On the Andrica and Cramer's Conjectures. Mathematical connections between Number Theory and some sectors of String Theory

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Abstract

In this paper we have described, in the **Section 1**, some mathematics concerning the Andrica's conjecture. In the **Section 2**, we have described the Cramer –Shank Conjecture. In the **Section 3**, we have described some equations concerning the possible proof of the Cramer's conjecture and the related differences between prime numbers, principally the Cramer's conjecture and Selberg's theorem. In the **Section 4**, we have described some equations concerning the p-adic strings and the zeta strings. In the **Section 5**, we have described some equations concerning the Ω-deformation in toroidal compactification for N = 2 gauge theory. In conclusion, in the **Section 6**, we have described some possible mathematical connections between various sectors of string theory and number theory.

1. The Andrica's Conjecture [1]

In this section we will show some mathematics related to the Andrica's conjecture:

$$\sqrt{p_{n+1}} - \sqrt{p_n} < 1$$

using some our results on Legendre's conjecture ([2]).

Andrica's conjecture

Andrica's conjecture is so defined:

"…*Andrica's Conjecture is a conjecture of Numbers' Theory, concerning the gaps between two successive prime numbers, formulated by romeno's mathematician Dorin Andrica in 1986. It affirms that, for every couple of consecutive numbers p_n and p_{n+1}, we have:*

$$\sqrt{p_{n+1}} - \sqrt{p_n} < 1$$

If we pose $g_n = p_{n+1} - p_n$, then the conjecture can be written as

$$g_n < 2\sqrt{p_n} + 1$$ ".

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Now we propose some mathematics concepts useful for a possible proof based on some demonstrations concerning the Legendre's Conjecture, and on square roots of numbers included in the numeric gap between a square and the successive one.

"*Legendre's Conjecture, by Adrien – Marie Legendre, affirms that exists always a prime number between n^2 and $(n+1)^2$. This conjecture is one of problems of Landau and, till now, it has not been demonstrated*".

Some observations about Legendre's conjectures are:

- between n^2 and $(n+1)^2$ don't exists always a prime number, but at least two.
- ERATOSTENE Group has developed it, see [1] and [2].

Difference between to perfect squares in the range I = $[n^2, (n+1)^2]$

To examine the connection between Legendre's conjecture and the Andrica 's conjecture, we must introduce some concepts.

Let I the closed range of integers definite as I = $[n^2, (n+1)^2]$.

Let D_{qp} the difference between to consecutive perfect squares, in the range I.

Lemma 1.

The difference D_{qp} *between two consecutive perfect squares, in a closed range of integers* I = $[n^2, (n+1)^2]$ *is always an odd number*.

Proof.

$D_{qp} = (n+1)^2 - n^2 = 2n + 1$

Since for every n, $D_{qp} = 2n + 1$, then is always odd.

Example:

n = 2, valid for all the n natural numbers.

$D_{qp} = 3^2 - 2^2 = 9 - 4 = 5 = 2*2 + 1$

Lemma 2.

The number n of integer included in a closed range of integers $I = [n^2, (n+1)^2]$ *is an even number*.

Proof.

For Lemma 1, since number of integer in I is:

$N = D_{qp} +1 = 2n + 2 = 2(n + 1)$, then N is an even number.

Example:

If $n = 2$ $N = 2(3) = 6$. Indeed the numbers included in gap I are : 4, 5, 6, 7, 8, 9; with 5 and 7 prime numbers.

Square roots of numbers in gap $I = [n^2, (n+1)^2]$.

Lemma 3.

The difference D_{rq} *of square roots of two numbers, also not consecutive (prime and composite), in a closed range of integers* $I = [n^2, (n+1)^2]$, *excepts the number* $(n +1)^2$, *is smaller than* 1.

Proof.

At the extremes of the range of integer, D_{rq} is :

$D_{rq} = n – n = 0$ if we consider at beginning of the interval the difference with itself or $D_{rq} = (n+1) – n = 1$. Therefore D_{rq} changes between 0 and 1.

Lemma 3 excludes the numbers $(n + 1)^2$, because in the second case the difference doesn't give a decimal part after the point. For this Lemma 3 is to check between n^2 and $(n +1)^2 -1$.

Since we think true the Lengendre's conjecture (see [2]), then between n^2 and $(n + 1)^2$ exists at least a prime number and therefore an integer, so between the values 0 and 1 assumed by D_{rq} exist some values smaller than 1.

Obviously since we make reference at integer numbers in the range I; it is indifferent that they are prime or composite. Therefore it is possible the applicability of Legendre's conjecture.

Example Square roots for n = 2.

$\sqrt{4} = 2,00$

$\sqrt{5} = 2,23$ with 5 prime number p_n

$\sqrt{6} = 2,44$

$\sqrt{7} = 2,64$ with 7 prime number p_{n+1}

$\sqrt{8} = 2{,}82$

$\sqrt{9} = 3{,}00$

Lemma 4.

In the range of integers I = $[n^2, (n+1)^2]$ *exist at least two prime numbers*.

Proof.

The Bertrand's postulate, that is true, says that "if n is an integer with $n > 1$, then there is always a prime number such that $n < p < 2n$".

If we define $a = n^2$ then the interval that we are considering is $[a, a + 1 + 2\sqrt{a}]$. Now for $n > 3$ the term $a + 1 + 2\sqrt{a} > 2a$; therefore certainly is applicable the Betrand's postulate but we observe also that this interval is bigger than of that used in Bertrand's postulate, therefore it increases the probability to find at least a second prime number; in fact for Prime Number Theorem is:

$$\pi((n+1)^2) - \pi(n^2) \approx \frac{2n+1}{\ln((n+1)^2)} > 1 \quad (1)$$

Note: the intervals that we consider are the smaller critical intervals where we could risk don't find the second prime number, but that the (1) guarantee. In the case of Andrica's conjecture, we think, moreover, that the two consecutive prime numbers exists also a notable distances or notable gaps.

Example:

In the interval I with $n = 2$ we have the two consecutive prime numbers 5 and 7.

$\sqrt{7} - \sqrt{5} = 2{,}64 - 2.23 = 0{,}41 < 1$

From (1) results $\pi((n+1)^2)-\pi(n^2) \approx 2{,}27$

We note that this value is related with the aurea ratio by the following expression:

$$2{,}27 \cong \frac{1}{2}\left[(\Phi)^{21/7} + (\Phi)^{-21/7} + (\Phi)^{-35/7} + (\Phi)^{-56/7}\right] = \frac{1}{2} \cdot 4{,}58359 = 2{,}29179$$

with $\Phi = \frac{\sqrt{5}+1}{2} = 1{,}61803398...$, i.e. the aurea ratio.

Lemma 5.

The difference of square roots of two consecutive prime numbers that are in a closed interval of integers I = $[n^2, (n+1)^2]$, *except the number* $(n+1)^2$, *is smaller of* 1.

Proof.

The Lemma 5 is a consequence of Lemmas 3 and 4.

It is not still a proof of Andrica's conjecture; because the consecutive prime numbers could belong to different square intervals.

Prime numbers in different square intervals.

Some prime numbers belong to successive square intervals, also being valid the Andrica's conjecture, for example 113 and 127: the first is included in the interval between 10^2 and 11^2, the second, 127, is included in the interval between 11^2 and 12^2. Really the square interval is always possible individualize only one: for example it is between 10^2 and 12^2.

Lemma 6.

The difference of square roots of two numbers included in a closed interval of integers $I = [n^2, (n+1)^2]$ *with* $k \geq 1$, *except the number* $(n+k)^2$, *is lowest of 1 provided that if* $k > 1$ *the difference* $(n+k)^2 - n^2 \neq 0 \bmod 3$.

Proof.

Lemma 6 can be demonstrated with all previous Lemmas, marking also that k tends only to increase certainly the interval of squares, Therefore the Lemma 6 is a generalization of Lemma 5. In particular if k = 1 we return at Lemma 5 and it doesn't occur to consider if the difference is a multiple of 3.

For example 131 and 137 are two prime numbers and their difference is multiple of 3, but it doesn't count because the interval is the same for both the prime numbers. In fact is $[11^2, 12^2]$ with k = 1.

Instead if we look 113 and 137 the interval to consider is different. That is k = 2, in fact is $[10^2, 12^2]$ but 137 – 113 = 24 that is multiple of 3.

Here the difference between the square roots is greater than 1 when the difference is even and multiple of 3 (it is the same to say that it is multiple of 6). But there is to say that 137 and 113 are not neither consecutive prime numbers. The problem that the difference between square roots of two consecutive prime numbers can be greater than 1 could be when the two square intervals are not adjacent, thence for example for k > 2.

With regard the prime numbers 113 and 137 and their difference, i.e. 24, we note that it is possible the following mathematical connection with the Ramanujan's modular function concerning the physical vibrations of the bosonic strings:

$$137-113=24=\frac{4\left[anti\log\frac{\int_0^{\infty}\frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}.$$

Lemma 7.

Two consecutive prime numbers, included in closed interval of integers I = $[n^2, (n+1)^2]$ with k > 2, haven't got a difference D = 6j when j > 1, or even and multiple of 3.

Proof.

If the difference of two consecutive prime numbers is even and multiple of 3, it could be:

D = 3m = 6j where m = 2j (even)

If j = 1 we have the situation k = 2 and of the prime numbers as 23 and 29 where D = 6 (j=1) and the difference of square roots is smaller than 1; thence the Lemma concentrate itself on cases j > 1. Now the prime numbers can be to build with generator form $p_n = 6n \pm 1$, therefore if there exist two consecutive prime numbers in intervals I with k > 2 and j > 1 they have never D = 6j for the same generator form.

However, if for absurd the consecutive prime numbers are such that:

$$p_{n+1} - p_n = 6j, \quad j > 1 \quad (2)$$

equivalent to:

$$\sqrt{p_{n+1}} = \sqrt{p_n + 6j}$$

Then we conclude that

$$\sqrt{p_{n+1}} - \sqrt{p_n} = \sqrt{p_n + 6j} - \sqrt{p_n} > 1 \quad (3)$$

Since the (2) is false, we cannot conclude the (3). In other words if the difference between two consecutive prime numbers is multiple of 6 we have always found that the difference of square roots of two consecutive prime numbers is greater than 1.

Andrica's conjecture

The Andrica's conjecture is true for consequence of Lemma 6 and Lemma 7.

Proof.

The conjecture supposes the existence of two consecutive prime numbers. If these are included in the same square interval already the Lemma 5 will gives true the conjecture. If the two prime numbers are included in different square intervals then Lemma 6 and Lemma 7 guarantee that the conjecture is true.

2. The Cramer –Shank Conjecture [2]

In this Section we have described the Cramer – Shank Conjecture, utilizing the mathematics used in the precedent Section on the Andrica's conjecture.

In the Cramer's conjecture, R(p) is the *Cramer – Shank ratio,* it doesn't to be greater of 1 so that the Cramer's conjecture is true, in other words Cramer's conjecture is true if

$$R(p) = \frac{p_{n+1} - p_n}{(\ln p_n)^2} < 1$$

The greatest value of R(p) known is 0,92 for p_n =1 693 182 318 746 370 with gap = 1132 between this number and the following one $p_{n+1} = p_n + 1132$.

It is interesting note that the value 0,92 is related to the aurea ratio by the following expression:

$$0{,}92 \cong 0{,}92705098 = (\Phi)^{-7/7} \cdot \frac{3}{2} = 0{,}61803399 \cdot \frac{3}{2} = \left(\frac{\sqrt{5}-1}{2}\right) \cdot \frac{3}{2},$$

$$\text{where } \Phi = \frac{\sqrt{5}+1}{2} = 1{,}61803398...$$

Lemma

If the Andrica's conjecture is true, then Cramer's conjecture is true.

Proof.

In the precedent Section we have described some results concerning the Andrica's conjecture. Here we have showed a consequence that influences on Cramer's conjectures

If the Andrica's conjecture is true, then is

$$\sqrt{p_{n+1}} - \sqrt{p_n} < 1$$

From here, if we raise to square both the members and we take into consideration the algebraic rule $(a+b)^2 = a^2 + 2ab + b^2$, we obtain

$$\left(\sqrt{p_{n+1}} - \sqrt{p_n}\right)^2 < 1$$

From here, we obtain:

$$p_{n+1} + p_n - 2\sqrt{p_{n+1}}\sqrt{p_n} < 1$$

Re – arranging the formula, subtracting to both the members p_n we obtain

$$p_{n+1} - p_n + p_n - 2\sqrt{p_{n+1}}\sqrt{p_n} < 1 - p_n$$
$$p_{n+1} - p_n < 1 - p_n + 2\sqrt{p_{n+1}}\sqrt{p_n}$$
$$R(p) = \frac{p_{n+1} - p_n}{(\ln p_n)^2} < \frac{1 - p_n + 2\sqrt{p_{n+1}}\sqrt{p_n}}{(\ln p_n)^2} < 1$$

This is the demonstration that

$$R(p) = \frac{p_{n+1} - p_n}{(\ln p_n)^2} < 1$$

3. On some equations concerning a proof of the Cramer's conjecture and the related differences between prime numbers, principally the Cramer's conjecture and Selberg's theorem. [3]

In number theory, Cramer's conjecture, formulated by the Swedish mathematician Harald Cramer in 1936, states that

$$p_{n+1} - p_n = O\left((\log p_n)^2\right), \quad (3.1)$$

where p_n denotes the n[th] prime number, O is big O notation, and "log" is the natural logarithm. Cramer also gave much weaker conditional proof that

$$p_{n+1} - p_n = \mathrm{O}\left(\sqrt{p_n}\log p_n\right)$$

on the assumption of the Riemann hypothesis.
In the big-oh notation the eq. (3.1) can be rewritten also as follows

$$d_n = O\left((\log p_n)^2\right). \quad (3.2)$$

Let us take $f(x) = \log x$. First we prove, that $\lim_{n\to\infty}(T_n - S_n)$ is exists. We have that

$$\lim_{n\to\infty} T_n - S_n = \sum_{i=p_1}^{\infty} \frac{d_i}{\log p_i} - \frac{d_i}{\log p_{i+1}}. \quad (3.3)$$

We use the root test to show that the limit exists and is finite. The k^{th} term is

$$a_k = \frac{d_k}{\log p_k} - \frac{d_k}{\log p_{k+1}}. \quad (3.4)$$

Let, $v_k = \frac{d_k}{p_k} - \frac{d_k}{p_{k+1}}$, we get using the Bertrand's Postulate,

$$\limsup_{k\to\infty} |v_k|^{1/k} = \limsup_{k\to\infty} \left| \frac{(p_{k+1}-p_k)^2}{p_k p_{k+1}} \right|^{1/k} < \limsup_{k\to\infty} \left| \frac{p_k}{p_{k+1}} \right|^{1/k} = 1. \quad (3.5)$$

Hence, looking at the conclusion of root test we can say, $\lim_{n\to\infty} T_n - S_n$ exists. Therefore, there exists $r_0 \in N$, such that for all $n > r_0$

$$\int_{p_n}^{p_{n+1}} \frac{dx}{x} = \frac{d_n}{p_n}. \quad (3.6)$$

Since, by the prime number theorem $\lim_{x\to\infty} \frac{\pi(x)}{x/\ln(x)} = 1$, we can show that, $\lim_{n\to\infty} p_{n+1}/p_n = 1$. Therefore, we get, as $n \to \infty$

$$d_n = p_{n+1} \log\left(\frac{p_{n+1}}{p_n} \right). \quad (3.7)$$

Therefore,

$$\limsup_{k\to\infty} |a_k|^{1/k} = \limsup_{k\to\infty} \left| \frac{d_k}{\log p_k} - \frac{d_k}{\log p_{k+1}} \right|^{1/k} = \limsup_{k\to\infty} \left| \frac{p_{k+1}\log\left(\frac{p_{k+1}}{p_k}\right)}{\log p_k} - \frac{p_{k+1}\log\left(\frac{p_{k+1}}{p_k}\right)}{\log p_{k+1}} \right|^{1/k}. \quad (3.8)$$

Now, we have, as $x \to \infty$

$$\frac{x}{\log x} < \pi(x). \quad (3.9)$$

Also, we have $p_n^{\frac{1}{n+1}} > p_n^{\frac{1}{p_n}}$. Also, as $n \to \infty$, $\lim_{n\to\infty} p_n^{\frac{1}{p_n}} = 1$. We know from the prime number theorem, $\lim_{n\to\infty} p_{n+1}/p_n = 1$. Hence, as $n \to \infty$

$$p_n^{\frac{1}{n+1}} \geq \frac{p_{n+1}}{p_n} \quad (3.10)$$

i.e,

$$(n+1)\frac{\log p_{n+1}}{\log p_n} \leq (n+2). \quad (3.11)$$

Therefore, we get,

$$\limsup_{k\to\infty}|a_k|^{1/k} = \limsup_{k\to\infty}\left|\frac{p_{k+1}}{\log p_{k+1}}\log\left(\frac{p_{k+1}}{p_k}\right)\left(\frac{\log p_{k+1}}{\log p_k}-1\right)\right|^{1/k} < \limsup_{k\to\infty}\left|\pi(p_{k+1})\log\left(\frac{p_{k+1}}{p_k}\right)\left(\frac{\log p_{k+1}}{\log p_k}-1\right)\right|^{1/k}$$

$$= \limsup_{k\to\infty}\left|(k+1)\log\left(\frac{p_{k+1}}{p_k}\right)\left(\frac{\log p_{k+1}}{\log p_k}-1\right)\right|^{1/k} \leq \limsup_{k\to\infty}\left|\log\left(\frac{p_{k+1}}{p_k}\right)\right|^{1/k} \leq 1. \quad (3.12)$$

This implies, $\lim_{n\to\infty} T_n - S_n$ exists. Hence, we get, there exists n_0, such that for all $n > n_0$, we have,

$$\frac{d_n}{\log p_n} = \int_{p_n}^{p_{n+1}} \frac{dx}{\log x}, \quad (3.13)$$

implies

$$\frac{d_n}{\log p_n} = Li(p_{n+1}) - Li(p_n). \quad (3.14)$$

Similarly, we consider $f(x) = (\log x)^2$. First we prove that $\lim_{n\to\infty}(T_n - S_n)$ exists. Here,

$$\lim_{n\to\infty} T_n - S_n = \sum_{i=p_1}^{\infty} \frac{d_i}{(\log p_i)^2} - \frac{d_i}{(\log p_{i+1})^2}. \quad (3.15)$$

We use the comparison test to show that the limit exists. Here, the k^{th} term is

$$b_k = \frac{d_k}{(\log p_k)^2} - \frac{d_k}{(\log p_{k+1})^2}. \quad (3.16)$$

We can easily check that, as $k \to \infty$

$$0 < b_k < a_k. \quad (3.17)$$

(Since as $n \to \infty$, $\log p_n + \log p_{n+1} < \log p_n \log p_{n+1}$).

Hence, the sum $\sum_{i=1}^{\infty} b_k$ converges. This again implies, $\lim_{n\to\infty} T_n - S_n$ exists, for $f(x) = (\log x)^2$. Hence, we get, there exists $n_1 \in N$, such that for all $n > n_1$, we have,

$$\frac{d_n}{(\log p_n)^2} = \int_{p_n}^{p_{n+1}} \frac{dx}{(\log x)^2}, \quad (3.18)$$

implies

$$\frac{d_n}{(\log p_n)^2} = Li(p_{n+1}) - Li(p_n) - \frac{p_{n+1}}{\log p_{n+1}} + \frac{p_n}{\log p_n}, \quad (3.19)$$

where, $Li(x)$ is the logarithmic integral function.
From equation (3.14) we get, there exists $n_2 \in N$ such that for all $n > n_2$

$$\frac{d_n}{(\log p_n)^2} = \frac{d_n}{\log p_n} - \frac{p_{n+1}}{\log p_{n+1}} + \frac{p_n}{\log p_n} = \frac{p_{n+1}}{\log p_n} - \frac{p_{n+1}}{\log p_{n+1}} = \frac{p_{n+1}}{\log p_{n+1}}\left(\frac{\log p_{n+1}}{\log p_n} - 1\right). \quad (3.20)$$

Now, we have $p_n^{\frac{1}{n+1}} > p_n^{\frac{1}{p_n}}$. Also, as $n \to \infty$, $\lim_{n\to\infty} p_n^{\frac{1}{p_n}} = 1$. We know from the prime number theorem, $\lim_{n\to\infty} p_{n+1} / p_n = 1$. Hence, as $n \to \infty$

$$p_n^{\frac{1}{n+1}} \geq \frac{p_{n+1}}{p_n} \quad (3.21)$$

i.e,

$$(n+1)\frac{\log p_{n+1}}{\log p_n} \leq (n+2). \quad (3.22)$$

Hence, as $n \to \infty$ from equation (3.20) we get

$$\frac{d_n}{(\log p_n)^2} < \pi(p_{n+1})\left(\frac{\log p_{n+1}}{\log p_n} - 1\right) \leq (n+2) - (n+1) = 1. \quad (3.23)$$

Thence, we obtain the following expression:

$$\frac{d_n}{(\log p_n)^2} = \int_{p_n}^{p_{n+1}} \frac{dx}{(\log x)^2} =$$

$$= \frac{p_{n+1}}{\log p_{n+1}}\left(\frac{\log p_{n+1}}{\log p_n} - 1\right) = \frac{d_n}{(\log p_n)^2} < \pi(p_{n+1})\left(\frac{\log p_{n+1}}{\log p_n} - 1\right) \leq (n+2) - (n+1) = 1; \quad (3.24)$$

that can be rewritten also as follows:

$$\frac{d_n}{(\log p_n)^2} = \int_{p_n}^{p_{n+1}} \frac{dx}{(\log x)^2} =$$

$$= \frac{p_{n+1}}{\log p_{n+1}}\left(\frac{\log p_{n+1}}{\log p_n} - 1\right) = \frac{d_n}{(\log p_n)^2} < \pi(p_{n+1})\left(\frac{\log p_{n+1}}{\log p_n} - 1\right) \leq (n+2) - (n+1) = \Phi - \phi, \quad (3.25)$$

where $\Phi = \frac{\sqrt{5}+1}{2} = 1{,}61803398...$ (i.e., the aurea ratio), and $\phi = \frac{\sqrt{5}-1}{2} = 0{,}61803398...$ (i.e., the aurea section).

3.1 The Cramer's conjecture and Selberg's theorem

Now we would like to consider a conjecture, due to H. Cramer, which is almost certainly true. The conjecture is

$$\limsup_{n\to\infty} \frac{p_{n+1}-p_n}{(\log n)^2} = 1. \quad (3.26)$$

Obviously the conjecture implies that if $k>1$ and $x>x_0(k)$, then the interval $\left(x, x+k\log^2 x\right]$ contains a prime number.

The following theorem and its corollaries, provide some mathematical support for a believe in Cramer's conjecture. For they imply that if the Riemann's hypothesis is true, then the number of primes for which $(P_{n+1}-P_n)$ is larger than $(\log n)^2$ is "small". Let us introduce the following notation

$$\ell_h(X) = \sum_{\substack{X<P_n\le 2X \\ d_n \ge h}} d_n, \qquad N_h(X) = \sum_{\substack{P_n \le X \\ d_n \ge h}} 1. \quad (3.27)$$

We can now state the principal result:

THEOREM 1

If the Riemann hypothesis is true, then

$$\ell_h(X) = O\left(\frac{X}{h}\log^2 X\right), \qquad N_h(X) = O\left(\frac{X}{h^2}\log^2 X\right). \quad (3.28)$$

COROLLARY

If the Riemann hypothesis is true, then

(i) $d_n = O\left(\sqrt{P_n}\log P_n\right)$, (ii) $\sum_{X<P_n\le 2X} d_n^2 = O\left(X\log^3 X\right)$, (iii) *If* $\lambda>4$, *then* $\sum_{n=1}^{\infty}\frac{d_n^2}{P_n}(\log P_n)^{-\lambda} < \infty$.

The above theorem is an elementary consequence of the following result.

THEOREM 2

Suppose that the Riemann hypothesis is true. If $\varepsilon>0$ *and* ω *is a function of* X *such that* $0<\omega<X^{-\varepsilon}$, *then as* X *tends to infinity*

$$\int_0^X \left[\frac{\theta(x+\omega^x)-\theta(x)}{\omega^x} - 1\right]^2 dx = O\left(\frac{\log^2 X}{\omega}\right). \quad (3.29)$$

Deduction of theorem 1 from theorem 2. – Let $\varepsilon>0$ be a fixed real number to be chosen later. We shall consider two cases: (i) $0<h\le X^{1-\varepsilon}$ and (ii) $X^{1-\varepsilon}<h\le X$. In case (i), we choose $\omega = h/4X$ and so $0<\omega<X^{-\varepsilon}$. Now suppose that

$$(p_n, p_{n+1}] \subseteq (X, 2X] \quad (3.30)$$

and that $d_n>h$. If x satisfies

$$p_n < x \le p_n + \frac{1}{2} d_n , \quad (3.31)$$

then

$$x + \omega^x < p_n + \frac{1}{2} d_n + 2\omega^X \le p_n + d_n = p_{n+1} \quad (3.32)$$

with the consequence

$$\theta(x + \omega^x) - \theta(x) = 0 . \quad (3.33)$$

Hence, we have

$$\frac{1}{2} d_n = \int_{p_n}^{p_n + \frac{1}{2} d_n} \left[\frac{\theta(x+\omega^x) - \theta(x)}{\omega^x} - 1 \right]^2 dx . \quad (3.34)$$

From theorem 2, we conclude that

$$\sum_{\substack{X < p_n \le 2X \\ d_n \ge h}} d_n = 2 \sum_{\substack{X < p_n \le 2X \\ d_n \ge h}} \int_{p_n}^{p_n + \frac{1}{2} d_n} \left[\frac{\theta(x+\omega^x) - \theta(x)}{\omega^x} - 1 \right]^2 dx \le 2 \int_X^{2X} \left[\frac{\theta(x+\omega^x) - \theta(x)}{\omega^x} - 1 \right]^2 dx =$$

$$= O\left(\frac{\log^2 X}{\omega} \right) = O\left(\frac{X}{h} \log^2 X \right) . \quad (3.35)$$

Thus if $0 < h \le X^{1-\varepsilon}$, we have proved that $\ell_h(X) = O\left(\frac{X}{h} \log^2 X \right)$. However, if we take $\varepsilon < 1/2$ and choose $h = X^{\alpha}$ with $1/2 < \alpha < 1 - \varepsilon$, then $\ell_h(X) = 0$ for $X > X_0$. For if $\ell_h(X) \ne 0$, then $\ell_h(X) \ge h$, and since we are still in case (i), we also have

$$h = O\left(\frac{X}{h} \log^2 X \right), \quad (3.36)$$

which leads to a contradiction if X is sufficiently large. Hence $\ell_h(X) = 0$ for $X^{\alpha} \le h \le X$. Thus for all h satisfying $0 < h \le X$ we have $\ell_n(X) = O\left(\frac{X}{h} \log^2 X \right)$. From the definition of $N_h(X)$, it is trivial that

$$\ell_h(X) \ge h \left\{ N_h(X) - N_h\left(\frac{1}{2} X \right) \right\} . \quad (3.37)$$

Upon replacing X by $X/2^r$, $r = 1,2,...$ and adding we deduce that

$$N_h(X) = O\left(\frac{1}{h} \ell_h(X) \right) = O\left(\frac{X}{h^2} \log^2 X \right) . \quad (3.38)$$

Proof of the corollaries.

(i) If we take $h = c\sqrt{X}\log X$ with c sufficiently large, it follows that $N_h(X) < 1$ and so $N_h(X) = 0$.

(ii) We have

$$\sum_{1\le h\le X}\sum_{\substack{X<p_n\le 2X\\ d_n>h}} d_n = \sum_{X<p_n\le 2X} d_n \sum_{h\le d_n} 1 = \sum_{X<p_n\le 2X} d_n^2 \quad (3.39)$$

and from theorem 1, we also have

$$\sum_{1\le h\le X}\sum_{\substack{X<p_n\le 2X\\ d_n>h}} d_n = O\left\{\sum_{h\le X}\frac{X}{h}\log^2 X\right\} = O\left(X\log^3 X\right). \quad (3.40)$$

(iii) From (i), it follows that

$$\sum_{X<p_n\le 2X}\frac{d_n^2}{p_n}(\log p_n)^{-\lambda} \le \frac{1}{X(\log X)^{\lambda}}\sum_{X<p_n\le 2X} d_n^2 \le \frac{A}{(\log X)^{\lambda-3}}. \quad (3.41)$$

Upon replacing X by $2^r X$ for $r = 1,2,...$ and adding, we obtain

$$\sum_{p_n>X}\frac{d_n^2}{p_n}(\log p_n)^{-\lambda} \le A\sum_{r=1}^{\infty}\left(\log 2^r X\right)^{3-\lambda} = O\left(\sum_{r=1}^{\infty} r^{3-\lambda}\right) \quad (3.42)$$

and this latter series is convergent if $\lambda > 4$.
With regard the proof of theorem 3, we starts from the well known formula:

$$\theta(x) = \frac{1}{2\pi i}\int_{(c)}\frac{Z^*(S)}{S}x^S dS, \quad (3.43)$$

where $Z^*(S) = \sum_p (\log p)p^{-S}$, and (c) denotes the line $c+it$, $c>1$. Now, being completely formal, we move the line of integration to $1/2+z+it$, where z will be chose later, and encounter a pole at $S=1$ with residue x. Taking a difference, we have

$$\theta\left(x+\omega^x\right)-\theta(x)-\omega^x = \frac{1}{2\pi i}\int_{\left(\frac{1}{2}+z\right)}\frac{Z^*(S)}{S}\left[(1+\omega)^S - 1\right]x^S dS, \quad (3.44)$$

thus

$$\frac{\theta\left(x+\omega^x\right)-\theta(x)-\omega^x}{x^{\frac{1}{2}+z}} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{Z^*\left(\frac{1}{2}+z+it\right)}{\left(\frac{1}{2}+it\right)}\left[(1+\omega)^{\frac{1}{2}+z+it}-1\right]x^{it}dt. \quad (3.45)$$

We now observe that the left hand side of the above equation is the formal Fourier transform of the right hand side. From the Parseval inequality, we have

$$\int_0^\infty \left[\frac{\theta(x+\omega^x)-\theta(x)-\omega^x}{x^{\frac{1}{2}+z}} \right]^2 \frac{dx}{x} \le \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left| \frac{Z^*\left(\frac{1}{2}+z+it\right)}{\left(\frac{1}{2}+z+it\right)} \left[(1+\omega)^{\frac{1}{2}+z+it} -1 \right] \right|^2 dt . \quad (3.46)$$

In fact, the above inequality does hold, but the rigorous argument, which closely parallels the above formal manipulations, starts not with $\theta(x)$ but with a more artificial function which approximates to $\theta(x)$. However, assuming that the inequality has been proved, we see

$$\int_0^\infty \left[\frac{\theta(x+\omega^x)-\theta(x)-\omega^x}{x^{\frac{1}{2}+z}} \right]^2 \frac{dx}{x} \ge \omega^2 \int_0^X \left[\frac{\theta(x+\omega^x)-\theta(x)}{\omega^x} -1 \right]^2 \frac{dx}{x^{2z}} \ge \frac{\omega^2}{X^{2z}} \int_0^X \left[\frac{\theta(x+\omega^x)-\theta(x)}{\omega^x} -1 \right]^2 dx \,, \quad (3.47)$$

and so

$$\int_0^X \left[\frac{\theta(x+\omega^x)-\theta(x)}{\omega^x} -1 \right]^2 dx \le \frac{x^{2z}}{2\pi\omega^2} \int_{-\infty}^{+\infty} \left| \frac{Z^*\left(\frac{1}{2}+z+it\right)}{\left(\frac{1}{2}+z+it\right)} \left[(1+\omega)^{\frac{1}{2}+z+it} -1 \right] \right|^2 dt . \quad (3.48)$$

Now we consider the integral on the right hand side of the above inequality. First of all, we note that

$$\left| (1+\omega)^S -1 \right| = \left| \int_1^{1+\omega} S u^{S-1} du \right| \le |S|\omega \quad (3.49)$$

and

$$\left| (1+\omega)^S -1 \right| \le (1+\omega)^\sigma +1 \le 3 , \quad (3.50)$$

since $\omega < 1$ and $\sigma < 1$. Thus, upon splitting the range of integration $(-\infty,+\infty)$ to the three parts $(-\infty,-T]$, $[-T,+T]$, (T,∞) and using the first estimate in the middle range and the second estimate in the end ranges we obtain as an upper bound for the integral:

$$\int_T^\infty \left| \frac{Z^*\left(\frac{1}{2}+z+it\right)}{\frac{1}{2}+z+it} \right|^2 dt + 2\omega^2 \int_0^T \left| Z^*\left(\frac{1}{2}+z+it\right) \right|^2 dt . \quad (3.51)$$

It is now a relatively straightforward technical lemma to show that

$$\int_T^\infty \left| \frac{Z^*\left(\frac{1}{2}+z+it\right)}{\frac{1}{2}+z+it} \right|^2 dt = O\left(\frac{1}{Tz^2}\right), \quad (3.52)$$

and

$$\int_0^T \left| Z^*\left(\frac{1}{2}+z+it\right) \right|^2 dt = O\left(\frac{T}{z^2}\right). \quad (3.53)$$

Thus we now have

$$\int_0^X \left[\frac{\theta(x+\omega^x)-\theta(x)}{\omega^x} - 1 \right]^2 dx = O\left(\frac{X^{2z}}{\omega^2 Tz^2} + \frac{X^{2z}T}{z^2} \right), \quad (3.54)$$

and if we choose $T = 3/\omega$ and $z = 4/\varepsilon \log X$, the upper bound becomes $O(\log^2 X/\omega)$, which completes the proof of theorem 2.

4. On some equations concerning the p-adic strings and the zeta strings. [4] [5]

Like in the ordinary string theory, the starting point of p-adic strings is a construction of the corresponding scattering amplitudes. Recall that the ordinary crossing symmetric Veneziano amplitude can be presented in the following forms:

$$A_\infty(a,b) = g^2 \int_R |x|_\infty^{a-1} |1-x|_\infty^{b-1} dx = g^2 \left[\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} + \frac{\Gamma(b)\Gamma(c)}{\Gamma(b+c)} + \frac{\Gamma(c)\Gamma(a)}{\Gamma(c+a)} \right] = g^2 \frac{\zeta(1-a)}{\zeta(a)} \frac{\zeta(1-b)}{\zeta(b)} \frac{\zeta(1-c)}{\zeta(c)} =$$

$$= g^2 \int DX \exp\left(-\frac{i}{2\pi} \int d^2\sigma \partial^\alpha X_\mu \partial_\alpha X^\mu \right) \prod_{j=1}^4 \int d^2\sigma_j \exp\left(ik_\mu^{(j)} X^\mu\right), \quad (4.1-4.4)$$

where $\hbar = 1$, $T = 1/\pi$, and $a = -\alpha(s) = -1 - \frac{s}{2}$, $b = -\alpha(t)$, $c = -\alpha(u)$ with the condition $s+t+u = -8$, i.e. $a+b+c = 1$.
The p-adic generalization of the above expression

$$A_\infty(a,b) = g^2 \int_R |x|_\infty^{a-1} |1-x|_\infty^{b-1} dx,$$

is:

$$A_p(a,b) = g_p^2 \int_{Q_p} |x|_p^{a-1} |1-x|_p^{b-1} dx, \quad (4.5)$$

where $|\ldots|_p$ denotes p-adic absolute value. In this case only string world-sheet parameter x is treated as p-adic variable, and all other quantities have their usual (real) valuation.
Now, we remember that the Gauss integrals satisfy adelic product formula

$$\int_R \chi_\infty\left(ax^2+bx\right)d_\infty x\prod_{p\in P}\int_{Q_p}\chi_p\left(ax^2+bx\right)d_p x=1,\quad a\in Q^\times,\quad b\in Q,\quad (4.6)$$

what follows from

$$\int_{Q_v}\chi_v\left(ax^2+bx\right)d_v x=\lambda_v(a)|2a|_v^{-\frac{1}{2}}\chi_v\left(-\frac{b^2}{4a}\right),\quad v=\infty,2,\ldots,p\ldots.\quad (4.7)$$

These Gauss integrals apply in evaluation of the Feynman path integrals

$$K_v(x'',t'';x',t')=\int_{x',t'}^{x'',t''}\chi_v\left(-\frac{1}{h}\int_{t'}^{t''}L(\dot q,q,t)dt\right)D_v q,\quad (4.8)$$

for kernels $K_v(x'',t'';x',t')$ of the evolution operator in adelic quantum mechanics for quadratic Lagrangians. In the case of Lagrangian

$$L(\dot q,q)=\frac{1}{2}\left(-\frac{\dot q^2}{4}-\lambda q+1\right),$$

for the de Sitter cosmological model one obtains

$$K_\infty(x'',T;x',0)\prod_{p\in P}K_p(x'',T;x',0)=1,\quad x'',x',\lambda\in Q,\ T\in Q^\times,\quad (4.9)$$

where

$$K_v(x'',T;x',0)=\lambda_v(-8T)|4T|_v^{-\frac{1}{2}}\chi_v\left(-\frac{\lambda^2T^3}{24}+[\lambda(x''+x')-2]\frac{T}{4}+\frac{(x''-x')^2}{8T}\right).\quad (4.10)$$

Also here we have the number 24 that correspond to the Ramanujan function that has 24 “modes”, i.e., the physical vibrations of a bosonic string. Hence, we obtain the following mathematical connection:

$$K_v(x'',T;x',0)=\lambda_v(-8T)|4T|_v^{-\frac{1}{2}}\chi_v\left(-\frac{\lambda^2T^3}{24}+[\lambda(x''+x')-2]\frac{T}{4}+\frac{(x''-x')^2}{8T}\right)\Rightarrow$$

$$\Rightarrow\frac{4\left[anti\log\frac{\int_0^\infty\frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}.\quad (4.10b)$$

The adelic wave function for the simplest ground state has the form

$$\psi_A(x)=\psi_\infty(x)\prod_{p\in P}\Omega\left(|x|_p\right)=\begin{cases}\psi_\infty(x),x\in Z\\0,x\in Q\setminus Z\end{cases},\quad (4.11)$$

where $\Omega(|x|_p)=1$ if $|x|_p \le 1$ and $\Omega(|x|_p)=0$ if $|x|_p > 1$. Since this wave function is non-zero only in integer points it can be interpreted as discreteness of the space due to p-adic effects in adelic approach. The Gel'fand-Graev-Tate gamma and beta functions are:

$$\Gamma_\infty(a)=\int_R |x|_\infty^{a-1}\chi_\infty(x)d_\infty x=\frac{\zeta(1-a)}{\zeta(a)}, \quad \Gamma_p(a)=\int_{Q_p}|x|_p^{a-1}\chi_p(x)d_p x=\frac{1-p^{a-1}}{1-p^{-a}}, \quad (4.12)$$

$$B_\infty(a,b)=\int_R |x|_\infty^{a-1}|1-x|_\infty^{b-1}d_\infty x=\Gamma_\infty(a)\Gamma_\infty(b)\Gamma_\infty(c), \quad (4.13)$$

$$B_p(a,b)=\int_{Q_p}|x|_p^{a-1}|1-x|_p^{b-1}d_p x=\Gamma_p(a)\Gamma_p(b)\Gamma_p(c), \quad (4.14)$$

where $a,b,c\in C$ with condition $a+b+c=1$ and $\zeta(a)$ is the Riemann zeta function. With a regularization of the product of p-adic gamma functions one has adelic products:

$$\Gamma_\infty(u)\prod_{p\in P}\Gamma_p(u)=1, \quad B_\infty(a,b)\prod_{p\in P}B_p(a,b)=1, \quad u\neq 0,1, \quad u=a,b,c, \quad (4.15)$$

where $a+b+c=1$. We note that $B_\infty(a,b)$ and $B_p(a,b)$ are the crossing symmetric standard and p-adic Veneziano amplitudes for scattering of two open tachyon strings. Introducing real, p-adic and adelic zeta functions as

$$\zeta_\infty(a)=\int_R \exp(-\pi x^2)|x|_\infty^{a-1}d_\infty x=\pi^{-\frac{a}{2}}\Gamma\left(\frac{a}{2}\right), \quad (4.16)$$

$$\zeta_p(a)=\frac{1}{1-p^{-1}}\int_{Q_p}\Omega(|x|_p)|x|_p^{a-1}d_p x=\frac{1}{1-p^{-a}}, \quad \mathrm{Re}\, a>1, \quad (4.17)$$

$$\zeta_A(a)=\zeta_\infty(a)\prod_{p\in P}\zeta_p(a)=\zeta_\infty(a)\zeta(a), \quad (4.18)$$

one obtains

$$\zeta_A(1-a)=\zeta_A(a), \quad (4.19)$$

where $\zeta_A(a)$ can be called adelic zeta function. We have also that

$$\zeta_A(a)=\zeta_\infty(a)\prod_{p\in P}\zeta_p(a)=\zeta_\infty(a)\zeta(a)=\int_R \exp(-\pi x^2)|x|_\infty^{a-1}d_\infty x\cdot\frac{1}{1-p^{-1}}\int_{Q_p}\Omega(|x|_p)|x|_p^{a-1}d_p x. \quad (4.19b)$$

Let us note that $\exp(-\pi x^2)$ and $\Omega(|x|_p)$ are analogous functions in real and p-adic cases. Adelic harmonic oscillator has connection with the Riemann zeta function. The simplest vacuum state of the adelic harmonic oscillator is the following Schwartz-Bruhat function:

$$\psi_A(x)=2^{\frac{1}{4}}e^{-\pi x_\infty^2}\prod_{p\in P}\Omega(|x_p|_p), \quad (4.20)$$

whose the Fourier transform

$$\psi_A(k)=\int\chi_A(kx)\psi_A(x)=2^{\frac{1}{4}}e^{-\pi k_\infty^2}\prod_{p\in P}\Omega\left(\left|k_p\right|_p\right) \quad (4.21)$$

has the same form as $\psi_A(x)$. The Mellin transform of $\psi_A(x)$ is

$$\Phi_A(a)=\int\psi_A(x)|x|^a d_A^\times x=\int_R\psi_\infty(x)|x|^{a-1}d_\infty x\prod_{p\in P}\frac{1}{1-p^{-1}}\int_{Q_p}\Omega\left(|x|_p\right)|x|^{a-1}d_p x=\sqrt{2}\Gamma\left(\frac{a}{2}\right)\pi^{-\frac{a}{2}}\zeta(a) \quad (4.22)$$

and the same for $\psi_A(k)$. Then according to the Tate formula one obtains (4.19).

The exact tree-level Lagrangian for effective scalar field φ which describes open p-adic string tachyon is

$$\mathcal{L}_p=\frac{1}{g^2}\frac{p^2}{p-1}\left[-\frac{1}{2}\varphi p^{-\frac{\Box}{2}}\varphi+\frac{1}{p+1}\varphi^{p+1}\right], \quad (4.23)$$

where p is any prime number, $\Box=-\partial_t^2+\nabla^2$ is the D-dimensional d'Alambertian and we adopt metric with signature $(-+\ldots+)$. Now, we want to show a model which incorporates the p-adic string Lagrangians in a restricted adelic way. Let us take the following Lagrangian

$$L=\sum_{n\geq 1}C_n\mathcal{L}_n=\sum_{n\geq 1}\frac{n-1}{n^2}\mathcal{L}_n=\frac{1}{g^2}\left[-\frac{1}{2}\phi\sum_{n\geq 1}n^{-\frac{\Box}{2}}\phi+\sum_{n\geq 1}\frac{1}{n+1}\phi^{n+1}\right]. \quad (4.24)$$

Recall that the Riemann zeta function is defined as

$$\zeta(s)=\sum_{n\geq 1}\frac{1}{n^s}=\prod_p\frac{1}{1-p^{-s}}, \quad s=\sigma+i\tau, \quad \sigma>1. \quad (4.25)$$

Employing usual expansion for the logarithmic function and definition (4.25) we can rewrite (4.24) in the form

$$L=-\frac{1}{g^2}\left[\frac{1}{2}\phi\zeta\left(\frac{\Box}{2}\right)\phi+\phi+\ln(1-\phi)\right], \quad (4.26)$$

where $|\phi|<1$. $\zeta\left(\frac{\Box}{2}\right)$ acts as pseudodifferential operator in the following way:

$$\zeta\left(\frac{\Box}{2}\right)\phi(x)=\frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk, \quad -k^2=k_0^2-\vec{k}^2>2+\varepsilon, \quad (4.27)$$

where $\tilde{\phi}(k)=\int e^{(-ikx)}\phi(x)dx$ is the Fourier transform of $\phi(x)$.

Dynamics of this field ϕ is encoded in the (pseudo)differential form of the Riemann zeta function. **When the d'Alambertian is an argument of the Riemann zeta function we shall call such string a "zeta string".** Consequently, the above ϕ is an open scalar zeta string. The equation of motion for the zeta string ϕ is

$$\zeta\left(\frac{\Box}{2}\right)\phi = \frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon} e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk = \frac{\phi}{1-\phi} \quad (4.28)$$

which has an evident solution $\phi = 0$.
For the case of time dependent spatially homogeneous solutions, we have the following equation of motion

$$\zeta\left(\frac{-\partial_t^2}{2}\right)\phi(t) = \frac{1}{(2\pi)}\int_{|k_0|>\sqrt{2}+\varepsilon} e^{-ik_0t}\zeta\left(\frac{k_0^2}{2}\right)\tilde{\phi}(k_0)dk_0 = \frac{\phi(t)}{1-\phi(t)}. \quad (4.29)$$

With regard the open and closed scalar zeta strings, the equations of motion are

$$\zeta\left(\frac{\Box}{2}\right)\phi = \frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk = \sum_{n\geq 1}\theta^{\frac{n(n-1)}{2}}\phi^n, \quad (4.30)$$

$$\zeta\left(\frac{\Box}{4}\right)\theta = \frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{4}\right)\tilde{\theta}(k)dk = \sum_{n\geq 1}\left[\theta^{n^2} + \frac{n(n-1)}{2(n+1)}\theta^{\frac{n(n-1)}{2}-1}\left(\phi^{n+1}-1\right)\right], \quad (4.31)$$

and one can easily see trivial solution $\phi = \theta = 0$.

5. On some equations concerning the Ω-deformation in toroidal compactification for N = 2 gauge theory.

We denote the torus as T_o and endow it with a constant metric G_{IJ}, $I, J = 1,2$:

$$ds_{T_o}^2 = G_{ij}d\theta^i d\theta^j, \qquad \theta^i \approx \theta^i + 2\pi. \quad (5.1)$$

The gauge theory probes the dual torus T_o^V, the moduli space of flat $U(1)$-connections on T_o. We write such a connection as

$$A = i\alpha_1 d\theta^1 + i\alpha_2 d\theta^2, \quad (5.2)$$

with constant hermitian matrices α_1, α_2. The gauge transformations generated by the $U(1)$-valued functions

$$u_{n_1,n_2} = \exp\left(in_1\theta^1 + in_2\theta^2\right) \quad (5.3)$$

shift the components $\alpha_{1,2}$ by $n_{1,2}$, respectively. The natural metric on T_o^V is given by:

$$ds_{T_o^V}^2 = \frac{1}{(2\pi i)^2}\int_{T_o} dA \wedge *dA = \sqrt{\det(G)}G^{ij}d\alpha_i d\alpha_j. \quad (5.4)$$

It depends only on the complex structure of T_o. It is convenient to parametrize G^{ij} by two complex numbers $\omega_{1,2}$,

$$G^{ij}d\alpha_i d\alpha_j = |\omega_1 d\alpha_1 + \omega_2 d\alpha_2|^2 \quad (5.5)$$

defined up to a simultaneous phase rotation, so that the invariants are:

$$|\omega_1|^2 = G^{11}, \quad |\omega_2|^2 = G^{22}, \quad \mathrm{Re}(\omega_1\overline{\omega}_2) = G^{12}. \quad (5.6)$$

Let us assume

$$\mathrm{Im}\left(\frac{\omega_2}{\omega_1}\right) > 0$$

We then have:

$$\sqrt{\det(G)} = \frac{1}{\mathrm{Im}(\omega_2\overline{\omega}_1)}. \quad (5.7)$$

For a rectangular torus,

$$\omega_1 = \frac{1}{\rho_1}, \quad \omega_2 = \frac{i}{\rho_2}$$

Let us first consider the case of gauge group $U(1)$. We take the Maxwell action to be

$$I = \frac{1}{8\pi}\int_{X^4} d^4x\sqrt{g}\left(\frac{4\pi}{g_4^2}F_{mn}F^{mn} + \frac{i\vartheta}{4\pi}\varepsilon_{mnpq}F^{mn}F^{pq}\right). \quad (5.8)$$

If we take the four-manifold to be $\Sigma \times T_o$, with the product metric $h \times G$, with h being the metric on Σ, and denote the Riemannian measure of Σ as $d\mu$, then, in the low-energy approximation, (5.8) reads as:

$$I = \frac{(2\pi)^2}{8\pi}\sqrt{\det(g)}\int_\Sigma d\mu\left(\frac{8\pi}{g_4^2}h^{ab}G^{ij}(\partial_a A_i \partial_b A_j) - \frac{i\vartheta}{\pi}\varepsilon^{ab}\varepsilon^{ij}(\partial_a A_i \partial_b A_j)\right) =$$

$$= -i\vartheta\int_\Sigma d\alpha_1 \wedge d\alpha_2 + \frac{4\pi^2\sqrt{\det(G)}}{g_4^2}\int_\Sigma d(\omega_1\alpha_1 + \omega_2\alpha_2) \wedge * d(\overline{\omega}_1\alpha_1 + \overline{\omega}_2\alpha_2). \quad (5.9)$$

We note that

$$\frac{(2\pi)^2}{8\pi} = \frac{39,4784176}{25,1327412} = 1,57096.$$

Now, we have that:

$$\left[(\Phi)^{14/7} + (\Phi)^{-7/7} + (\Phi)^{-28/7} + (\Phi)^{-42/7}\right] \cdot \frac{4}{9} = 3,437694 \cdot \frac{4}{9} = 1,52786405;$$

$$(1,52786405 + \Phi) \cdot \frac{1}{2} = (1,52786405 + 1,61803399) \cdot \frac{1}{2} = 3,14589804 \cdot \frac{1}{2} = 1,57294902.$$

where $\Phi = \frac{\sqrt{5}+1}{2} = 1{,}61803399$ is the aurea ratio.

Thence, we can rewrite the eq. (5.9) also as follows:

$$\left\{\left[(\Phi)^{14/7} + (\Phi)^{-7/7} + (\Phi)^{-28/7} + (\Phi)^{-42/7}\right] \cdot \frac{4}{9} + \Phi\right\} \cdot \frac{1}{2} \times$$
$$\times \sqrt{\det(g)} \int_\Sigma d\mu \left(\frac{8\pi}{g_4^2} h^{ab} G^{ij} \left(\partial_a A_i \partial_b A_j\right) - \frac{i\vartheta}{\pi} \varepsilon^{ab} \varepsilon^{ij} \left(\partial_a A_i \partial_b A_j\right) \right) =$$
$$= -i\vartheta \int_\Sigma d\alpha_1 \wedge d\alpha_2 + \frac{4\pi^2 \sqrt{\det(G)}}{g_4^2} \int_\Sigma d\left(\omega_1 \alpha_1 + \omega_2 \alpha_2\right) \wedge * d\left(\overline{\omega}_1 \alpha_1 + \overline{\omega}_2 \alpha_2\right). \quad (5.9b)$$

The bosonic part of the pure $\mathcal{N} = 2$ gauge theory Lagrangian reduced on the torus T_o is given at low energies by

$$L = \frac{8\pi^2}{2g_4^2} \sqrt{\det(G)}\, tr\left\{\left(\omega_1 d\alpha_1 + \omega_2 d\alpha_2\right) \wedge *\left(\overline{\omega}_1 d\alpha_1 + \overline{\omega}_2 d\alpha_2\right) + d\phi \wedge * d\overline{\phi}\right\} - i\vartheta\, tr\, d\alpha_1 \wedge d\alpha_2, \quad (5.10)$$

where "tr" denotes the induced metric on t.
We note that

$$\frac{8\pi^2}{2} \cong 39{,}478 \cong 39{,}624 \quad ;$$

$\left[(\Phi)^{35/7} + \Phi\right] \cdot \frac{3}{2} = 12{,}708204 \cdot \frac{3}{2} = 19{,}06230590$;

$\left[(\Phi)^{35/7} + (\Phi)^{14/7}\right] \cdot \frac{3}{2} = 13{,}708204 \cdot \frac{3}{2} = 20{,}56230590$; 19,06230590 + 20,56230590 = 39,6246118;

where $\Phi = \frac{\sqrt{5}+1}{2} = 1{,}61803399$ is the aurea ratio.

Thence, we can rewrite the eq. (5.10) also as follows:

$$L = \left[(\Phi)^{35/7} + \Phi\right] \cdot \frac{3}{2} + \left[(\Phi)^{35/7} + (\Phi)^{14/7}\right] \cdot \frac{3}{2} \frac{1}{g_4^2} \sqrt{\det(G)}\, tr\left\{\left(\omega_1 d\alpha_1 + \omega_2 d\alpha_2\right) \wedge *\left(\overline{\omega}_1 d\alpha_1 + \overline{\omega}_2 d\alpha_2\right) +\right.$$
$$\left. + d\phi \wedge * d\overline{\phi}\right\} - i\vartheta\, tr\, d\alpha_1 \wedge d\alpha_2 . \quad (5.10b)$$

The gauge theory part of this Lagrangian can be borrowed from (5.9). We view here $\alpha_{1,2} \in t \otimes R / \left(\Lambda_{cwt} \otimes Z\right)$ as real, and $\phi \in t \otimes C$ as complex, with "tr" defining a positive definite inner product on t. The Euclidean path integral measure is given by

$$e^{-\int L} . \quad (5.11)$$

The condition for a field configuration to be invariant under the supercharge Q that is relevant to Donaldson theory and the Ω-deformation is

$$d\phi = 0\,, \quad \bar{\partial}(\omega_1\alpha_1 + \omega_2\alpha_2) = 0\,, \quad (5.12)$$

where the second equation is anti-selfduality of the gauge field in our low energy approximation. For such fields, (5.11) evaluates to:

$$\exp\left(-\int L\right) = \exp\left(2\pi i\,\tau_0 \int tr d\alpha_1 \wedge d\alpha_2\right), \quad (5.13)$$

where the complexified gauge coupling is equal to

$$\tau_0 = \frac{\vartheta}{2\pi} + \frac{4\pi i}{g_4^2}\,. \quad (5.14)$$

For the eq. (5.10), thence, we can rewrite the eq. (5.13) also as follows:

$$\exp\left(-\int \frac{8\pi^2}{2g_4^2}\sqrt{\det(G)}\,tr\left\{(\omega_1 d\alpha_1 + \omega_2 d\alpha_2) \wedge *(\bar{\omega}_1 d\alpha_1 + \bar{\omega}_2 d\alpha_2) + d\phi \wedge * d\bar{\phi}\right\} - i\vartheta tr d\alpha_1 \wedge d\alpha_2\right) =$$
$$= \exp\left(2\pi i\left(\frac{\vartheta}{2\pi} + \frac{4\pi i}{g_4^2}\right)\int tr d\alpha_1 \wedge d\alpha_2\right). \quad (5.14b)$$

Now we use the following notation:

$$\mu_0 = \frac{8\pi^2}{g_4^2}\sqrt{\det(G)} = 2\pi \frac{\mathrm{Im}(\tau_0)}{\mathrm{Im}(\omega_2\bar{\omega}_1)}\,. \quad (5.15)$$

The Lagrangian (5.10) describes a sigma-model with target the product of a torus and $t \otimes C$, all divided by the Weyl group. Upon T-duality along the α_1 direction, we map it to a sigma-model on $\mathcal{M}_H$, after taking into account the nonlinear corrections. The T-duality is performed in the standard fashion. The first step is to replace $d\alpha_1$ in (5.10) by an independent t-valued one-form p_1 and add the term $-2\pi i tr(p_1 \wedge d\tilde{\alpha}_1)$ to L, with the understanding that $\tilde{\alpha}_1$ takes values in a circle of circumference 1:

$$L' = \frac{\mu_0}{2} tr\left\{(\omega_1 p_1 + \omega_2 d\alpha_2) \wedge *(\bar{\omega}_1 p_1 + \bar{\omega}_2 d\alpha_2) + d\phi \wedge * d\bar{\phi}\right\} + 2\pi i tr\left\{\left(d\tilde{\alpha}_1 + \frac{\vartheta}{2\pi} d\alpha_2\right) \wedge p_1\right\}. \quad (5.16)$$

Integrating over $\tilde{\alpha}_1$ would lead us back to (5.10). Instead, we integrate over p_1. The path integral over p_1 is Gaussian, with the saddle point for p_1 at:

$$p_1 = -\mathrm{Re}\left(\frac{\omega_2}{\omega_1}\right) d\alpha_2 + i * \frac{2\pi}{\mu_0 |\omega_1|^2}\left(d\tilde{\alpha}_1 + \frac{\vartheta}{2\pi} d\alpha_2\right). \quad (5.17)$$

In terms of the left- and right-moving components of α_1, (5.17) reads as follows:

$$\alpha_1^L = \frac{2\pi}{\mu_0|\omega_1|^2}\left(\tilde{\alpha}_1 + \frac{\vartheta}{2\pi}\alpha_2\right) - \mathrm{Re}\left(\frac{\omega_2}{\omega_1}\right)\alpha_2; \quad \alpha_1^R = -\frac{2\pi}{\mu_0|\omega_1|^2}\left(\tilde{\alpha}_1 + \frac{\vartheta}{2\pi}\alpha_2\right) - \mathrm{Re}\left(\frac{\omega_2}{\omega_1}\right)\alpha_2. \quad (5.18)$$

The T-dual Lagrangian is given by:

$$L^T = \frac{\mu_0}{2}\left\{tr\frac{|\mathrm{Im}(\omega_2\overline{\omega}_1)|^2}{|\omega_1|^2}d\alpha_2 \wedge * d\alpha_2 + d\phi \wedge * d\phi + (2\pi)^2\frac{\left(d\tilde{\alpha}_1 + \frac{\vartheta}{2\pi}d\alpha_2\right)\wedge *\left(d\tilde{\alpha}_1 + \frac{\vartheta}{2\pi}d\alpha_2\right)}{\mu_0^2|\omega_1|^2}\right\} +$$

$$-2\pi i\,\mathrm{Re}\left(\frac{\omega_2}{\omega_1}\right)tr\left(d\tilde{\alpha}_2 + \frac{\vartheta}{2\pi}d\alpha_2\right)\wedge d\alpha_2. \quad (5.19)$$

Introduce the $t \otimes C$-valued dimensionless coordinates Z, W :

$$Z = \tilde{\alpha}_1 + \frac{\vartheta}{2\pi}\alpha_2 + \frac{4\pi i}{g_4^2}\alpha_2; \qquad W = \frac{1}{2\pi}\mu_0\overline{\omega}_1\phi. \quad (5.20)$$

In terms of W and Z , eq. (5.19) takes the form:

$$L^T = \frac{(2\pi)^2}{2\mu_0|\omega_1|^2}tr\{dZ \wedge * d\overline{Z} + dW \wedge * d\overline{W}\} - 2\pi i\,\mathrm{Re}\left(\frac{\omega_2}{\omega_1}\right)tr\left(d\tilde{\alpha}_1 + \frac{\vartheta}{2\pi}d\alpha_2\right)\wedge d\alpha_2. \quad (5.21)$$

Note that

$$\frac{2\pi^2}{\mu_0|\omega_1|^2} = \pi\frac{\mathrm{Im}\left(\frac{\omega_2}{\omega_1}\right)}{\mathrm{Im}(\tau_0)}. \quad (5.22)$$

We note also that π can be expressed also in the following form (Ramanujan modular equation):

$\pi = \frac{24}{\sqrt{142}}\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]$. We observe that 24 is the number concerning the modes of the physical vibrations of the bosonic strings. Furthermore, we observe also that the Fibonacci zeta function is $\zeta_F(s) = \sum_{n=1}^{\infty} f_n^{-s}$, where the nth Fibonacci number can be expressed as $f_n = \frac{\phi^n - (-\phi)^{-n}}{\sqrt{5}}$ and where $\phi = (1+\sqrt{5})/2$ is the aurea ratio. The derivative of the Fibonacci zeta function is:

$$\zeta'_F(s) = -\frac{1}{\ln(\phi)s^2} + \frac{1}{24}\left(2\ln(\phi) + \frac{3\ln^2(5)}{\ln(\phi)} - 6\ln(5)\right) - \ln(c) + O(s).$$

Also here, we note that there is the number 24, i.e. the modes corresponding to the physical vibrations of the bosonic strings.
Thence, from (5.21) and (5.22) we obtain the following mathematical connections with the Ramanujan modular equation, the Fibonacci zeta function and the Palumbo-Nardelli model:

$$\frac{2\pi^2}{\mu_0|\omega_1|^2} = \frac{24}{\sqrt{142}} \log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right] \frac{\mathrm{Im}\left(\frac{\omega_2}{\omega_1}\right)}{\mathrm{Im}(\tau_0)} \Rightarrow$$

$$\Rightarrow -\frac{1}{\ln(\phi)s^2} + \frac{1}{24}\left(2\ln(\phi) + \frac{3\ln^2(5)}{\ln(\phi)} - 6\ln(5)\right) - \ln(c) + \mathrm{O}(s) \Rightarrow$$

$$\Rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr(G_{\mu\nu}G_{\rho\sigma})f(\phi) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right] =$$
$$= \int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2}Tr_\nu\left(|F_2|^2\right)\right]. \quad (5.22b)$$

$$L^T = \frac{(2\pi)^2}{2\mu_0|\omega_1|^2} tr\{dZ \wedge *d\overline{Z} + dW \wedge *d\overline{W}\} - 2\pi i\,\mathrm{Re}\left(\frac{\omega_2}{\omega_1}\right) tr\left(d\tilde{\alpha}_1 + \frac{\vartheta}{2\pi}d\alpha_2\right) \wedge d\alpha_2 \Rightarrow$$

$$\Rightarrow \frac{2\pi^2}{\mu_0|\omega_1|^2} = \frac{24}{\sqrt{142}} \log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right] \frac{\mathrm{Im}\left(\frac{\omega_2}{\omega_1}\right)}{\mathrm{Im}(\tau_0)} \Rightarrow$$

$$\Rightarrow -\frac{1}{\ln(\phi)s^2} + \frac{1}{24}\left(2\ln(\phi) + \frac{3\ln^2(5)}{\ln(\phi)} - 6\ln(5)\right) - \ln(c) + \mathrm{O}(s) \Rightarrow$$

$$\Rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr(G_{\mu\nu}G_{\rho\sigma})f(\phi) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right] =$$
$$= \int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2}Tr_\nu\left(|F_2|^2\right)\right]. \quad (5.22c)$$

We deduce from (5.21) the target space metric

$$ds^2_{\mathcal{M}_H} = 2\pi\frac{\mathrm{Im}(\omega_2/\omega_1)}{\mathrm{Im}(\tau_0)}\left(dZd\overline{Z} + dWd\overline{W}\right). \quad (5.23)$$

In our approximation, the target space metric is flat; in the exact theory, it is a complete hyper-Kahler metric on what we usually call $\mathcal{M}_H$. We also deduce from (5.21) a B-field, which, up to exact terms, is given by:

$$B = \frac{2\pi}{2i}\frac{\mathrm{Re}(\omega_2/\omega_1)}{\mathrm{Im}(\tau_0)}\left(dZ \wedge d\overline{Z} + dW \wedge d\overline{W}\right) = \left(\mathrm{Re}(\omega_2/\omega_1)\right)\omega_I . \quad (5.24)$$

Here ω_I is the topologically normalized symplectic form on $\mathcal{M}_H$, which is Kahler in the complex structure I. The functions of Z, W are holomorphic in complex structure I.

Also here we can note that there exists the mathematical connection with the Aurea section. Indeed, we remember that π that is present in many equations of this chapter, is related to the Aurea section $\phi = \frac{\sqrt{5}-1}{2}$ by the following simple but fundamental relation:

$$\arccos\phi = 0{,}2879\pi . \quad (5.25)$$

6. Mathematical connections

Now we take the eq. (3.25). We obtain the following connections with the eqs. (4.19b) and (5.9):

$$\frac{d_n}{(\log p_n)^2} = \int_{p_n}^{p_{n+1}} \frac{dx}{(\log x)^2} =$$

$$= \frac{p_{n+1}}{\log p_{n+1}}\left(\frac{\log p_{n+1}}{\log p_n} - 1\right) = \frac{d_n}{(\log p_n)^2} < \pi(p_{n+1})\left(\frac{\log p_{n+1}}{\log p_n} - 1\right) \le (n+2) - (n+1) = \Phi - \phi \Rightarrow$$

$$\Rightarrow \zeta_A(a) = \zeta_\infty(a)\prod_{p\in P}\zeta_p(a) = \zeta_\infty(a)\zeta(a) = \int_R \exp\left(-\pi x^2\right)\left|x\right|_\infty^{a-1} d_\infty x \cdot \frac{1}{1-p^{-1}}\int_{Q_p}\Omega\left(\left|x\right|_p\right)\left|x\right|_p^{a-1} d_p x \Rightarrow$$

$$\Rightarrow \frac{(2\pi)^2}{8\pi}\sqrt{\det(g)}\int_\Sigma d\mu\left(\frac{8\pi}{g_4^2}h^{ab}G^{ij}\left(\partial_a A_i \partial_b A_j\right) - \frac{i\vartheta}{\pi}\varepsilon^{ab}\varepsilon^{ij}\left(\partial_a A_i \partial_b A_j\right)\right) =$$

$$= -i\vartheta\int_\Sigma d\alpha_1 \wedge d\alpha_2 + \frac{4\pi^2\sqrt{\det(G)}}{g_4^2}\int_\Sigma d\left(\omega_1\alpha_1 + \omega_2\alpha_2\right) \wedge * d\left(\overline{\omega}_1\alpha_1 + \overline{\omega}_2\alpha_2\right). \quad (6.1)$$

Now we take the eqs. (3.46), (3.48) and (3.54). We obtain the following connections with the eq. (4.28):

$$\int_0^\infty \left[\frac{\theta(x+\omega^x) - \theta(x) - \omega^x}{x^{\frac{1}{2}+z}}\right]^2 \frac{dx}{x} \le \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left|\frac{Z^*\left(\frac{1}{2}+z+it\right)}{\left(\frac{1}{2}+z+it\right)}\left[(1+\omega)^{\frac{1}{2}+z+it} - 1\right]\right|^2 dt \Rightarrow$$

$$\Rightarrow \zeta\left(\frac{\Box}{2}\right)\phi = \frac{1}{(2\pi)^D}\int_{k_0^2 - \vec{k}^2 > 2+\varepsilon} e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk = \frac{\phi}{1-\phi} ; \quad (6.2)$$

$$\int_0^X \left[\frac{\theta(x+\omega^x)-\theta(x)}{\omega^x}-1\right]^2 dx \le \frac{x^{2z}}{2\pi\omega^2}\int_{-\infty}^{+\infty}\left|\frac{Z^*\left(\frac{1}{2}+z+it\right)}{\left(\frac{1}{2}+z+it\right)}\left[(1+\omega)^{\frac{1}{2}+z+it}-1\right]\right|^2 dt \Rightarrow$$

$$\Rightarrow \zeta\left(\frac{\square}{2}\right)\phi = \frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon} e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk = \frac{\phi}{1-\phi}; \quad (6.3)$$

$$\int_0^X \left[\frac{\theta(x+\omega^x)-\theta(x)}{\omega^x}-1\right]^2 dx = O\left(\frac{X^{2z}}{\omega^2 Tz^2}+\frac{X^{2z}T}{z^2}\right) \Rightarrow$$

$$\Rightarrow \zeta\left(\frac{\square}{2}\right)\phi = \frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon} e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk = \frac{\phi}{1-\phi}. \quad (6.4)$$

In conclusion, we take the relationship (5.22c) that can be connected with the eq. (4.28) as follows:

$$L^T = \frac{(2\pi)^2}{2\mu_0|\omega_1|^2} tr\left\{dZ \wedge *d\overline{Z} + dW \wedge *d\overline{W}\right\} - 2\pi i \,\mathrm{Re}\left(\frac{\omega_2}{\omega_1}\right) tr\left(d\tilde{\alpha}_1 + \frac{\vartheta}{2\pi}d\alpha_2\right)\wedge d\alpha_2 \Rightarrow$$

$$\Rightarrow \frac{2\pi^2}{\mu_0|\omega_1|^2} = \frac{24}{\sqrt{142}}\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]\frac{\mathrm{Im}\left(\frac{\omega_2}{\omega_1}\right)}{\mathrm{Im}(\tau_0)} \Rightarrow$$

$$\Rightarrow -\frac{1}{\ln(\phi)s^2}+\frac{1}{24}\left(2\ln(\phi)+\frac{3\ln^2(5)}{\ln(\phi)}-6\ln(5)\right)-\ln(c)+O(s) \Rightarrow$$

$$\Rightarrow \zeta\left(\frac{\square}{2}\right)\phi = \frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon} e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk = \frac{\phi}{1-\phi}. \quad (6.5)$$

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Sites and various Blog

Dr. Michele Nardelli (various papers on the string theory)

http://xoomer.virgilio.it/stringtheory/

http://nardelli.xoom.it/virgiliowizard/

http://michelenardelli.blogspot.com/

CNR SOLAR

http://150.146.3.132/perl/user_eprints?userid=36

ERATOSTENE group

http://www.gruppoeratostene.com

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