

The Fibonacci's zeta function. Mathematical connections with some sectors of String Theory

Michele Nardelli^{1,2} and Rosario Turco

¹ Dipartimento di Scienze della Terra

Università degli Studi di Napoli Federico II, Largo S. Marcellino, 10

80138 Napoli, Italy

² Dipartimento di Matematica ed Applicazioni "R. Caccioppoli"

Università degli Studi di Napoli "Federico II" – Polo delle Scienze e delle Tecnologie

Monte S. Angelo, Via Cintia (Fuorigrotta), 80126 Napoli, Italy

Abstract

In this paper we have described, in the **Section 1**, the Fibonacci's zeta function and the Euler-Mascheroni constant and in the **Section 2**, we have described some sectors of the string theory: zeta strings, zeta nonlocal scalar fields and some Lagrangians with zeta Function nonlocality. In conclusion, in the **Section 3**, we have described some possible mathematical connections.

1. The Fibonacci's zeta function [1], [2], [3], [4], [5], [6]

The Fibonacci's zeta function is defined as:

$$F(s) = \sum_{k=1}^{\infty} \frac{1}{Fib(k)^s} \quad (1)$$

where $s=a+jb$.

The logarithm of the Fibonacci's zeta and the Fiborial

Let $F(s)$ the Fibonacci's zeta, then is:

$$-\frac{\ln F(s)}{s} = \ln(\text{Fib}\#), \text{ dove } \text{Fib}\# = \prod_{k=1}^{\infty} \text{Fib}(k) \quad (2)$$

We have that Fib# is the Fiborial, i.e. the product to the infinity of the Fibonacci's numbers.

We can write the eq. (1) also as follows:

$$F(s) = 1^{-s} + 1^{-s} + 2^{-s} + 3^{-s} + 5^{-s} + 8^{-s} + \dots$$

$$F(s) = e^{-s \ln 1} + e^{-s \ln 1} + e^{-s \ln 2} + e^{-s \ln 3} + e^{-s \ln 5} + e^{-s \ln 8} + \dots$$

$$\ln F(s) = -s(\ln 1 + \ln 1 + \ln 2 + \ln 3 + \ln 5 + \ln 8 + \dots)$$

$$-\frac{\ln F(s)}{s} = \ln \prod_{k=1}^{\infty} \text{Fib}(k) = \ln(\text{Fib}\#)$$

where we have applied a note property of the logarithms.

Generalization of the Gamma function for various series

If we want to reach a possible functional equation for the Fibonacci's zeta, then we need to ask first how to express the "Fibonacci's Gamma".

We know that the Gamma function $\Gamma(x)$ is an extension of factorial, by an integer k to a real number x . The Gamma function was studied by Euler (see [1]) and has a significant importance in Number Theory and in the study of the Riemann zeta.

In general we know that $\Gamma(x)$ enjoys some of the following properties:

1. $\Gamma(k+1) = k!, k \geq 0$;
2. $\Gamma(1) = 1, \Gamma(x+1) = x \Gamma(x)$ e $\Gamma(x)$ is the unique solution of this functional equation for $x > 0$;
3. $\Gamma(x) = \lim_{k \rightarrow \infty} (k k^x) / (x(x+1) \dots (x+k)) = \lim_{k \rightarrow \infty} (k^x/x) \prod_{i=1}^k (i/(x+i)), x \neq 0, -1, -2 \dots$ (Euler form);
4. $1/\Gamma(x) = x e^{\gamma x} \prod_{i=1}^{\infty} (1 + x/i) e^{-x/i}$ (Weierstrass form) where γ is the Euler-Mascheroni constant defined by $\gamma = \lim_{k \rightarrow \infty} \sum_{i=1}^k (1/i) - \log k$;
5. $\Gamma(x)\Gamma(-x) = -\pi/(x \sin \pi x)$ (property of reflection), $\Gamma(x)\Gamma(1-x) = \pi/\sin \pi x$ (property of extension) and $\Gamma(1/2)^2 = \pi$.

From above we can tell which should be considered in the follow, both the Euler form and the Weierstrass form.

Generalized Euler form

We consider a series $g_k > 0, k > 0$ and an extension $g(x) > 0, x > 0$ with the property that:

$$g(k) = g_k \text{ and } l(x) = \lim_{k \rightarrow \infty} g(x+k)/g(k) \text{ there exists and isn't zero.}$$

From the above property 3, we have:

$$\begin{aligned} \Gamma_g(x) &= \lim_{k \rightarrow \infty} g(x) \Gamma_{g,k}(x) \\ &= \lim_{k \rightarrow \infty} \lambda(x) \frac{g(k)^x}{g(x)} \prod_{i=1}^k \frac{g(i)}{g(x+i)}, \quad g(x+i) \neq 0, i=0,1,2,\dots \end{aligned} \quad (3)$$

where $\lambda(x)$ is to establish.

Thence

$$\Gamma_{g,k}(x+1) = \frac{g(k)}{g(x+k+1)} \frac{\lambda(x+1)}{\lambda(x)} g(x) \lambda(x) \frac{g(k)^x}{g(x)} \prod_{i=1}^k \frac{g(i)}{g(x+i)}$$

we obtain

$$\Gamma_g(x+1) = \frac{\lambda(x+1)}{l(x+1)\lambda(x)} g(x) \Gamma_g(x)$$

If we choose $\lambda(x)$ such that

$$\frac{\lambda(x+1)}{l(x+1)\lambda(x)} = 1,$$

for example:

$$\lambda(x+1) = l(x+1)\lambda(x),$$

we obtain

$$\Gamma_g(x+1) = g(x) \Gamma_g(x) \quad (4)$$

If we take $\Gamma_g(1) = 1$, we obtain $\Gamma_g(k+1) = \prod_{i=1}^k g_i$.

Furthermore, from eq. (3) we have:

$$\begin{aligned} \Gamma_g(1) = 1 &\Rightarrow \lim_{k \rightarrow \infty} \lambda(1) \frac{g(k)}{g(k+1)} \\ &= 1 \Rightarrow \lambda(1) = l(1) \end{aligned}$$

such that

$$\lambda(k) = \prod_{i=1}^k l(i).$$

With this restriction, $\lambda(k)$ can be extended to $\lambda(x)$ or to have $\Gamma_g(x)$ with additional properties.

Using the eq. (4), $\Gamma_g(x)$, $x > 0$ satisfy the relation

$$\ln \Gamma_g(x + 1) - \ln \Gamma_g(x) = \ln g(x)$$

Example A

A simple generalization of $g_k = k$ is $g_k = ak + b$ with a, b choose such that $g_k > 0, k > 0$. Thence, we obtain:

$$g(x) = ax + b, \quad l(x) = \lim_{k \rightarrow \infty} \frac{a(x+k) + b}{ak + b} = 1 \quad \text{and} \quad \lambda(k) = 1.$$

If we take $\lambda(x) = 1$, we obtain:

$$\Gamma_g(x) = \lim_{k \rightarrow \infty} \frac{(ak + b)^x}{(ax + b)} \prod_{i=1}^k \frac{ai + b}{a(x + i) + b}$$

with

$$\Gamma_g(k + 1) = \prod_{i=1}^k (ai + b).$$

The Figure 1 concerning the product of the odd numbers.

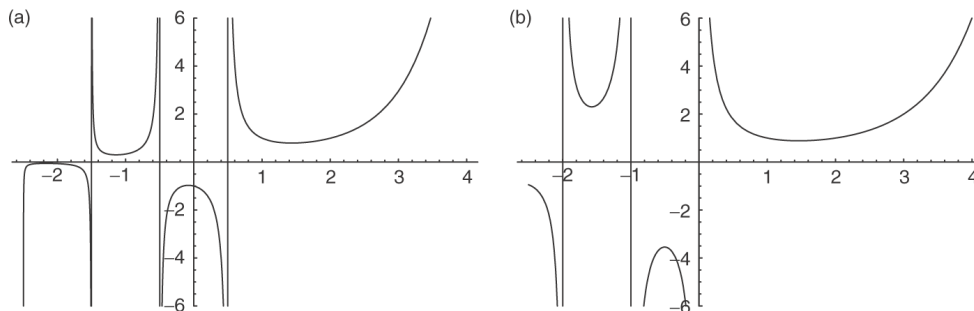


Figure 1

(a) show $\Gamma_g(x)$, an extension of the product $\Gamma_g(k + 1) = \prod_{i=1}^k (ai + b)$ to the real numbers, for $a = 2, b = -1$. (b) show the comparison with the normal Gamma function $\Gamma(x)$.

Example B

For the Fibonacci's numbers is:

$$g(x) = F_x = (\phi^x - \phi^{-x} \cos(\pi x)) / \sqrt{c}$$

where $\phi = (1 + \sqrt{5})/2$, i.e. the aurea ratio, and $c = 5$, and thence:

$$l(x) = \lim_{k \rightarrow \infty} \phi^{x+k} / \phi^k = \phi^x$$

and

$$\lambda(k) = \prod_{i=1}^k \phi^i =.$$

The most simple and effective extension is:

$$\lambda(x) = \phi^{(x+1)x/2}$$

and we obtain

$$\Gamma_F(x) = \phi^{(x+1)x/2} \lim_{k \rightarrow \infty} \frac{F_k^x}{F_x} \prod_{i=1}^k \frac{F_i}{F_{x+i}} \quad (5)$$

where $\Gamma_F(k+1) = (k!)_F$.

The Figure 2 show $\Gamma_F(x)$.

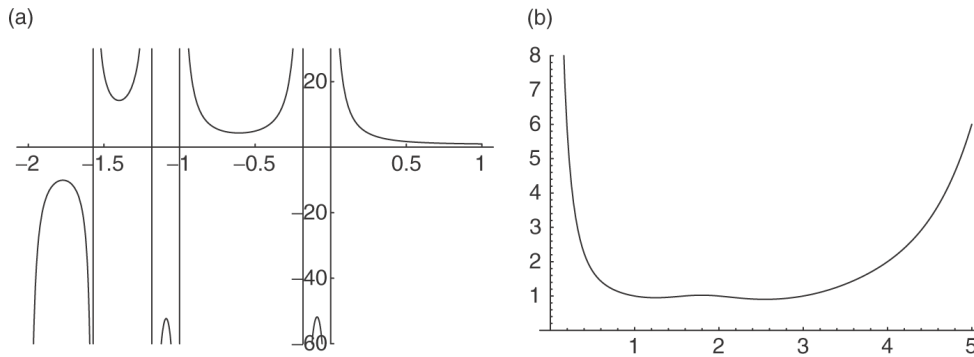


Figure 2

$\Gamma_F(x)$, an extension of the Fiborial to the real numbers with the property that $\Gamma_F(k+1) = (k!)_F$. (a) show the values for $-2 \leq x \leq 1$ (b) show the values for $0 \leq x \leq 5$.

Weierstrass generalization form

Using the eq. (3) and assuming that $\lambda(x)$ and $g(x)$ are differentiable, we obtain:

$$\frac{\Gamma'_g(x)}{\Gamma_g(x)} = \frac{\lambda'(x)}{\lambda(x)} - \frac{g'(x)}{g(x)} - \lim_{x \rightarrow \infty} \left(\sum_{i=1}^k \frac{g'(x+i)}{g(x+i)} - \ln g(k) \right) \quad (6)$$

which suggests an *Euler-Mascheroni generalized constant*:

$$\gamma_g = \lim_{x \rightarrow \infty} \left(\sum_{i=1}^k \frac{g'(i)}{g(i)} - \ln g(k) \right) \quad (7)$$

(we remember that the Euler-Mascheroni constant can be written also as follow

$$\gamma = \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \frac{1}{k} - \ln(n) \right) = \int_1^{\infty} \left(\frac{1}{[x]} - \frac{1}{x} \right) dx, \quad (7b))$$

thence γ becomes the associated constant to the series $g_k = k$.

$$g(k)^x = \exp\left(x\left(\ln g(k) - \sum_{i=1}^k \frac{g'(i)}{g(i)}\right) + x \sum_{i=1}^k \frac{g'(i)}{g(i)}\right) = \exp\left(-x\gamma_g + x \sum_{i=1}^k \frac{g'(i)}{g(i)}\right)$$

Thence the *Weierstrass generalized form* (see property 4) becomes:

$$\frac{1}{\Gamma_g(x)} = \frac{g(x)}{\lambda(x)} e^{x\gamma_g} \prod_{i=1}^{\infty} \frac{g(x+i)}{g(i)} \exp\left(-x \frac{g'(i)}{g(i)}\right) \quad (8)$$

While the *generalized reflection formula* becomes:

$$\Gamma_g(x)\Gamma_g(-x) = \frac{\lambda(x)\lambda(-x)}{g(x)g(-x)} \prod_{i=1}^{\infty} \frac{g(i)^2}{g(x+i)g(-x+i)} \quad (9)$$

and since $\Gamma_g(1-x) = g(-x)\Gamma_g(-x)$,

we obtain

$$\Gamma_g(x)\Gamma_g(1-x) = \frac{\lambda(x)\lambda(-x)}{g(x)} \prod_{i=1}^{\infty} \frac{g(i)^2}{g(x+i)g(-x+i)} \quad (10)$$

$$\Gamma_g\left(\frac{1}{2}\right)^2 = \frac{\lambda(1/2)\lambda(-1/2)}{g(1/2)} \prod_{i=1}^{\infty} \frac{g(i)^2}{g(1/2+i)g(-1/2+i)} \quad (11)$$

Example A

For $g_k = ak + b$, as before, the *Weierstrass form* is:

$$\frac{1}{\Gamma_g(x)} = (ax+b)e^{x\gamma_g} \prod_{i=1}^{\infty} \left(1 + \frac{ax}{ai+b}\right) e^{-xa/(ai+b)}$$

where

$$\begin{aligned} \gamma_g &= \lim_{k \rightarrow \infty} \left(\sum_{i=1}^k \frac{a}{ai+b} - \ln(ak+b) \right) \\ &= -a/b - \ln a - \psi_0\left(\frac{b}{a}\right) \end{aligned}$$

(ψ_0 is the *digamma function*), and the eq. (11) gives:

$$\begin{aligned}\Gamma_g\left(\frac{1}{2}\right) &= \frac{1}{\sqrt{a/2+b}} \prod_{i=1}^{\infty} \frac{ai+b}{\sqrt{(ai+b)^2 - a^2/4}} \\ &= \frac{\Gamma(b/a + 1/2)}{\sqrt{a}\Gamma(b/a + 1)}\end{aligned}$$

The functional equation $\Gamma_g(x+1) = (ax+b)\Gamma_g(x)$, thence, becomes

$$\begin{aligned}\Gamma_g(k+1/2) &= \prod_{i=1}^k (a(i-1/2)+b)\Gamma_g\left(\frac{1}{2}\right) \\ &= (a/2-b)^k k! \Gamma_g\left(\frac{1}{2}\right) \quad \text{for } k > 0\end{aligned}$$

Example B

For the Fibonacci's numbers the *Weierstrass form* is:

$$\frac{1}{\Gamma_F(x)} = \phi^{-(x+1)x/2} F_x e^{x\gamma_F} \prod_{i=1}^{\infty} \frac{F_{x+i}}{F_i} e^{-x F_i / F_i} \quad (12)$$

where $\gamma_F = 0.6676539532 \dots$, and the eq. (11) gives

$$\Gamma_F\left(\frac{1}{2}\right) = \frac{\phi^{1/8}}{\sqrt{F_{1/2}}} \prod_{i=1}^{\infty} \frac{F_i}{\sqrt{F_{1/2+i} F_{-1/2+i}}}$$

since $F_{\pm 1/2+i} = \phi^{i \pm 1/2} / \sqrt{c}$ for i integer.

We have that:

$$\Gamma_F\left(\frac{1}{2}\right) = \left(\frac{c^2}{\phi}\right)^{1/8} \prod_{i=1}^{\infty} (1 - (-\phi^{-2})^i) = 1.7273179426 \dots$$

Alternatively, can be considered a closed form for the constant of the Fiborial (Sloane's A062073 [4])

$$\prod_{i=1}^{\infty} (1 - (-\phi^{-2})^i) = \left(\frac{\phi}{c^2}\right)^{1/8} \Gamma_F\left(\frac{1}{2}\right) \quad (=1.2267420107 \dots)$$

(If we replace $\Gamma_F(1/2)$ with $\Gamma_g(1/2)$ where $g(x) = (\phi^x - \phi^{-x} \cos(\pi x)) / \sqrt{c}$ and ϕ is the aurea ratio, the precedent result can give a closed form for the infinite product where $\phi > 1$. This product is linked to the partitions functions and the q -series)

$$\Gamma_F(k + 1/2) = \prod_{i=1}^k F_{i-1/2} \Gamma_F\left(\frac{1}{2}\right) = \frac{\phi^{k^2/2}}{c^{k/2}} \Gamma_F\left(\frac{1}{2}\right) \quad \text{for } k > 0$$

Other series and generalizations

Generalized Gamma functions can be obtained also for series as: $g(x) = \sum_i c_i x^i$, $g(x) = (\sum_i c_i x^i)^{-1}$ and $g(x) = \sum_i c_i r_i^x$ etc. The Euler-Mascheroni generalized constant can also be used to introduce sequences that are well known. If $g'(i)/g(i) = f(i)$ then $g(i) = A e^{\int f(i) di}$ (A constant) such that:

$$\gamma_g = \lim_{k \rightarrow \infty} \left(\sum_{i=1}^k f(i) - \ln g(k) \right)$$

For example, $f(i) = 1/i^z$, $z > 0$ from $g(x) = e^{x^{1-z}/1-z}$ with $\gamma_g = \zeta(z)$, the Riemann zeta.

$\binom{n}{m}_g = \frac{\prod_{i=1}^n g_i}{\prod_{i=1}^m g_i \prod_{i=1}^{n-m} g_i}$ can be extended to real numbers using:

$$\binom{x}{y}_g = \frac{\Gamma_g(x+1)}{\Gamma_g(y+1) \Gamma_g(x-y+1)}$$

The *Beta function* can be also generalized to:

$$B_g(x, y) = \frac{\Gamma_g(x) \Gamma_g(y)}{\Gamma_g(x+y)}$$

The logarithmic derivative, analyzed previously, generalizes the *digamma function*

$$\psi_0(x) = \frac{d}{dx} \ln \Gamma(x) = -\gamma + \sum_{i=1}^{\infty} \frac{x-1}{i(x+i-1)} \quad \text{with } \psi_0(1) = -\gamma. \quad \text{Thence:}$$

$$\begin{aligned} \psi_{g,0}(x) &= \frac{d}{dx} \ln \Gamma_g(x) = \frac{\Gamma'_g(x)}{\Gamma_g(x)} \\ &= \frac{\lambda'(x)}{\lambda(x)} - \gamma_g + \sum_{i=1}^{\infty} \frac{g'(i)g(x+i-1) - g(i)g'(x+i-1)}{g(i)g(x+i-1)}, \end{aligned}$$

$g(x+i) \neq 0$, $i = 0, 1, 2, \dots$, using the *Weierstrass form*.

This satisfies also other relations:

$$\psi_{g,0}(x+1) - \psi_{g,0}(x) = \frac{g'(x)}{g(x)}.$$

Considering

$$\psi_0(k) = -\gamma + H_{k-1},$$

where $H_k = \sum_{i=1}^k 1/i$ is the k-th harmonic number, the 'g-harmonic' number can be defined as $H_{g,k} = \sum_{i=1}^k g'(i)/g(i)$.

Analogously, if we assume that g(x) is differentiable, also the properties of the *polygamma function* $\psi_{g,k}(x) = d^k/dx^k \psi_{g,0}(x)$ can be considered.

For example:

$$\psi_k(1) = \sum_{i=1}^{\infty} (-\ln i)^{(k+1)} = (-1)^{k+1} k! \zeta(k+1), \quad k > 0,$$

while

$$\psi_{g,k}(1) = (\ln \lambda(x))^{(k+1)} \Big|_{x=1} + \sum_{i=1}^{\infty} (-\ln g(i))^{(k+1)}$$

Thence a 'g-zeta' can be defined as

$$\zeta_g(k) = \frac{(-1)^k}{(k-1)!} \sum_{i=1}^{\infty} (-\ln g(i))^{(k)} \quad \text{with } k > 1$$

when $g(i) = ai + b$, we obtain

$$\zeta_g(k) = \sum_{i=1}^{\infty} \frac{1}{(i + b/a)^k}$$

that can be extended to the real numbers and becomes the *Hurwitz zeta function*:

$$\zeta(b/a + 1, x) = \sum_{i=0}^{\infty} \frac{1}{(i + b/a + 1)^x}$$

For the Fibonacci's zeta, thence, we can take in consideration the eq. (5) or the eq. (12).

1.1 The Euler-Mascheroni constant

The Euler-Mascheroni constant (see eq. (7b)) is a mathematical constant, used principally in number theory and in the mathematical analysis. It is defined as the limit of the difference between the truncated harmonic series and the natural logarithm:

$$\gamma = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n} - \ln(n) \right),$$

that we can rewrite in the following form:

$$\gamma = \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \frac{1}{k} - \ln(n) \right) = \int_1^{\infty} \left(\frac{1}{[x]} - \frac{1}{x} \right) dx, \quad (7b)$$

where $[x]$ is the part integer function.

We have that γ has the following approximated value $\gamma \cong 0,5772156649\dots$. The constant can be defined in various modes by the integrals:

$$\gamma = - \int_0^{\infty} \frac{\ln(x)}{e^x} dx = - \int_0^1 \ln \ln \left(\frac{1}{x} \right) dx = \int_0^{\infty} \left(\frac{1}{1 - e^{-x}} - \frac{1}{x} \right) e^{-x} dx = \int_0^{\infty} \frac{1}{x} \left(\frac{1}{1 + x} - e^{-x} \right) dx. \quad (7c)$$

The Euler-Mascheroni constant is related to the zeta function from the following expressions:

$$\gamma = \ln \left(\frac{4}{\pi} \right) + \sum_{m=1}^{\infty} \frac{(-1)^{m-1} \zeta(m+1)}{2^m (m+1)}; \quad \gamma = \lim_{s \rightarrow 1^+} \sum_{n=1}^{\infty} \left(\frac{1}{n^s} - \frac{1}{s^n} \right) = \lim_{s \rightarrow 1} \left(\zeta(s) - \frac{1}{s-1} \right). \quad (7d)$$

The constant is also linked to the gamma function:

$$\gamma = \lim_{x \rightarrow \infty} x - \Gamma \left(\frac{1}{x} \right) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \left(\left[\frac{n}{k} \right] - \frac{n}{k} \right), \quad (7e)$$

and to the beta function:

$$\gamma = \lim_{n \rightarrow \infty} \frac{\Gamma \left(\frac{1}{n} \right) \Gamma(n+1) n^{1+1/n}}{\Gamma \left(2 + n + \frac{1}{n} \right)} - \frac{n^2}{n+1}. \quad (7f)$$

Now we describe two theorems related to the Euler-Mascheroni constant γ .

Theorem (1).

For positive integers a we have

$$\lim_{a \rightarrow \infty} \frac{1}{a} \int_1^a \{a/x\} dx = 1 - \gamma \quad \text{where } \gamma \text{ is the Euler-Mascheroni constant.}$$

We have:

$$\begin{aligned}\int_1^a \{a/x\} dx &= \int_1^a (a/x) dx - \int_1^a [a/x] dx = \int_1^a (a/x) dx - \left(\int_1^{(a/a-1)} \left\lfloor \frac{a}{x} \right\rfloor dx + \int_{(a/a-1)}^{(a/a-2)} \left\lfloor \frac{a}{x} \right\rfloor dx + \dots + \int_{(a/2)}^a \left\lfloor \frac{a}{x} \right\rfloor dx \right) \\ &= \int_1^a \left(\frac{a}{x} \right) dx - \sum_{i=1}^{a-1} \left(\frac{a}{(a-i)} - \frac{a}{(a-i+1)} \right) (a-i) = a \ln(a) - \sum_{i=1}^a \frac{a}{i} + a.\end{aligned}$$

Therefore,

$$\begin{aligned}\lim_{a \rightarrow \infty} \frac{1}{a} \int_1^a \{a/x\} dx &= \lim_{a \rightarrow \infty} \frac{1}{a} \left(a \ln(a) - \sum_{i=1}^a \frac{a}{i} + a \right) = \lim_{a \rightarrow \infty} \left(\ln(a) - \sum_{i=1}^a \frac{1}{i} + 1 \right) \\ &= 1 - \lim_{a \rightarrow \infty} (H_a - \ln(a)) = 1 - \gamma. \quad (7g)\end{aligned}$$

Theorem (2).

If p_n denotes the n^{th} prime number, then we get as $n \rightarrow \infty$

$$\lim_{n \rightarrow \infty} \frac{1}{n-1} \sum_{i=1}^{n-1} \frac{d(i)}{p_i} - \frac{\ln(p_n)}{n-1} = 0 \quad \text{where } d(i) = p_{i+1} - p_i.$$

We start by using the result of theorem (1), as $n \rightarrow \infty$

$$\begin{aligned}1 - \gamma &= \frac{1}{p_n} \int_1^{p_n} \left\{ \frac{p_n}{x} \right\} dx = \frac{1}{p_n} \int_1^{p_1} \left\{ \frac{p_n}{x} \right\} dx + \frac{1}{p_n} \int_{p_1}^{p_2} \left\{ \frac{p_n}{x} \right\} dx + \dots + \frac{1}{p_n} \int_{p_{n-2}}^{p_{n-1}} \left\{ \frac{p_n}{x} \right\} dx + \frac{1}{p_n} \int_{p_{n-1}}^{p_n} \left\{ \frac{p_n}{x} \right\} dx \\ &= \frac{1}{p_n} \sum_{i=0}^{n-1} \int_{p_i}^{p_{i+1}} \left\{ \frac{p_n}{x} \right\} dx = \frac{1}{p_n} \sum_{i=0}^{n-1} \int_{p_i}^{p_{i+1}} \frac{p_n}{x} dx - \frac{1}{p_n} \sum_{i=0}^{n-1} \int_{p_i}^{p_{i+1}} \left\lfloor \frac{p_n}{x} \right\rfloor dx \\ &= \ln(p_n) - \frac{1}{p_n} \sum_{i=0}^{n-1} \int_{p_i}^{p_{i+1}} \left\lfloor \frac{p_n}{x} \right\rfloor dx. \quad (7h)\end{aligned}$$

Now we proceed in two directions:

1)

$$\begin{aligned}1 - \gamma &< \ln(p_n) - \frac{1}{p_n} \sum_{i=0}^{n-1} (p_{i+1} - p_i) \left(\frac{p_n}{p_i} - 1 \right) = \ln(p_n) - \sum_{i=0}^{n-1} \frac{d(i)}{p_i} + \frac{1}{p_n} \sum_{i=0}^{n-1} (p_{i+1} - p_i) \\ &= \ln(p_n) - \sum_{i=0}^{n-1} \frac{d(i)}{p_i} + \frac{p_n - 1}{p_n} < \ln(p_n) - \sum_{i=0}^{n-1} \frac{d(i)}{p_i} + 1 = \ln(p_n) - \sum_{i=1}^{n-1} \frac{d(i)}{p_i}.\end{aligned}$$

Hence we get

$$\sum_{i=1}^{n-1} \frac{d(i)}{p_i} < \ln(p_n) - 1 + \gamma.$$

2)

$$\begin{aligned}
1 - \gamma &= \ln(p_n) - \frac{1}{p_n} \sum_{i=0}^{n-1} \int_{p_i}^{p_{i+1}} \left[\frac{p_n}{x} \right] dx > \ln(p_n) - \frac{1}{p_n} \sum_{i=0}^{n-1} (p_{i+1} - p_i) \frac{p_n}{p_{i+1}} = \ln(p_n) - \sum_{i=0}^{n-1} \frac{d(i)}{p_{i+1}} = \\
&= \ln(p_n) - \sum_{i=1}^{n-1} \frac{d(i)}{p_{i+1}} - 1. \quad (7i)
\end{aligned}$$

Hence we get

$$\sum_{i=1}^{n-1} \frac{d(i)}{p_{i+1}} > \ln(p_n) - 2 + \gamma.$$

Combining, the above two cases, we get

$$\ln(p_n) - 2 + \gamma < \sum_{i=1}^{n-1} \frac{d(i)}{p_{i+1}} < \sum_{i=1}^{n-1} \frac{d(i)}{p_i} < \ln(p_n) - 1 + \gamma.$$

Dividing by $n - 1$ and taking limits as $n \rightarrow \infty$, we get the following result, i.e.,

$$\lim_{n \rightarrow \infty} \frac{1}{n-1} \sum_{i=1}^{n-1} \frac{d(i)}{p_i} - \frac{\ln(p_n)}{n-1} = 0.$$

2. Zeta Strings, Zeta Nonlocal Scalar Fields and Some Lagrangians with Zeta Function Nonlocality [7], [8], [9]

The exact tree-level Lagrangian for effective scalar field ϕ which describes open p-adic string tachyon is

$$\mathcal{L}_p = \frac{1}{g^2} \frac{p^2}{p-1} \left[-\frac{1}{2} \phi P^{-\frac{\square}{2}} \phi + \frac{1}{p+1} \phi^{p+1} \right], \quad (13)$$

where P is any prime number, $\square = -\partial_t^2 + \nabla^2$ is the D-dimensional d'Alambertian and we adopt metric with signature $(- + \dots +)$. Now, we want to show a model which incorporates the p-adic string Lagrangians in a restricted adelic way. Let us take the following Lagrangian

$$L = \sum_{n \geq 1} C_n \mathcal{L}_n = \sum_{n \geq 1} \frac{n-1}{n^2} \mathcal{L}_n = \frac{1}{g^2} \left[-\frac{1}{2} \phi \sum_{n \geq 1} n^{-\frac{\square}{2}} \phi + \sum_{n \geq 1} \frac{1}{n+1} \phi^{n+1} \right]. \quad (14)$$

Recall that the Riemann zeta function is defined as

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \prod_p \frac{1}{1 - p^{-s}}, \quad s = \sigma + it, \quad \sigma > 1. \quad (15)$$

Employing usual expansion for the logarithmic function and definition (15) we can rewrite (14) in the form

$$L = -\frac{1}{g^2} \left[\frac{1}{2} \phi \zeta \left(\frac{\square}{2} \right) \phi + \phi + \ln(1 - \phi) \right], \quad (16)$$

where $|\phi| < 1$. $\zeta \left(\frac{\square}{2} \right)$ acts as pseudodifferential operator in the following way:

$$\zeta \left(\frac{\square}{2} \right) \phi(x) = \frac{1}{(2\pi)^D} \int e^{ixk} \zeta \left(-\frac{k^2}{2} \right) \tilde{\phi}(k) dk, \quad -k^2 = k_0^2 - \vec{k}^2 > 2 + \varepsilon, \quad (17)$$

where $\tilde{\phi}(k) = \int e^{-ikx} \phi(x) dx$ is the Fourier transform of $\phi(x)$.

Dynamics of this field ϕ is encoded in the (pseudo)differential form of the Riemann zeta function. **When the d'Alembertian is an argument of the Riemann zeta function we shall call such string a "zeta string"**. Consequently, the above ϕ is an open scalar zeta string. The equation of motion for the zeta string ϕ is

$$\zeta \left(\frac{\square}{2} \right) \phi = \frac{1}{(2\pi)^D} \int_{k_0^2 - \vec{k}^2 > 2 + \varepsilon} e^{ixk} \zeta \left(-\frac{k^2}{2} \right) \tilde{\phi}(k) dk = \frac{\phi}{1 - \phi} \quad (18)$$

which has an evident solution $\phi = 0$.

For the case of time dependent spatially homogeneous solutions, we have the following equation of motion

$$\zeta \left(\frac{-\partial_t^2}{2} \right) \phi(t) = \frac{1}{(2\pi)} \int_{|k_0| > \sqrt{2 + \varepsilon}} e^{-ik_0 t} \zeta \left(\frac{k_0^2}{2} \right) \tilde{\phi}(k_0) dk_0 = \frac{\phi(t)}{1 - \phi(t)}. \quad (19)$$

With regard the open and closed scalar zeta strings, the equations of motion are

$$\zeta\left(\frac{\square}{2}\right)\phi = \frac{1}{(2\pi)^D} \int e^{ixk} \zeta\left(-\frac{k^2}{2}\right) \tilde{\phi}(k) dk = \sum_{n \geq 1} \theta^{\frac{n(n-1)}{2}} \phi^n, \quad (20)$$

$$\zeta\left(\frac{\square}{4}\right)\theta = \frac{1}{(2\pi)^D} \int e^{ixk} \zeta\left(-\frac{k^2}{4}\right) \tilde{\theta}(k) dk = \sum_{n \geq 1} \left[\theta^{n^2} + \frac{n(n-1)}{2(n+1)} \theta^{\frac{n(n-1)}{2}-1} (\phi^{n+1} - 1) \right], \quad (21)$$

and one can easily see trivial solution $\phi = \theta = 0$.

The exact tree-level Lagrangian of effective scalar field ϕ , which describes open p-adic string tachyon, is:

$$\mathcal{L}_p = \frac{m_p^D}{g_p^2} \frac{p^2}{p-1} \left[-\frac{1}{2} \phi p^{-\frac{\square}{2m_p^2}} + \frac{1}{p+1} \phi^{p+1} \right], \quad (22)$$

where P is any prime number, $\square = -\partial_t^2 + \nabla^2$ is the D-dimensional d'Alambertian and we adopt metric with signature $(- + \dots +)$, as above. Now, we want to introduce a model which incorporates all the above string Lagrangians (22) with P replaced by $n \in \mathbb{N}$. Thence, we take the sum of all Lagrangians \mathcal{L}_n in the form

$$L = \sum_{n=1}^{+\infty} C_n \mathcal{L}_n = \sum_{n=1}^{+\infty} C_n \frac{m_n^D}{g_n^2} \frac{n^2}{n-1} \left[-\frac{1}{2} \phi n^{-\frac{\square}{2m_n^2}} + \frac{1}{n+1} \phi^{n+1} \right], \quad (23)$$

whose explicit realization depends on particular choice of coefficients C_n , masses m_n and coupling constants g_n .

Now, we consider the following case

$$C_n = \frac{n-1}{n^{2+h}}, \quad (24)$$

where h is a real number. The corresponding Lagrangian reads

$$L_h = \frac{m^D}{g^2} \left[-\frac{1}{2} \phi \sum_{n=1}^{+\infty} n^{-\frac{\square}{2m^2}-h} \phi + \sum_{n=1}^{+\infty} \frac{n^{-h}}{n+1} \phi^{n+1} \right] \quad (25)$$

and it depends on parameter h . According to the Euler product formula one can write

$$\sum_{n=1}^{+\infty} n^{-\frac{\square}{2m^2}-h} = \prod_p \frac{1}{1 - p^{-\frac{\square}{2m^2}-h}}. \quad (26)$$

Recall that standard definition of the Riemann zeta function is

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - p^{-s}}, \quad s = \sigma + i\tau, \quad \sigma > 1, \quad (27)$$

which has analytic continuation to the entire complex s plane, excluding the point $s = 1$, where it has a simple pole with residue 1. Employing definition (27) we can rewrite (25) in the form

$$L_h = \frac{m^D}{g^2} \left[-\frac{1}{2} \phi \zeta \left(\frac{\square}{2m^2} + h \right) \phi + \sum_{n=1}^{+\infty} \frac{n^{-h}}{n+1} \phi^{n+1} \right]. \quad (28)$$

Here $\zeta \left(\frac{\square}{2m^2} + h \right)$ acts as a pseudodifferential operator

$$\zeta \left(\frac{\square}{2m^2} + h \right) \phi(x) = \frac{1}{(2\pi)^D} \int e^{ixk} \zeta \left(-\frac{k^2}{2m^2} + h \right) \tilde{\phi}(k) dk, \quad (29)$$

where $\tilde{\phi}(k) = \int e^{-ikx} \phi(x) dx$ is the Fourier transform of $\phi(x)$.

We consider Lagrangian (28) with analytic continuations of the zeta function and the power series

$$\sum \frac{n^{-h}}{n+1} \phi^{n+1}, \text{ i.e.}$$

$$L_h = \frac{m^D}{g^2} \left[-\frac{1}{2} \phi \zeta \left(\frac{\square}{2m^2} + h \right) \phi + AC \sum_{n=1}^{+\infty} \frac{n^{-h}}{n+1} \phi^{n+1} \right], \quad (30)$$

where AC denotes analytic continuation.

Potential of the above zeta scalar field (30) is equal to $-L_h$ at $\square = 0$, i.e.

$$V_h(\phi) = \frac{m^D}{g^2} \left(\frac{\phi^2}{2} \zeta(h) - AC \sum_{n=1}^{+\infty} \frac{n^{-h}}{n+1} \phi^{n+1} \right), \quad (31)$$

where $h \neq 1$ since $\zeta(1) = \infty$. The term with ζ -function vanishes at $h = -2, -4, -6, \dots$. The equation of motion in differential and integral form is

$$\zeta \left(\frac{\square}{2m^2} + h \right) \phi = AC \sum_{n=1}^{+\infty} n^{-h} \phi^n, \quad (32)$$

$$\frac{1}{(2\pi)^D} \int_{R^D} e^{ixk} \zeta \left(-\frac{k^2}{2m^2} + h \right) \tilde{\phi}(k) dk = AC \sum_{n=1}^{+\infty} n^{-h} \phi^n, \quad (33)$$

respectively.

Now, we consider five values of h , which seem to be the most interesting, regarding the Lagrangian (30): $h = 0$, $h = \pm 1$, and $h = \pm 2$. For $h = -2$, the corresponding equation of motion now read:

$$\zeta \left(\frac{\square}{2m^2} - 2 \right) \phi = \frac{1}{(2\pi)^D} \int_{R^D} e^{ixk} \zeta \left(-\frac{k^2}{2m^2} - 2 \right) \tilde{\phi}(k) dk = \frac{\phi(\phi+1)}{(1-\phi)^3}. \quad (34)$$

This equation has two trivial solutions: $\phi(x) = 0$ and $\phi(x) = -1$. Solution $\phi(x) = -1$ can be also shown taking $\tilde{\phi}(k) = -\delta(k)(2\pi)^D$ and $\zeta(-2) = 0$ in (34).

For $h = -1$, the corresponding equation of motion is:

$$\zeta\left(\frac{\square}{2m^2} - 1\right)\phi = \frac{1}{(2\pi)^D} \int_{R^D} e^{ixk} \zeta\left(-\frac{k^2}{2m^2} - 1\right) \tilde{\phi}(k) dk = \frac{\phi}{(1-\phi)^2}. \quad (35)$$

where $\zeta(-1) = -\frac{1}{12}$.

The equation of motion (35) has a constant trivial solution only for $\phi(x) = 0$.

For $h = 0$, the equation of motion is

$$\zeta\left(\frac{\square}{2m^2}\right)\phi = \frac{1}{(2\pi)^D} \int_{R^D} e^{ixk} \zeta\left(-\frac{k^2}{2m^2}\right) \tilde{\phi}(k) dk = \frac{\phi}{1-\phi}. \quad (36)$$

It has two solutions: $\phi = 0$ and $\phi = 3$. The solution $\phi = 3$ follows from the Taylor expansion of the Riemann zeta function operator

$$\zeta\left(\frac{\square}{2m^2}\right) = \zeta(0) + \sum_{n \geq 1} \frac{\zeta^{(n)}(0)}{n!} \left(\frac{\square}{2m^2}\right)^n, \quad (37)$$

as well as from $\tilde{\phi}(k) = (2\pi)^D 3\delta(k)$.

For $h = 1$, the equation of motion is:

$$\frac{1}{(2\pi)^D} \int_{R^D} e^{ixk} \zeta\left(-\frac{k^2}{2m^2} + 1\right) \tilde{\phi}(k) dk = -\frac{1}{2} \ln(1-\phi)^2, \quad (38)$$

where $\zeta(1) = \infty$ gives $V_1(\phi) = \infty$.

In conclusion, for $h = 2$, we have the following equation of motion:

$$\frac{1}{(2\pi)^D} \int_{R^D} e^{ixk} \zeta \left(-\frac{k^2}{2m^2} + 2 \right) \tilde{\phi}(k) dk = - \int_0^\phi \frac{\ln(1-w)^2}{2w} dw. \quad (39)$$

Since holds equality

$$- \int_0^1 \frac{\ln(1-w)}{w} dw = \sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2)$$

one has trivial solution $\phi = 1$ in (39).

Now, we want to analyze the following case: $C_n = \frac{n^2 - 1}{n^2}$. In this case, from the Lagrangian (23), we obtain:

$$L = \frac{m^D}{g^2} \left[-\frac{1}{2} \phi \left\{ \zeta \left(\frac{\square}{2m^2} - 1 \right) + \zeta \left(\frac{\square}{2m^2} \right) \right\} \phi + \frac{\phi^2}{1 - \phi} \right]. \quad (40)$$

The corresponding potential is:

$$V(\phi) = -\frac{m^D}{g} \frac{31 - 7\phi}{24(1 - \phi)} \phi^2. \quad (41)$$

We note that 7 and 31 are prime natural numbers, i.e. $6n \pm 1$ with $n=1$ and 5, with 1 and 5 that are Fibonacci's numbers. Furthermore, the number 24 is related to the Ramanujan function that has 24 "modes" that correspond to the physical vibrations of a bosonic string. Thence, we obtain:

$$V(\phi) = -\frac{m^D}{g} \frac{31 - 7\phi}{24(1 - \phi)} \phi^2 \Rightarrow \frac{4 \left[\text{anti log} \frac{\int_0^\infty \frac{\cos \pi tx w'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi t^2 w'}{4}} \phi_{w'}(itw')} \right] \cdot \frac{\sqrt{142}}{t^2 w'}}{\log \left[\sqrt{\left(\frac{10 + 11\sqrt{2}}{4} \right)} + \sqrt{\left(\frac{10 + 7\sqrt{2}}{4} \right)} \right]}. \quad (41b)$$

The equation of motion is:

$$\left[\zeta \left(\frac{\square}{2m^2} - 1 \right) + \zeta \left(\frac{\square}{2m^2} \right) \right] \phi = \frac{\phi [(\phi - 1)^2 + 1]}{(\phi - 1)^2}. \quad (42)$$

Its weak field approximation is:

$$\left[\zeta \left(\frac{\square}{2m^2} - 1 \right) + \zeta \left(\frac{\square}{2m^2} \right) - 2 \right] \phi = 0, \quad (43)$$

which implies condition on the mass spectrum

$$\zeta \left(\frac{M^2}{2m^2} - 1 \right) + \zeta \left(\frac{M^2}{2m^2} \right) = 2. \quad (44)$$

From (44) it follows one solution for $M^2 > 0$ at $M^2 \approx 2.79m^2$ and many tachyon solutions when $M^2 < -38m^2$.

We note that the number 2.79 is connected with $\phi = \frac{\sqrt{5} - 1}{2}$ and $\Phi = \frac{\sqrt{5} + 1}{2}$, i.e. the “aurea” section and the “aurea” ratio. Indeed, we have that:

$$\left(\frac{\sqrt{5} + 1}{2} \right)^2 + \frac{1}{2^2} \left(\frac{\sqrt{5} - 1}{2} \right) = 2,772542 \cong 2,78.$$

Furthermore, we have also that:

$$(\Phi)^{14/7} + (\Phi)^{-25/7} = 2,618033989 + 0,179314566 = 2,79734$$

With regard the extension by ordinary Lagrangian, we have the Lagrangian, potential, equation of motion and mass spectrum condition that, when $C_n = \frac{n^2 - 1}{n^2}$, are:

$$L = \frac{m^D}{g^2} \left[\frac{\phi}{2} \left\{ \frac{\square}{m^2} - \zeta \left(\frac{\square}{2m^2} - 1 \right) - \zeta \left(\frac{\square}{2m^2} \right) - 1 \right\} \phi + \frac{\phi^2}{2} \ln \phi^2 + \frac{\phi^2}{1-\phi} \right], \quad (45)$$

$$V(\phi) = \frac{m^D \phi^2}{g^2} \left[\zeta(-1) + \zeta(0) + 1 - \ln \phi^2 - \frac{1}{1-\phi} \right], \quad (46)$$

$$\left[\zeta \left(\frac{\square}{2m^2} - 1 \right) + \zeta \left(\frac{\square}{2m^2} \right) - \frac{\square}{m^2} + 1 \right] \phi = \phi \ln \phi^2 + \phi + \frac{2\phi - \phi^2}{(1-\phi)^2}, \quad (47)$$

$$\zeta \left(\frac{M^2}{2m^2} - 1 \right) + \zeta \left(\frac{M^2}{2m^2} \right) = \frac{M^2}{m^2}. \quad (48)$$

In addition to many tachyon solutions, equation (48) has two solutions with positive mass: $M^2 \approx 2.67m^2$ and $M^2 \approx 4.66m^2$.

We note also here, that the numbers 2.67 and 4.66 are related to the “aureo” numbers. Indeed, we have that:

$$\left(\frac{\sqrt{5}+1}{2} \right)^2 + \frac{1}{2 \cdot 5} \left(\frac{\sqrt{5}-1}{2} \right) \cong 2.6798,$$

$$\left(\frac{\sqrt{5}+1}{2} \right)^2 + \left(\frac{\sqrt{5}+1}{2} \right) + \frac{1}{2^2} \left(\frac{\sqrt{5}+1}{2} \right) \cong 4.64057.$$

Furthermore, we have also that:

$$(\Phi)^{14/7} + (\Phi)^{-41/7} = 2,618033989 + 0,059693843 = 2,6777278;$$

$$(\Phi)^{22/7} + (\Phi)^{-30/7} = 4,537517342 + 0,1271565635 = 4,6646738.$$

Now, we describe the case of $C_n = \mu(n) \frac{n-1}{n^2}$. Here $\mu(n)$ is the Mobius function, which is defined for all positive integers and has values 1, 0, -1 depending on factorization of n into prime numbers P . It is defined as follows:

$$\mu(n) = \begin{cases} 0, & n = p^2 m \\ (-1)^k, & n = p_1 p_2 \dots p_k, p_i \neq p_j \\ 1, & n = 1, (k = 0). \end{cases} \quad (49)$$

The corresponding Lagrangian is

$$L_\mu = C_0 \mathcal{L}_0 + \frac{m^D}{g^2} \left[-\frac{1}{2} \phi \sum_{n=1}^{+\infty} \frac{\mu(n)}{n^{\frac{\square}{2m^2}}} \phi + \sum_{n=1}^{+\infty} \frac{\mu(n)}{n+1} \phi^{n+1} \right] \quad (50)$$

Recall that the inverse Riemann zeta function can be defined by

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{+\infty} \frac{\mu(n)}{n^s}, \quad s = \sigma + it, \quad \sigma > 1. \quad (51)$$

Now (50) can be rewritten as

$$L_\mu = C_0 \mathcal{L}_0 + \frac{m^D}{g^2} \left[-\frac{1}{2} \phi \frac{1}{\zeta\left(\frac{\square}{2m^2}\right)} \phi + \int_0^\infty \mathcal{M}(\phi) d\phi \right], \quad (52)$$

where $\mathcal{M}(\phi) = \sum_{n=1}^{+\infty} \mu(n) \phi^n = \phi - \phi^2 - \phi^3 - \phi^5 + \phi^6 - \phi^7 + \phi^{10} - \phi^{11} - \dots$. The corresponding potential, equation of motion and mass spectrum formula, respectively, are:

$$V_\mu(\phi) = -L_\mu(\square = 0) = \frac{m^D}{g^2} \left[\frac{C_0}{2} \phi^2 (1 - \ln \phi^2) - \phi^2 - \int_0^\phi \mathcal{M}(\phi) d\phi \right], \quad (53)$$

$$\frac{1}{\zeta\left(\frac{\square}{2m^2}\right)} \phi - \mathcal{M}(\phi) - C_0 \frac{\square}{m^2} \phi - 2C_0 \phi \ln \phi = 0, \quad (54)$$

$$\frac{1}{\zeta\left(\frac{M^2}{2m^2}\right)} - C_0 \frac{M^2}{m^2} + 2C_0 - 1 = 0, \quad |\phi| \ll 1, \quad (55)$$

where usual relativistic kinematic relation $k^2 = -k_0^2 + \vec{k}^2 = -M^2$ is used.

Now, we take the pure numbers concerning the eqs. (44) and (48). They are: 2.79, 2.67 and 4.66. We note that all the numbers are related with $\Phi = \frac{\sqrt{5}+1}{2}$, thence with the aurea ratio, by the following expressions:

$$2,79 \cong (\Phi)^{15/7}; \quad 2,67 \cong (\Phi)^{13/7} + (\Phi)^{-21/7}; \quad 4,66 \cong (\Phi)^{22/7} + (\Phi)^{-30/7}. \quad (56)$$

3. Mathematical connections

Now we take the eqs. (5) and (12) of the **Section 1**, and the eqs. (18), (21) and (36) of **Section 2**. We obtain the following mathematical connections:

$$\begin{aligned} \Gamma_F(x) &= \phi^{(x+1)x/2} \lim_{k \rightarrow \infty} \frac{F_k^x}{F_x} \prod_{i=1}^k \frac{F_i}{F_{x+i}} \Rightarrow \frac{1}{(2\pi)^D} \int_{k_0^2 - \vec{k}^2 > 2+\epsilon} e^{ixk\zeta} \left(-\frac{k^2}{2}\right) \tilde{\phi}(k) dk = \frac{\phi}{1-\phi} \Rightarrow \\ &\Rightarrow \frac{1}{(2\pi)^D} \int e^{ixk\zeta} \left(-\frac{k^2}{4}\right) \tilde{\theta}(k) dk = \sum_{n \geq 1} \left[\theta^{n^2} + \frac{n(n-1)}{2(n+1)} \theta^{\frac{n(n-1)}{2}-1} (\phi^{n+1} - 1) \right] \Rightarrow \\ &\Rightarrow \frac{1}{(2\pi)^D} \int_{R^D} e^{ixk\zeta} \left(-\frac{k^2}{2m^2}\right) \tilde{\phi}(k) dk = \frac{\phi}{1-\phi}; \quad (57) \end{aligned}$$

$$\begin{aligned}
\frac{1}{\Gamma_F(x)} &= \phi^{-(x+1)x/2} F_x e^{x\gamma_F} \prod_{i=1}^{\infty} \frac{F_{x+i}}{F_i} e^{-x F_i / F_i} \Rightarrow \\
&\Rightarrow \frac{1}{(2\pi)^D} \int_{k_0^2 - \bar{k}^2 > 2+\varepsilon} e^{ixk} \zeta \left(-\frac{k^2}{2} \right) \tilde{\phi}(k) dk = \frac{\phi}{1-\phi} \Rightarrow \\
&\Rightarrow \frac{1}{(2\pi)^D} \int e^{ixk} \zeta \left(-\frac{k^2}{4} \right) \tilde{\theta}(k) dk = \sum_{n \geq 1} \left[\theta^{n^2} + \frac{n(n-1)}{2(n+1)} \theta^{\frac{n(n-1)}{2}-1} (\phi^{n+1} - 1) \right] \Rightarrow \\
&\Rightarrow \frac{1}{(2\pi)^D} \int_{R^D} e^{ixk} \zeta \left(-\frac{k^2}{2m^2} \right) \tilde{\phi}(k) dk = \frac{\phi}{1-\phi}. \quad (58)
\end{aligned}$$

Now we take the eqs. (7h) and (7i) of the **Section 1**, and the eqs. (18), (21) and (36) of **Section 2**. We obtain the following mathematical connections:

$$\begin{aligned}
1 - \gamma &= \frac{1}{p_n} \int_1^{p_n} \left\{ \frac{p_n}{x} \right\} dx = \ln(p_n) - \frac{1}{p_n} \sum_{i=0}^{n-1} \int_{p_i}^{p_{i+1}} \left\{ \frac{p_n}{x} \right\} dx \Rightarrow \\
&\Rightarrow \frac{1}{(2\pi)^D} \int_{k_0^2 - \bar{k}^2 > 2+\varepsilon} e^{ixk} \zeta \left(-\frac{k^2}{2} \right) \tilde{\phi}(k) dk = \frac{\phi}{1-\phi} \Rightarrow \\
&\Rightarrow \frac{1}{(2\pi)^D} \int e^{ixk} \zeta \left(-\frac{k^2}{4} \right) \tilde{\theta}(k) dk = \sum_{n \geq 1} \left[\theta^{n^2} + \frac{n(n-1)}{2(n+1)} \theta^{\frac{n(n-1)}{2}-1} (\phi^{n+1} - 1) \right] \Rightarrow \\
&\Rightarrow \frac{1}{(2\pi)^D} \int_{R^D} e^{ixk} \zeta \left(-\frac{k^2}{2m^2} \right) \tilde{\phi}(k) dk = \frac{\phi}{1-\phi}; \quad (59)
\end{aligned}$$

$$\begin{aligned}
1 - \gamma &= \ln(p_n) - \frac{1}{p_n} \sum_{i=0}^{n-1} \int_{p_i}^{p_{i+1}} \left\{ \frac{p_n}{x} \right\} dx = \ln(p_n) - \sum_{i=1}^{n-1} \frac{d(i)}{p_{i+1}} - 1 \Rightarrow \\
&\Rightarrow \frac{1}{(2\pi)^D} \int_{k_0^2 - \bar{k}^2 > 2+\varepsilon} e^{ixk} \zeta \left(-\frac{k^2}{2} \right) \tilde{\phi}(k) dk = \frac{\phi}{1-\phi} \Rightarrow
\end{aligned}$$

$$\Rightarrow \frac{1}{(2\pi)^D} \int e^{ixk} \zeta \left(-\frac{k^2}{4} \right) \tilde{\theta}(k) dk = \sum_{n \geq 1} \left[\theta^{n^2} + \frac{n(n-1)}{2(n+1)} \theta^{\frac{n(n-1)}{2}-1} (\phi^{n+1} - 1) \right] \Rightarrow$$

$$\Rightarrow \frac{1}{(2\pi)^D} \int_{R^D} e^{ixk} \zeta \left(-\frac{k^2}{2m^2} \right) \tilde{\phi}(k) dk = \frac{\phi}{1-\phi}. \quad (60)$$

Acknowledgments

The co-author **Nardelli Michele** would like to thank Prof. **Branko Dragovich** of Institute of Physics of Belgrade (Serbia) for his availability and friendship.

References

- [1] Sulle spalle dei giganti - Rosario Turco, Maria Colonnese, Michele Nardelli, Giovanni Di Maria, Francesco Di Noto, Annarita Tulumello
- [2] C'è solo un'acca tra pi e phi - Rosario Turco, Maria Colonnese
- [3] Gamma and related functions generalized for sequences -R. L. Ollerton
- [4] Ward, M. (1936), un calcolo di sequenze. American Journal of Mathematics 58: 2, pp. 255-266.
- [5] Sloane, NJA (2006) On-Line Encyclopedia of Integer Sequences - Disponibile online all'indirizzo: www.research.att.com/~njas/sequence/ (accessed 23 giugno 2005)
- [6] Andrews, GE, Askey, R. Ranjan, R. (1999) Funzioni speciali. Encyclopedia of Mathematics and its Applications 71, Cambridge University Press, Cambridge
- [7] Branko Dragovich – “Zeta Strings” – arXiv:hep-th/0703008v1 – 1 Mar 2007.
- [8] Branko Dragovich – “Zeta Nonlocal Scalar Fields” – arXiv:0804.4114v1 – [hep-th] – 25 Apr 2008.
- [9] Branko Dragovich – “Some Lagrangians with Zeta Function Nonlocality” – arXiv:0805.0403 v1 – [hep-th] – 4 May 2008.