for} \quad -Y_0 < y < -1.57; \qquad D_{imh}(y) = D_{imh}^{>}(y) \quad \text{for} \quad -1.57 < y < 0, \quad (1.31)$$

where

$$D_{imh}^{<}(y) = -\frac{1}{2}\overline{B}_0(y+Y_0)^{\frac{5}{6}} + \frac{1}{2}\overline{B}_1(y+Y_0)^{\frac{11}{6}} + \frac{\sqrt{3}}{2}\overline{C}_0(y+Y_0)^{\frac{5}{3}}, \quad (1.32)$$

with

$$\overline{B}_0 = -1.3693, \quad \overline{B}_1 = -0.74378, \quad \overline{C}_0 = 0.42446, \quad (1.33)$$

and $D_{imh}^{>}(y)$ is just the first eight terms of the y-expansion,

$$D_{imh}^{>}(y) = \overline{\Phi}_0 + \overline{\Phi}_1 y + \overline{\Phi}_2 y^2 + \overline{\Phi}_3 y^3 + \overline{\Phi}_4 y^4 + \overline{\Phi}_5 y^5 + \overline{\Phi}_6 y^6 + \overline{\Phi}_7 y^7, \quad (1.34)$$

with $\overline{\Phi}_n$ related to $\Phi_n$ from the following relation:

$$\overline{\Phi}_n = \Phi_n \sin\frac{4\pi(n-2)}{15} \quad \text{for} \quad n \neq 2; \qquad \overline{\Phi}_2 = -\frac{1}{15}. \quad (1.35)$$

Here we accept $Y_0 \approx 2.4295$ as the value of $Y_0$, while the coefficients $\overline{B}_0, \overline{B}_1, \overline{C}_0$ in (1.33) are obtained by exact matching (1.32) with (1.34) at $y = -1.57$, and fine-tuning the remaining one parameter to achieve better agreement with the data.
We note that the value 2.4295 is very near to the value 2.4270 that is connected with the aurea ratio as follows:

$$2.4270 = \left[(\Phi)^{14/7} + (\Phi)^{-7/7}\right], \text{ with } \Phi = \frac{\sqrt{5}+1}{2} = 1.618033987...$$

The results for the first few coefficients $G_{2n}$ in the following expression

$$G_{high}(\xi) = G_2\xi^2 + G_4\xi^4 + G_6\xi^6 + \ldots, \quad (1.36)$$

are obtained through the following dispersion relation

$$G_{2n} = (-)^n \int_{\xi_0}^{\infty} \frac{2\Im mG_{imh}(t)}{t^{2n+1}} \frac{dt}{\pi}, \quad (1.37)$$

with the use of (1.31).
According to the following dispersion relation

$$G_{low}(\xi) = \tilde{G}_1\xi - \xi^2 \int_0^{\infty} \frac{\Im mG_{meta}(t)}{t^2(t+\xi)} \frac{1}{\pi} dt, \quad (1.38)$$

in the low-T wedge $-\frac{8\pi}{15} < \arg(\eta) < \frac{8\pi}{15}$ the scaling function $\Phi(\eta)$ is represented in terms of the function $D_{meta}(y)$ as

$$\Phi(\eta) = \tilde{G}_1\eta^{\frac{1}{8}} - \frac{15\eta^{-\frac{7}{4}}}{8\pi} \int_0^{\infty} \frac{D_{meta}(y) y^{-\frac{9}{8}}}{y^{-\frac{15}{8}} + \eta^{-\frac{15}{8}}} dy. \quad (1.30b)$$

We note that $\frac{8\pi}{15} = 1{,}675516082 \cong 1{,}618033987 + 0{,}05676330$; where we have that:

$$0{,}05676330 = \left[(\Phi)^{-35/7} + (\Phi)^{-49/7} + (\Phi)^{-84/7}\right] \cdot \frac{4}{9}; \text{ with } \Phi = \frac{\sqrt{5}+1}{2} = 1.618033987...$$

It is interesting to check how the above information about $D_{meta}(y)$ agrees with the direct data on this scaling function. Furthermore, we have the following connection between the eq. (1.30) and the eq. (1.30b):

$$\Phi(\eta) = -\frac{15\eta^2}{4\pi}\int_0^{Y_0}\frac{D_{imh}(-y)y^{\frac{3}{4}}dy}{y^{\frac{15}{4}}+(-\eta)^{\frac{15}{4}}} = \widetilde{G}_1\eta^{\frac{1}{8}} - \frac{15\eta^{-\frac{7}{4}}}{8\pi}\int_0^{\infty}\frac{D_{meta}(y)y^{-\frac{9}{8}}}{y^{-\frac{15}{8}}+\eta^{-\frac{15}{8}}}dy. \quad (1.39)$$

The dispersion integral (1.30b) with

$$D_{meta}(y) = D_{meta}^{<}(y) \quad \text{for} \quad 0 < y < 1.1 \qquad D_{meta}(y) = D_{meta}^{\text{int}\,erp}(y) \quad \text{for} \quad 1.1 < y < 2.72$$
$$D_{meta}(y) = D_{meta}^{>}(y) \quad \text{for} \quad 2.72 < y < \infty, \quad (1.40)$$

reproduces the direct data very accurately.
The dispersion integral of the scaling function $\widetilde{\Phi}(\eta)$ is:

$$\widetilde{\Phi}(\eta) = \Phi_0 + \widetilde{\Phi}_{\log}(\eta) - \frac{\eta}{\pi}\int_{Y_0}^{\infty}\frac{y\,\mathrm{Re}\left(e^{\frac{4\pi i}{15}}\Delta(y)\right)+\eta\,\mathrm{Re}\left(e^{\frac{8\pi i}{15}}\Delta(y)\right)}{y^2 2y\eta\cos\left(\frac{4\pi}{15}\right)+\eta^2}\frac{dy}{y}, \quad (1.41)$$

where the term

$$\widetilde{\Phi}_{\log}(\eta) = -\frac{Y_0\eta}{4\pi}\cos\left(\frac{4\pi}{15}\right)+\frac{\eta^2}{8\pi}\log\left(Y_0^2+2\eta Y_0\cos\left(\frac{4\pi}{15}\right)+\eta^2\right) \quad (1.42)$$

comes from the domain $0 \le y < Y_0$ in which $\Delta(y)$ coincides with $\Delta_{\log}(y) = \frac{1}{4}y^2$.

From (1.41) one readily derives the following expression for the coefficients of the Taylor expansion $\widetilde{\Phi}(\eta) = \Phi_0 + \Phi_1\eta + \Phi_2\eta^2 + \ldots$ :

$$\Phi_0 = \mathrm{Re}\left[\frac{e^{\frac{8\pi i}{15}}}{\pi}\int_0^{\infty}\frac{\Delta(y)-\Delta_{ass}(y)}{y}dy\right]. \quad (1.43)$$

With regard the results from high-T and low-T dispersion relations, eqs. (1.30) and (1.30b), with the use of approximations (1.32) and (1.40), and the results from extended dispersion relation, eq. (1.41), with following approximation $\Delta(y) = \Delta^{<}(y)$ for 0 < y < 3; $\Delta(y) = \Delta^{>}(y)$ for 3 < y, we have that with regard $\Phi(\eta)$, for $\eta = 4$, we obtain the following values: from TFFSA data = 1.6188506; from low-T disp. relation = –1.6188510; from extended disp. relation = –1.6187275.

We note that these values are all well connected with the aurea ratio: $\frac{1+\sqrt{5}}{2} = 1.618033987\ldots$

**2. On some equations concerning the AdS/CFT and condensed matter physics to describe the superconductivity. Mathematical connections between matter condensed and string theory [4]**

In the AdS/CFT correspondence, plasma or fluid-like phases of the field theory at nonzero temperature are described by black hole solutions to the bulk gravitational action. A Reissner-Nordstrom AdS black hole, when coupled to a neutral scalar with $m^2 = -\frac{2}{L^2}$, becomes unstable near extremality. The instability produces the hairy black hole.
The Euclidean action for the hairy black hole is:

$$S_E = -\int d^4x\sqrt{-g_B}L \quad , \quad (2.1)$$

where $L$ is given from the following equation

$$L = R + \frac{6}{L^2} - \frac{1}{4}F^{ab}F_{ab} - V(|\psi|) - |\nabla\psi - iqA\psi|^2 \quad , \quad (2.2)$$

and $g_B$ is the determinant of the bulk metric. From the symmetries of the following solutions

$$ds^2 = -g(r)e^{-\chi(r)}dt^2 + \frac{dr^2}{g(r)} + r^2(dx^2 + dy^2) \quad , \quad (2.3)$$

$$A = \phi(r)dt \ , \quad \psi = \psi(r) , \quad (2.4)$$

the $xx$ components of the stress energy tensor only has a contribution from the terms proportional to the metric. Thus, Einstein's equation

$$R_{ab} - \frac{g_{ab}R}{2} - \frac{3g_{ab}}{L^2} = \frac{1}{2}F_{ac}F_b^c - \frac{g_{ab}}{8}F^{cd}F_{cd} - \frac{g_{ab}}{2}V(|\psi|) - \frac{g_{ab}}{2}|\nabla\psi - iqA\psi|^2 +$$
$$+\frac{1}{2}[(\nabla_a\psi - iqA_a\psi)(\nabla_b\psi^* + iqA_b\psi^*) + a \leftrightarrow b] \quad , \quad (2.5)$$

where $q$ is the charge of the scalar field, implies that the Einstein tensor satisfies

$$G_{xx} = \frac{1}{2}r^2(L - R) \quad . \quad (2.6)$$

This implies

$$-R = G_a^a = G_t^t + G_r^r + L - R \quad , \quad (2.7)$$

Or

$$L = -G_t^t - G_r^r = -\frac{1}{r^2}[(rg)' + (rge^{-\chi})'e^{\chi}] \quad . \quad (2.8)$$

The Euclidean action is then a total derivative

$$S_E = \int d^3x \int_{r_+}^{r_\infty} dr \left[2rge^{-\frac{\chi}{2}}\right]' \quad . \quad (2.9)$$

The London equation

$$J_i(\omega, k) = -n_s A_i(\omega, k) \quad (2.10)$$

was proposed to explain both the infinite conductivity and the Meissner effect of superconductors. This equation is understood to be valid where $\omega$ and k are small compared to the scale at which the system loses its superconductivity. In the limit k = 0 and $\omega \to 0$, we can take a time derivative of both sides to find

$$J_i(\omega, 0) = \frac{i n_s}{\omega} E_i(\omega, 0) \quad (2.11)$$

explaining the infinite DC conductivity observed in superconductors. On the other hand, in the limit $\omega = 0$ and $k \to 0$, we can instead consider the curl of the London equation, yielding

$$i\varepsilon_{ijl} k^j J^l(0, k) = -n_s B_i(0, k). \quad (2.12)$$

Together with Maxwell's equation $\varepsilon^{ijl}\partial_j B_l = 4\pi J^i$, this other limit of the London equation implies that the magnetic field lines are excluded from superconductors. By decoupling the metric fluctuations, we will remove the additional divergence in the conductivity at $\omega \to 0$ due to translation invariance. The background metric is

$$ds^2 = -g(r)dt^2 + \frac{dr^2}{g(r)} + r^2(dx^2 + dy^2), \quad (2.13)$$

where $g(r) = r^2 - \frac{r_+^3}{r}$. Assume that we have solved self-consistently for $A_t$ and $\psi$ in this Schwarzschild background. We then allow for perturbations in $A_x$ that have both momentum and frequency dependence of the form $A_x \sim e^{-i\omega t + iky}$. We have taken the momentum in a direction orthogonal to $A_\mu$. This allows us to consistently perturb the gauge field without sourcing any other fields. With these assumptions, the differential equation for $A_x$ reduces to

$$\left(\frac{\omega^2}{g} - \frac{k^2}{r^2}\right)A_x + (gA'_x)' = 2q^2\psi^2 A_x, \quad (2.14)$$

where ' denotes differentiation with respect to $r$. Introducing a new radial variable $z = \frac{1}{r}$, eq. (2.14) reduces to the Klein-Gordon equation with mass proportional to $\langle O_1 \rangle$:

$$(\omega^2 - k^2 - q^2\langle O_1\rangle^2)A_x + \ddot{A}_x = 0. \quad (2.15)$$

We are implicitly working at low frequencies where $\omega^2, k^2 \ll q^2\langle O_1\rangle^2$. Since the horizon is at large $z$, we impose the boundary condition that $A_x$ be well behaved there to find

$$A_x = a_x e^{-i\omega t + iky - \lambda z}, \quad (2.16)$$

where $\lambda^2 = q^2\langle O_1\rangle^2 + k^2 - \omega^2 \approx q^2\langle O_1\rangle^2$. To obtain the conductivity we expand $A_x$ near the boundary $z = 0$ in the low frequency case:

$$A_x = a_x(1 - \lambda z + O(z^2)). \quad (2.17)$$

From the AdS/CFT dictionary, described several times above, we can interpret the zeroth order term as an external field strength and the linear term as a current $J_x$. Thus, this expansion gives us a modified London equation:

$$J_x = -\sqrt{q^2\langle O_1\rangle^2 + k^2 - \omega^2}\, a_x. \quad (2.18)$$

In the limit $\omega, k \ll q\langle O_1\rangle$, we get precisely the London equation:

$$J_x = -q\langle O_1\rangle a_x. \quad (2.19)$$

We have verified numerically that the strength of the pole in the imaginary part of the conductivity is indeed very close to $q\langle O_1\rangle$ at low temperatures.The London equation leads to the magnetic penetration depth

$$\lambda^2 = \frac{1}{4\pi n_s}, \quad (2.20)$$

via the Maxwell equation for the curl of the magnetic field:

$$-\nabla^2 B = \nabla \times (\nabla \times B) = 4\pi \nabla \times J = -4\pi n_s \nabla \times A = -4\pi n_s B. \quad (2.21)$$

Therefore:

$$\nabla^2 B = \frac{1}{\lambda^2} B, \quad (2.22)$$

implying that static magnetic fields can penetrate a distance $\lambda$ into the superconductor.
Now we give a proof of the following claim: when coupled to a neutral scalar field with $m^2 = -2$, the Reissner-Nordstrom AdS black hole becomes unstable near extremality. We shall prove this using test functions and the Rayleigh-Ritz method. Let us write the black hole metric in terms of the coordinate $z = \frac{1}{r}$

$$ds^2 = \frac{1}{z^2}\left[-f(z)dt^2 + \frac{dz^2}{f(z)} + dx^2 + dy^2\right], \quad (2.23)$$

Where

$$f = 1 - (1 + c^2)z^3 + c^2 z^4. \quad (2.24)$$

Here $c$ denotes a dimensionless charge density, obtained by rescaling the horizon to $z = 1$. It is related to the physical charge density $\rho$ and temperature $T$ by

$$\frac{\rho}{T^2} = \frac{16\pi^2 c}{(3 - c^2)^2}. \quad (2.25)$$

We now want to write the equation of motion for the neutral scalar field $\psi$ in Schrodinger form. This form will help us gain intuition about the behaviour of the field. The rewriting requires rescaling the field and changing variables from $z$ to a new coordinate $s$. Let

$$\psi = z\Psi, \quad \frac{ds}{dz} = \frac{1}{f}. \qquad (2.26)$$

Then, taking $\Psi(s,t)$ with a time dependence $e^{-i\omega t}$, we obtain the Schrodinger equation

$$-\frac{d^2\Psi}{ds^2} + V(s)\Psi = \omega^2\Psi, \qquad (2.27)$$

with the potential, written in terms of the $z(s)$ variable,

$$V = -f\left(\frac{2}{z^2} + \frac{f'}{z} - \frac{2f}{z^2}\right). \qquad (2.28)$$

This potential (2.28) is positive everywhere unless the charge density $c$ is close to the extremal value $c = \sqrt{3}$ (i.e. $T = 0$). More specifically, the potential develops a negative region in the vicinity of the horizon for $c > 1$. So any instability is restricted to the charge values $1 < c \leq \sqrt{3}$, i.e. $4\pi^2 \leq \frac{\rho}{T^2} < \infty$. The action to use depends on the boundary conditions of the field $\Psi$. The general allowed falloff at the boundary $z \to 0$ is

$$\Psi \sim a + bz + \cdots. \qquad (2.28b)$$

If we impose the boundary condition $\delta a = 0$ (i.e. $\delta\Psi(0) = 0$) the following action is stationary on solutions to the Schrodinger equation

$$S_{\delta a=0} = \int ds\left[\left(\frac{d\Psi}{ds}\right)^2 + (V(s) - \omega^2)\Psi^2\right]. \qquad (2.29)$$

However, if we wish to impose $\delta b = 0$ (i.e. $\delta\Psi'(0) = 0$) then we must add a boundary term

$$S_{(\delta b = 0)} \int \left[ ds[(d\Psi/ds)^2 + (V(s) - \omega^2)\Psi^2] - 2\Psi\, d\Psi/ds\, |_{(s \to 0)} \right]. \qquad (2.30)$$

Both of these actions are finite on solutions to the Schrodinger equation, partly due to a cancellation in the potential (2.28) as $z \to 0$ which only occurs at the mass we have chosen, $m^2 = -2$. In fact, the boundary term in (2.30) vanishes on shell for "normalisable" modes. To show an instability, we need to find test functions such that $S(\omega = 0) < 0$. Let us start with the second of the boundary conditions above. Normalisable modes therefore have $b = 0$. A simple test function that satisfies this boundary condition is

$$\Psi_{test} = 1 - az^2. \qquad (2.31)$$

It is easy to check that the $(\omega = 0)$ action is minimised by

$$\alpha = \frac{21(3c^2 - 5)}{10(9c^2 - 35)} . \quad (2.32)$$

This function then leads to an action which is negative for $1.609 \leq c \leq \sqrt{3} \approx 1.732$ . Also here is very evident that c can be equal to $\frac{\sqrt{5}+1}{2} = 1.618 \ldots$ , i.e. the **aurea ratio**.
Now we attempt to calculate $n_s$ analytically at low temperatures for the dimension two case. If we assume that at very low temperature, $\sqrt{2}\,\psi \approx \frac{\langle O_2 \rangle}{r^2}$ , then eq. (2.14) becomes

$$(\omega^2 - k^2 - q^2 \langle O_2 \rangle^2 z^2) A_x + \ddot{A}_x = 0 . \quad (2.33)$$

where $z = \frac{1}{r}$ and a dot denotes $\frac{d}{dz}$. This differential equation can be solved in terms of parabolic cylinder functions, $D_\nu(cz)$ where the choice

$$\nu = -\frac{1}{2} + \frac{k^2 - \omega^2}{2q\langle O_2 \rangle} \quad \text{and} \quad c = \sqrt{2q\langle O_2 \rangle} \quad (2.34)$$

gives the proper exponential fall-off as $z$ gets large. Here the condition that $k$ and $\omega$ are small is more precisely $k^2, \omega^2 \ll q\langle O_2 \rangle$. Expanding $D_\nu(cz)$ near the boundary, we find

$$A_x = a_x \left( 1 - \frac{2\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} \sqrt{q\langle O_2 \rangle}\, z + O(z^3) \right) , \quad (2.35)$$

where we have suppressed corrections in $\frac{k^2 - \omega^2}{q\langle O_2 \rangle}$ . The London equation here is then

$$J_x = -\frac{2\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} \sqrt{q\langle O_2 \rangle}\, a_x . \quad (2.36)$$

Numerically, this estimate of $n_s$ appears to be wrong by about 25% at low temperatures. While $\frac{2\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} \approx 0.676$ , the real constant of proportionality appears to be about 0.546
We note that $\frac{(0.676 + 0.546)}{2} \cong 0.611 \approx 0.618 \ldots = \frac{\sqrt{5}-1}{2}$ , thence the mathematical connection with the **aurea ratio**.

Also in this paper in many expressions is very evident the link between $\pi$ and $\phi = \frac{\sqrt{5}-1}{2}$ , i.e. the Aurea ratio, by the simple formula

$$\arccos\phi = 0{,}2879\pi . \quad (2.37)$$

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