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An inversion procedure for the recovery of propagation speed and damping of the medium(*)

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Summary. — We consider the one-dimensional inverse problem of determining variations in propagation speed taking into account damping of the medium. We also consider the inverse problem of recovering variations in damping from observations of signals which pass through the medium of interest. Our method is based on the linearized inversion associated with Born's approximation. Thus we assume wave speed and damping are well approximated by the background plus the perturbation. We exploit the high-frequency character of seismic data. Therefore, we use WKBJ Green's function in deriving our inversion representation.

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1. – Introduction

The objective of this work is to study the problem of mapping of the interior of the earth as an inverse problem and to develop methods which yield increasingly more accurate results of that inverse problem. The methods we use are classical, employing perturbation techniques, transform methods and asymptotic analysis to get information about the interior of the earth. We assume the perturbation in wave speed and damping have parallel form and it is this perturbation we seek to recover. One or more signals are introduced near the surface of the earth in a region of interest and responses of irregularities in the interior of the earth are recorded.

An approximate solution of the above inverse problem, for the velocity, was obtained by Claerbout [1] under the assumption of constant density. The one-dimensional problem has been discussed in detail by Gerver [2]. He demonstrated that velocity of propagation

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can be determined uniquely from the observations at one point. An inverse problem of determining small variations in propagation speed through the medium of interest was considered by Cohen and Bleistein [3,4]. They have shown that closed-form approximate solutions for the velocity profile can be obtained for a wide variety of wave propagation equations. The linearization used in the derivation of inversion procedure is often referred to as the Born approximation [5].

In this paper we introduce a damping term in the wave equation [6] and study its effects on the inversion. The damping may be caused by impurities in the medium, the presence of fluid saturated rocks in the medium, distributed boundary frictions or small viscous effects.

One-dimensional inversion theories are of practical interest in situations where data or material parameters have only one-dimension of variability. These data may be measurements of a time-independent quantity made in one spatial dimension, or may consist of measurements of a temporally variable quantity that is spatially independent. Our primary interest is the improvement of the current inversion techniques with applications to the problem of seismic exploration. For instance, the petroleum industry relies heavily on seismic imaging techniques for location of hydrocarbons. We believe that the improved technique presented in this paper may help to provide more reliable information of the subsurface.

The velocity inversion without damping is summarized in the second section and is based upon Bleistein, Cohen and Stockwell [7]. The velocity inversion in the presence of damping is discussed in the third section. The damping parameter is recovered in the fourth section, and an iteration procedure to improve the results is also described. Finally the results are summarized in the fifth section, and further research interest is also outlined.

2. – Inversion without damping

Assume that the propagation of the field $u\,(x,\omega)$ is governed by the scalar Helmholtz equation

$$\pounds u = \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + \frac{\omega^2}{v^2\,(x)} u = -\delta\,(x)\,, \tag{1}$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} \mp \frac{i\omega}{v\,(x)} u \to 0, \quad \text{as } x \to \pm\infty. \tag{2}$$

Suppose $v\,(x)$ is a perturbation on some reference or background speed, $c\,(x)$

$$\frac{1}{v^2\,(x)} = \frac{1}{c^2\,(x)}\,[1+\alpha\,(x)] \qquad \alpha\,(x) \ll 1. \tag{3}$$

The total field $u\,(x,\omega)$ can be separated into the incident part $u_{\mathrm{I}}\,(x,\omega)$ in the absence of the perturbation and $u_{\mathrm{S}}\,(x,\omega)$ in the presence of the perturbation, $\alpha\,(x)$. Thus, set

$$u\,(x,\omega) = u_{\mathrm{I}}\,(x,\omega) + u_{\mathrm{S}}\,(x,\omega)\,, \tag{4}$$

and require that $u_{\mathrm{I}}(x,\omega)$ and $u_{\mathrm{S}}(x,\omega)$ are solutions of the following problems:

$$\pounds_0 u_{\mathrm{I}} = \frac{\mathrm{d}^2 u_{\mathrm{I}}}{\mathrm{d}x^2} + \frac{\omega^2}{c^2} u_{\mathrm{I}} = -\delta(x), \tag{5}$$

$$\pounds_0 u_{\mathrm{S}} = -\frac{\omega^2 \alpha(x)}{c^2} \left[u_{\mathrm{I}}(x,\omega) + u_{\mathrm{S}}(x,\omega) \right]. \tag{6}$$

Next, using Green's function representation to write down the solution of eq. (6) as

$$u_{\mathrm{S}}(\zeta,\omega) = \omega^2 \int_0^\infty \frac{\alpha(x)}{c^2(x)} \left[u_{\mathrm{I}}(x,\omega) + u_{\mathrm{S}}(x,\omega) \right] g(x,\zeta,\omega)\,\mathrm{d}x. \tag{7}$$

The product, $\alpha(x)\,u_{\mathrm{S}}(x,\omega)$, appearing under the integral in (7) is significantly smaller than the product, $\alpha(x)\,u_{\mathrm{I}}(x,\omega)$, and this leads to the Born approximation

$$u_{\mathrm{S}}(\zeta,\omega) = \omega^2 \int_0^\infty \frac{\alpha(x)}{c^2(x)} u_{\mathrm{I}}(x,\omega)\, g(x,\zeta,\omega)\,\mathrm{d}x. \tag{8}$$

Since for the inverse problem, the total field is observed at the origin, *i.e.* $\zeta = 0$

$$u_{\mathrm{S}}(0,\omega) = \omega^2 \int_0^\infty \frac{\alpha(x)}{c^2(x)} u_{\mathrm{I}}(x,\omega)\, g(x,0,\omega)\,\mathrm{d}x. \tag{9}$$

The WKBJ approximation of the Green's function has the following form:

$$g(x,0,\omega) = -\frac{A(x)}{2i\omega} \exp\left[i\omega\phi(x,0) \right] \qquad \phi(x,y) = \int_y^x \frac{\mathrm{d}t}{c(t)}. \tag{10}$$

In the simplest case, when $c(x)$ is continuous, the WKBJ amplitude $A(x)$ is given by

$$A(x) = \sqrt{c(0)\,c(x)}.$$

Since we are concerned here entirely with high-frequency solutions, so it is desirable to use $u_{\mathrm{I}}(x,\omega) = F(\omega)\,g(x,0,\omega)$. Where $F(\omega)$ is some frequency domain filter. With this modification for $u_{\mathrm{I}}(x,\omega)$, and using (10) for $g(x,0,\omega)$ in (9) leads to the integral equation

$$u_{\mathrm{S}}(0,\omega) = -\int_0^\infty F(\omega)\, \frac{\alpha(x)\,A^2(x)}{4c^2(x)} \exp\left[2i\omega\phi(x,0) \right] \mathrm{d}x. \tag{11}$$

Since $\alpha(x) = 0$ for $x < 0$, this is the Fourier-type integral because the lower limit can be extended to $-\infty$. However, the amplitude in this more general Fourier integral should be calculated separately, see Appendix A. The inversion operator corresponding to this has the form

$$\alpha(y) = -\frac{4c(y)}{\pi A^2(y)} \int_{-\infty}^\infty u_{\mathrm{S}}(0,\omega) \exp\left[-2i\omega\phi(y,0) \right] \mathrm{d}\omega. \tag{12}$$

The reflectivity function of a surface is defined to be the normal reflection strength multiplied by the singular function. The reflectivity function locates the boundary of the scattering object and characterizes the change in medium properties through the normal reflection coefficient. If the discontinuities are our primary interest, then band-limited delta-functions are easier to identify as compared to band-limited step functions. Some examples applied to layered media, where the artifacts are produced by the reflectivity function, are nicely demonstrated in Bleistein *et al.* [7]. The reflectivity function $\beta(y)$ can be obtained by differentiating (12) with respect to y and dividing by -4, keeping only leading-order terms in ω. Thus multiplying (12) by the factor $i\omega/2c(y)$, we obtain the result

$$\beta(y) = -\frac{2}{\pi A^2(y)} \int_{-\infty}^{\infty} i\omega u_{\mathrm{S}}(0,\omega) \exp[-2i\omega\phi(y,0)]\, \mathrm{d}\omega. \tag{13}$$

3. – Velocity inversion in the presence of damping

In this section, the one-dimensional problem of the variable velocity in the presence of variable damping will be considered. The propagation of the field is governed by the scalar Helmholtz equation

$$\pounds u = \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + \left[\frac{\omega^2 + i\omega\gamma(x)}{v^2(x)}\right] u = -\delta(x - x_{\mathrm{s}}), \tag{14}$$

with the damping $\gamma(x)$ and the source placed at a point x_{s}. It is assumed that the source point is to the left of the region where $v(x)$ and $\gamma(x)$ are unknown. The impulse response will be observed at a point x_{g}, which is to the left of the region of unknown $v(x)$ and $\gamma(x)$. The objective is to see what can be recovered about the perturbations from the observed response.

Introduce variations in damping and sound speed to have the parallel form

$$\gamma(x) = \gamma_0(x) + \epsilon\gamma(x), \tag{15}$$

$$v(x) = v_0(x) + \epsilon v(x), \qquad \frac{1}{v^2(x)} = \frac{1}{v_0^2(x)}\left[1 - \frac{2\epsilon v(x)}{v_0(x)}\right]. \tag{16}$$

These representations are substituted into (14) and only linear terms in ϵ are retained. The resulting equation is

$$\begin{aligned}\pounds_0 u = \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + \left[\frac{\omega^2 + i\omega\gamma_0(x)}{v_0^2(x)}\right] u = -\delta(x - x_{\mathrm{s}}) + \\ + \left[-i\epsilon\omega\gamma(x) + \frac{2\epsilon\omega^2 v(x)}{v_0(x)} + \frac{2i\epsilon\omega\gamma_0(x) v(x)}{v_0(x)}\right] \frac{u}{v_0^2(x)}.\end{aligned} \tag{17}$$

Introduce $u_{\mathrm{I}}(x, x_{\mathrm{s}}, \omega)$ as the response to the delta-function in the unperturbed medium and $u_{\mathrm{S}}(x, x_{\mathrm{s}}, \omega)$ as everything else, we can write $u = u_{\mathrm{I}} + u_{\mathrm{S}}$ with

$$\pounds_0 u_{\mathrm{I}}(x, x_{\mathrm{s}}, \omega) = -\delta(x - x_{\mathrm{s}}), \tag{18}$$

and

$$\pounds_0 u_{\mathrm{S}}(x, x_{\mathrm{s}}, \omega) = \left[-i\epsilon\omega\gamma(x) + \frac{2\epsilon\omega^2 v(x)}{v_0(x)} + \frac{2i\epsilon\omega\gamma_0(x)\, v(x)}{v_0(x)}\right] \frac{u_{\mathrm{I}}(x, x_{\mathrm{s}}, \omega)}{v_0^2(x)}. \tag{19}$$

We now construct the Green's function representation of (19), observed at the point x_{g} (geophone location). It satisfies the following equation:

$$\pounds_0 g(x, x_{\mathrm{g}}, \omega) = -\delta(x - x_{\mathrm{g}}). \tag{20}$$

Note that this Green's function differs from u_{I}, defined by (18), only by subscripts "s" and "g". Since we are concerned only with the high-frequency inversion, we will use WKBJ approximations for u_{I} and g to the leading order in ω. These leading-order approximations are given by

$$u_{\mathrm{I}}(x, x_{\mathrm{s}}, \omega) = -\frac{F(\omega)\, A(x_{\mathrm{s}}, x)}{2i\omega} \exp\left[i\omega\phi(x_{\mathrm{s}}, x) - \frac{1}{2}\psi(x_{\mathrm{s}}, x)\right], \tag{21}$$

$$g(x, x_{\mathrm{g}}, \omega) = -\frac{A(x_{\mathrm{g}}, x)}{2i\omega} \exp\left[i\omega\phi(x_{\mathrm{g}}, x) - \frac{1}{2}\psi(x_{\mathrm{g}}, x)\right], \tag{22}$$

where for the case of continuous $v_0(x)$ and $\gamma_0(x)$ the WKBJ amplitude and phase are given by

$$A(x_{\mathrm{g}}, x) = \sqrt{v_0(x_{\mathrm{g}})\, v_0(x)}, \quad \phi(x_{\mathrm{g}}, x) = \int_{x_{\mathrm{g}}}^{x} \frac{\mathrm{d}t}{v_0(t)}, \quad \text{and} \quad \psi(x_{\mathrm{g}}, x) = \int_{x_{\mathrm{g}}}^{x} \frac{\gamma_0(t)\,\mathrm{d}t}{v_0(t)}. \tag{23}$$

The solution of (19) in terms of Green's function is given by

$$u_{\mathrm{S}}(x_{\mathrm{g}}, x_{\mathrm{s}}, \omega) = -\int_0^{\infty} \left[-i\epsilon\omega\gamma(x) + \frac{2\epsilon\omega^2 v(x)}{v_0(x)} + \frac{2i\epsilon\omega\gamma_0(x)\, v(x)}{v_0(x)}\right] \cdot \cdot \frac{u_{\mathrm{I}}(x, x_{\mathrm{s}}, \omega)\, g(x, x_{\mathrm{g}}, \omega)}{v_0^2(x)} \mathrm{d}x. \tag{24}$$

Now using the WKBJ representations (21) and (22) in (24), and retaining only the leading-order terms in ω, we get

$$\begin{aligned} &u_{\mathrm{S}}(x_{\mathrm{g}}, x_{\mathrm{s}}, \omega) = \\ &\frac{\epsilon}{2} \int_0^{\infty} F(\omega) \frac{v(x)}{v_0^3(x)} A(x_{\mathrm{g}}, x)\, A(x_{\mathrm{s}}, x) \exp\left[i\omega\left[\phi(x_{\mathrm{g}}, x) + \phi(x_{\mathrm{s}}, x)\right]\right] \cdot \\ &\cdot \exp\left[-\frac{1}{2}\left[\psi(x_{\mathrm{g}}, x) + \psi(x_{\mathrm{s}}, x)\right]\right] \mathrm{d}x. \end{aligned} \tag{25}$$

This can be treated as a Fourier transform of $\epsilon v(x)$ and the inversion can be performed in the same way as (11), see Appendix A. The result is

$$\epsilon v(y) = \frac{2v_0^2(y) \exp\left[\frac{1}{2}\left[\psi(x_{\mathrm{g}}, y) + \psi(x_{\mathrm{s}}, y)\right]\right]}{\pi A(x_{\mathrm{g}}, y)\, A(x_{\mathrm{s}}, y)} \int_{-\infty}^{\infty} u_{\mathrm{S}}(x_{\mathrm{g}}, x_{\mathrm{s}}, \omega) \cdot \tag{26}$$

$$\cdot \exp\left[-i\omega\left[\phi\left(x_{\rm g}, y\right)+\phi\left(x_{\rm s}, y\right)\right]\right] {\rm d}\omega.$$

The reflectivity function $\beta(y)$ can be computed in the same way as for (12) and is given by

$$\beta(y)=\frac{-2\exp\left[\frac{1}{2}\left[\psi\left(x_{\rm g}, y\right)+\psi\left(x_{\rm s}, y\right)\right]\right]}{\pi A\left(x_{\rm g}, y\right) A\left(x_{\rm s}, y\right)} \int_{-\infty}^{\infty} i\omega u_{\rm S}\left(x_{\rm g}, x_{\rm s}, \omega\right) \cdot \cdot \exp\left[-i\omega\left[\phi\left(x_{\rm g}, y\right)+\phi\left(x_{\rm s}, y\right)\right]\right] {\rm d}\omega. \tag{27}$$

¿From expressions (26) and (27), it is clear that the perturbation in velocity and reflectivity function also depend on the background damping $\gamma_0(x)$. Hence these results demonstrate an improvement on previous results. These expressions reduce to (12) and (13), if we take $\gamma(x)=0$ and $x_{\rm s}=x_{\rm g}=0$.

4. – Recovery of the damping effect

Consider again (24) and retain terms of order $1/\omega$ to get

$$u_{\rm S}\left(x_{\rm g}, x_{\rm s}, \omega\right)=-\int_0^{\infty} F(\omega)\left[\frac{i\epsilon\gamma(x)}{4\omega}-\frac{\epsilon v(x)}{2v_0(x)}-\frac{i\epsilon\gamma_0(x) v(x)}{2\omega v_0(x)}\right] \cdot \cdot \frac{A\left(x_{\rm g}, x\right) A\left(x_{\rm s}, x\right)}{v_0^2(x)} \exp\left[i\omega\left[\phi\left(x_{\rm g}, x\right)+\phi\left(x_{\rm s}, x\right)\right]\right] \cdot \cdot \exp\left[-\frac{1}{2}\left[\psi\left(x_{\rm g}, x\right)+\psi\left(x_{\rm s}, x\right)\right]\right] {\rm d}x. \tag{28}$$

Since $\epsilon v(x)$ is known from (26), therefore set

$$V\left(x_{\rm g}, x_{\rm s}, \omega\right)=\int_0^{\infty} F(\omega)\left[\frac{\epsilon v(x)}{2v_0(x)}+\frac{i\epsilon\gamma_0(x) v(x)}{2\omega v_0(x)}\right] \frac{A\left(x_{\rm g}, x\right) A\left(x_{\rm s}, x\right)}{v_0^2(x)} \cdot \cdot \exp\left[i\omega\left[\phi\left(x_{\rm g}, x\right)+\phi\left(x_{\rm s}, x\right)\right]\right] \exp\left[-\frac{1}{2}\left[\psi\left(x_{\rm g}, x\right)+\psi\left(x_{\rm s}, x\right)\right]\right] {\rm d}x, \tag{29}$$

and

$$W\left(x_{\rm g}, x_{\rm s}, \omega\right)=4i\omega\left[u_{\rm S}\left(x_{\rm g}, x_{\rm s}, \omega\right)-V\left(x_{\rm g}, x_{\rm s}, \omega\right)\right]. \tag{30}$$

Using (29) and (30) in (28), we get the following integral equation:

$$W\left(x_{\rm g}, x_{\rm s}, \omega\right)= \int_0^{\infty} F(\omega) \frac{\epsilon\gamma(x) A\left(x_{\rm g}, x\right) A\left(x_{\rm s}, x\right)}{v_0^2(x)} \exp\left[i\omega\left[\phi\left(x_{\rm g}, x\right)+\phi\left(x_{\rm s}, x\right)\right]\right] \cdot \cdot \exp\left[-\frac{1}{2}\left[\psi\left(x_{\rm g}, x\right)+\psi\left(x_{\rm s}, x\right)\right]\right] {\rm d}x. \tag{31}$$

Since $\epsilon\gamma(x)=0$ for $x<0$, therefore the lower integral limit can be extended to $-\infty$. This is a Fourier integral and it can be inverted in the same way as (11), see appendix A.

The result is

$$\epsilon\gamma\left(y\right)=\frac{v_{0}\left(y\right)\exp\left[\frac{1}{2}\left[\psi\left(x_{\mathrm{g}},y\right)+\psi\left(x_{\mathrm{s}},y\right)\right]\right]}{\pi A\left(x_{\mathrm{g}},y\right)A\left(x_{\mathrm{s}},y\right)}\int_{-\infty}^{\infty}W\left(x_{\mathrm{g}},x_{\mathrm{s}},\omega\right)\cdot \cdot\exp\left[-i\omega\left[\phi\left(x_{\mathrm{g}},y\right)+\phi\left(x_{\mathrm{s}},y\right)\right]\right]\mathrm{d}\omega. \tag{32}$$

Substitute the approximation obtained for $\epsilon\gamma\left(y\right)$ from (32) in the expression (28), and set

$$X\left(x_{\mathrm{g}},x_{\mathrm{s}},\omega\right)= \int_{0}^{\infty}F\left(\omega\right)\left[\frac{i\epsilon\gamma\left(x\right)}{4\omega}\right]\frac{A\left(x_{\mathrm{g}},x\right)A\left(x_{\mathrm{s}},x\right)}{v_{0}^{2}\left(x\right)}\exp\left[i\omega\left[\phi\left(x_{\mathrm{g}},x\right)+\phi\left(x_{\mathrm{s}},x\right)\right]\right]\cdot \cdot\exp\left[-\frac{1}{2}\left[\psi\left(x_{\mathrm{g}},x\right)+\psi\left(x_{\mathrm{s}},x\right)\right]\right]\mathrm{d}x. \tag{33}$$

Assume constant background damping, that is, $\gamma_0\left(x\right)=\gamma_0$, and write

$$Y\left(x_{\mathrm{g}},x_{\mathrm{s}},\omega\right)=\frac{2\omega}{\left(\omega+i\gamma_{0}\right)}\left[X\left(x_{\mathrm{g}},x_{\mathrm{s}},\omega\right)+u_{\mathrm{S}}\left(x_{\mathrm{g}},x_{\mathrm{s}},\omega\right)\right]. \tag{34}$$

Therefore the next approximation to $\epsilon v\left(y\right)$ is given by

$$\epsilon v\left(y\right)=\frac{v_{0}^{2}\left(y\right)\exp\left[\frac{1}{2}\left[\psi\left(x_{\mathrm{g}},y\right)+\psi\left(x_{\mathrm{s}},y\right)\right]\right]}{\pi A\left(x_{\mathrm{g}},y\right)A\left(x_{\mathrm{s}},y\right)}\int_{-\infty}^{\infty}Y\left(x_{\mathrm{g}},x_{\mathrm{s}},\omega\right)\cdot \cdot\exp\left[-i\omega\left[\phi\left(x_{\mathrm{g}},y\right)+\phi\left(x_{\mathrm{s}},y\right)\right]\right]\mathrm{d}\omega. \tag{35}$$

Now from (35), $\epsilon v\left(y\right)$ can be computed and the above procedure from (29) to (35) can be repeated to get the next approximations for $\epsilon\gamma\left(y\right)$ and $\epsilon v\left(y\right)$. The results of this section demonstrate the dependence of the perturbation in wave speed on the perturbation in damping and vice versa. We have derived an approximate solution to the inverse problem for the wave speed in the presence of damping. The perturbation in damping is also recovered in an inhomogeneous medium. Finally an iterative procedure is presented to get increasingly better approximations.

5. – Conclusions

We have derived approximate solutions to the inverse problem of the velocity and damping in a medium which supports acoustic waves. The approximations made are often used in modeling the inverse problem in seismic exploration. It is established in this work that the damping of the medium plays a role in getting a more accurate map of the subsurface. An iterative procedure to improve velocity and damping profiles is also presented.

We have presented a procedure to determine wavespeed and damping profiles of a medium with one-dimension of parameter variability. We also have assumed constant-background wavespeed and damping. Nevertheless, the inversion procedure presented in this paper may provide a launching pad to attack more general problems:

1) The derivation of inversion formulas for the three-dimensional problem.

2) The derivation of inversion formulas for a variety of source-receiver geometries.

3) The derivation of inversion formulas for two-dimensional parameter variability.

4) The derivation of inversion formulas for variable-background and a variety of source-receiver geometries.

Appendix A.

The inversion operator for (11) has the form

$$\alpha(y) = \int_{-\infty}^{\infty} b(y,\omega)\, u_{\mathrm{S}}(0,\omega) \exp\left[2i\omega\phi(y,0)\right] \mathrm{d}\omega, \tag{A.1}$$

where $b(y,\omega)$ is to be determined. To see the form of $b(y,\omega)$, substitute (11) in the above expression to get

$$\begin{aligned} \alpha(y) &= -\int_0^{\infty} \mathrm{d}x \frac{\alpha(x) A^2(x)}{4c(x)} \int_{-\infty}^{\infty} F(\omega)\, b(y,\omega) \exp\left[2i\omega\phi(x,y)\right] \mathrm{d}\omega \\ &= \int_0^{\infty} \alpha(x) f(x,y)\, \mathrm{d}x. \end{aligned} \tag{A.2}$$

Equation (A.2) will be at least approximately satisfied if we set

$$f(x,y) = \delta_{\mathrm{B}}(x-y) = \frac{1}{\pi} \int_{-\infty}^{\infty} F(\omega) \exp\left[i\omega(x-y)\right] \mathrm{d}\omega, \tag{A.3}$$

where $\delta_{\mathrm{B}}(x-y)$ is the band-limited delta-function. If $b(y,\omega)$ is independent of ω, that is $b(y,\omega) = b(y)$. Then

$$\begin{aligned} f(x,y) &= -\frac{2\pi A^2(x)\, b(y)}{4c^2(x)} \delta_{\mathrm{B}}\left[2\phi(x,y)\right] \\ &= -\frac{\pi A^2(y)\, b(y)}{4c(y)} \delta_{\mathrm{B}}(x-y). \end{aligned} \tag{A.4}$$

The second line in the above expression follows from the first because $2/c(x)$ is the derivative of the argument of the delta-function and the support of the delta-function is $x = y$. These are asymptotic equalities depending upon sufficiently high frequencies. The choice of $b(y)$ to make (A.2) true is now apparent

$$b(y) = -\frac{4c(y)}{\pi A^2(y)}. \tag{A.5}$$

This leads to the inversion formula given by (12).

The inversion amplitude for (25) can easily be deduced by comparing it with (11). We have to make the following replacements:

– replace $-A^2(x)/4c^2(x)$ by $A(x_{\mathrm{g}},x)\, A(x_{\mathrm{s}},x) \exp\left[-\frac{1}{2}\left[\psi(x_{\mathrm{g}},x) + \psi(x_{\mathrm{s}},x)\right]\right] / 2v_0^3(x)$;

– replace the argument $2\phi(x,0)$ by $[\phi(x_{\mathrm{g}},x)+\phi(x_{\mathrm{s}},x)]$.

With these changes in place the amplitude $b(y)$ is given by

$$b(y)=\frac{2v_0^2(y)\exp\left[\frac{1}{2}\left[\psi(x_{\mathrm{g}},y)+\psi(x_{\mathrm{s}},y)\right]\right]}{\pi A(x_{\mathrm{g}},y)\,A(x_{\mathrm{s}},y)}. \tag{A.6}$$

Similarly the inversion amplitude for (31) can be found by making appropriate changes.

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