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A model based on Heisenberg's theory for the eddy diffusivity in decaying turbulence applied to the residual layer

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Summary. — The problem of the theoretical derivation of a parameterization for the eddy diffusivity in decaying turbulence is addressed. This derivation makes use of the dynamical equation for the energy spectrum density and the classical statistical diffusion theory. The starting point is Heisenberg's elementary decaying turbulence theory. The main assumption is related to the identification of a frequency, lying in the inertial subrange, characterizing the inertial energy transfer among eddies of different size. The resulting eddy diffusivity parameterization is then applied to the decay of convective turbulence in the residual layer. Besides the intrinsic scientific interest, this topic has relevance for mesoscale transport and diffusion simulations. The resulting expression for the eddy diffusivity cannot be solved analytically. For this reason an algebraic approximated formulation, giving nearly the same results as the exact expression, is also proposed.

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1. – Introduction

The dispersion of contaminants by turbulent flows is of central importance in a number of environmental problems. In the recent years a great deal of work has been done to study the airborne pollutant dispersion in Convective and Stable Boundary Layers (CBL and SBL). However, less attention has been paid to the dispersion in the Residual Layer (RL), the nearly adiabatic remnant of the daytime boundary layer, where the dispersion of contaminants occurs in condition of decaying turbulence.

About half hour before sunset, over land, the surface heat flux (positive during the day) begins to decrease and then the thermals cease to form and along the time the turbulence tends to disappear in the CBL. The new resulting layer of air separated of the surface by the stable nocturnal boundary layer is sometimes called the Residual Layer [1].

The decay of energy-containing eddies in the CBL is the physical mechanism that can maintain the dispersion process in the RL efficient. Besides this mechanism, a second turbulence process, the shear-driven turbulence generation, may be present in the RL [2]. However it will not be considered in this paper.

Unfortunately, to our knowledge, there are no conclusive turbulence observations helping in studying the status of the turbulence and its decaying characteristics in the residual layer.

Freedman and Bornstein [3] simulated with a 1D formulation of TVM mesoscale model [4] the structure and evolution of the turbulence characteristics for Wangara Day 33 case. Turbulence was modeled by a prognostic equation for the turbulent kinetic energy (TKE).

Nieuwstadt and Brost [5] (NB) and Sorbjan [6] studied the first stage (about one hour) of the decay of convective turbulence using large-eddy simulations. NB considered the case of surface heat flux abruptly decreased to zero at sunset, whereas Sorbjan [6] considered the case of a gradual decrease with time of surface heat flux. In particular, NB found that TKE decay scales with the dimensionless time t/t_*, where $t_* = h/w_*$ and w_* is the convective velocity scale and h is the CBL depth. These authors also showed evidence of a decoupling of large and small scales during the decay. On the other hand, Sorbjan [6] concluded that during the decay, turbulent eddies continue to exist even when the surface heat flux becomes negative and, consequently, the ground-based inversion develops.

Desiato *et al.* [7], simulating the ETEX I long-range tracer dispersion experiment with two Lagrangian particle models, obtained the best results when the dispersion in the RL was also included.

It is the aim of the present paper to propose a general semi-empirical model to derive eddy diffusivities in a decaying turbulence and to apply it to the case of the decay of convective turbulence in the RL. The starting point is to employ the Heisenberg decaying turbulence theory to parameterize the energy-transfer spectrum function in the dynamical equation for the energy spectrum density in a homogeneous isotropic turbulence. From a physical point of view the novelty in this study is related to the identification of a frequency in the inertial subrange which is statistically independent of the frequency characterizing the energy-containing eddies. With this assumption the main result of the paper is the elaboration of a solution to the objection made by Batchelor which concerns to statistical independence between large (energy-containing frequencies) and small (inertial effects containing frequencies). Finally, we are confident to stress that the derived eddy diffusivity from Heisenberg theory is relevant and interesting in the study of the turbulent dispersion parameterization in a decaying CBL.

2. – Dynamical equation for the energy density spectrum

Homogeneous isotropic turbulence, when buoyant and shear production terms are not important, satisfies the following energy transfer relation [8, 9]:

$$\frac{\partial \mathbf{E}(k,t)}{\partial t} = \mathbf{W}(k,t) - 2\nu\, k^2 \mathbf{E}(k,t) \ , \tag{1}$$

where k is the wave number, $\mathbf{E}(k,t)$is the three-dimensional (3D) energy density spectrum function (EDS); $\mathbf{W}(k,t)$ is referred to as the energy-transfer-spectrum function and represents the contribution due to the inertial transfer of energy among different wave numbers or the time-rate-of-change per unit wave number of the energy spectrum, due to non-linear interactions [10-12], the second term on the r.h.s. of eq. (1) is the energy loss due to ordinary viscous dissipation.

One who is concerned basically with the practical and applied aspects of fluid flow might be inclined to ask why the idealized type of isotropy turbulence is considered in the present analysis. The answer is that, despite its hypothetical character, a knowledge of its characteristics may still form a fundamental basis for the study of actual, nonisotropic turbulent flows [8].

Following the tradition in micrometeorology (see, for instance, Kaimal *et al.* [13]) space spectra can be replaced by frequency spectra, since frequency n, not wave number, is measured. Consequently, eq. (1) becomes

$$\frac{\partial \mathbf{S}(n,t)}{\partial t} = \mathbf{T}(n,t) - \frac{8\pi^2 n^2 \nu}{U^2}\mathbf{S}(n,t) \ , \tag{2}$$

where $n = kU/2\pi$, U is the mean wind speed, $\mathbf{T}(n,t) = \mathbf{W}(k,t)2\pi/U$ represents the energy transfer among different frequencies and $\mathbf{S}(n,t) = \mathbf{E}(k,t)2\pi/U$.

Heisenberg [9,14,15] explained the mechanism of inertial transfer of energy from large to small eddies in terms of an additional eddy viscosity, called kinematic turbulence viscosity (KTV). Thus, the effect of the inertia term can be regarded as equivalent to a virtual turbulent friction, ν_{T}, produced by the small-scale turbulence (small eddies) and acting on the large-scale turbulence (larger eddies). ν_{T} represents the kinematic turbulence viscosity caused by the eddies with frequency ranging from n to infinity. Therefore, following Heisenberg, $\mathbf{T}(n,t)$ can be represented as

$$\mathbf{T}(n,t) = -\frac{8\pi^2 n^2}{U^2}\nu_{\mathrm{T}}\mathbf{S}(n,t) \ . \tag{3}$$

The correct choice of KTV is the major difficulty in Heisenberg's approach. Indeed, from a physical point of view, the introduction of KTV to account for the energy transfer from the larger to the smaller eddies, can be correct only if the small eddies, responsible for the existence of such a turbulence viscosity, are statistically independent of the large eddies [16].

The starting point of our analysis consists in considering the structure of the 3D turbulent spectrum $\mathbf{S}(n)$ in geophysical flows, such as the PBL, where Reynolds's numbers are very large (limit of infinite Reynolds's number [17-19]). In such situations, the turbulent energy spectra can be subdivided in three major spectral regions: energy-containing, inertial and dissipation subranges [8]. In the energy-containing subrange, where eddies make the main contribution to the TKE, EDS shows its maximum, so it is possible to choose its corresponding frequency n_e to characterize the size of these eddies. In the dissipation subrange, where the viscosity plays its major role, it is also possible to associate a frequency n_d with the size of the eddies that provide the main contribution to the dissipation. Following a modern viewpoint with emphasis on postulated symmetries rather than on postulated universality (*i.e.* independence of the particular mechanism by which the turbulence is generated [12]), if the Reynolds number is infinitely large, all the possible symmetries of the turbulent flow are restored and self-similarity occurs at small scales (where small scale is here understood as scales small compared to the energy-containing eddies scale) and away from boundaries.

Therefore, for fully developed turbulence ($\nu \to 0$) the finite positive dissipation in the region of frequencies very far below the region of maximum dissipation will be negligibly small compared with the flux of energy transferred by inertial effects. In such an inertial subrange, the effect of molecular viscosity would then vanish ($\nu_\mathrm{T} \gg \nu$). As a consequence we may associate a frequency n_I with the size of the eddies that provide the main contribution to this inertial energy flux, that is

$$n_\mathrm{e} \ll n_\mathrm{I} \ll n_\mathrm{d}\,. \tag{4}$$

With this assumption, the turbulence, in this subrange, is statistically independent of the range of energy-containing eddies and a relationship for KTV can be obtained. This means that, in the inertial range, the energy flux at scales $\sim l$ involves predominantly scales of comparable size. The traditional argument in favour of localness can be found in the literature ([8, 12], p. 232 and p. 105, respectively).

ν_T can be calculated directly from Taylor's statistical diffusion theory for large travel times τ [20, 21] as

$$\nu_\mathrm{T} = \frac{\beta}{6}\frac{\sigma_\mathrm{I}^2}{n_\mathrm{I}}\,, \tag{5}$$

where σ_I^2 is the turbulent velocity variance calculated in the inertial subrange and β is defined as the ratio of the Lagrangian to the Eulerian time scales. Equation (2) can thus be solved (setting $\mathbf{S}_0(n) = \mathbf{S}(n, t = 0)$), obtaining

$$\mathbf{S}\,(n,t) = \mathbf{S}_0\,(n) \exp\left[-8\pi^{\,2}\nu_\mathrm{T}\frac{n^2}{U^2}t\right]. \tag{6}$$

Having obtained an expression for the time evolution of EDS in decaying turbulence, an equation for the eddy diffusivity in decaying turbulence may be easily derived by using the same theoretical framework used to derive $\nu_\mathrm{T}(n)$. Let us assume that in the initial stage of the decay process, this last is predominantly determined by the decay of the energy-containing eddies [5, 6, 8], so that these large eddies decay according to their intrinsic time scales. Therefore, as the energy-containing spectral range is characterized by n_e, an equation like (5) can be written for $\mathbf{K}\,(z,t)$, namely

$$\mathbf{K}\,(z,t) = \frac{\beta}{6}\frac{\sigma_\mathrm{e}^2\,(z,t)}{n_\mathrm{e}}\,, \tag{7a}$$

where

$$\frac{1}{2}\sigma_{\mathrm{e}}^2\left(z,t\right)=\int\limits_{n_{\mathrm{e}}}^{\infty}\mathbf{S}_0\left(n\right)\exp\left[-8\pi^2\nu_{\mathrm{T}}\frac{n^2}{U^2}t\right]\mathrm{d}n\,. \tag{7b}$$

Both eqs. (5) and (7a) are derived from Taylor statistical diffusion theory for large diffusion travel times. In fact, these expressions are calculated from the asymptotic product $\sigma^2 T_{\mathrm{L}}$. Here σ^2 and T_{L} represent, respectively, a turbulent velocity variance and a Lagrangian decorrelation time scale. In this study the Lagrangian decorrelation time scales are determined according to Hanna's parameterization [20].

As in the case of eq. (5), for large travel time, $\mathbf{K}$ can be expressed as a function of local turbulence properties and in terms of the characteristic discrete spectral frequencies n_{e} (energy-containing eddies) and n_{I} (inertial-transferring eddies). Equation (7a) represents the general semi-empirical method to derive eddy diffusivities in a decaying turbulence that we propose. Looking at this equation suggests that the decrease of eddy diffusivity $\mathbf{K}$ with time depends on n_{e}. As n_{e} refers to the large eddies, this means that a certain level of turbulence can be sustained for a long time. On the other hand, from eq. (7b) it can be seen that the decrease of $\mathbf{K}$ with time depends also on ν_{T}. This, in turn, depends on n_{I}, that describes the dimension of eddies providing the major contribution to the inertial energy flux. Consequently, this additional viscosity may be estimated from observed spectra. Since the inertial transfer of energy is the dominant factor in the inertial subrange, n_{I} will be considered here as the initial frequency of this range [12,22].

3. – Comparison between eq. (5) and the Heinsenberg model for the 3D KTV

As a test for our approach, which has been derived from Taylor diffusion statistical theory, we compare at this point eq. (5) with the classical Heisenberg model [14] that allows evaluation of the 3D KTV. Firstly, to proceed this comparison, we consider that the 3D turbulence energy spectrum in the inertial subrange can be written as [14, 22]

$$\mathbf{S}\left(n\right)=\frac{\alpha}{\left(2\pi\right)^{2/3}}U^{2/3}\left(\frac{\psi_\varepsilon}{h}\right)^{2/3}w_*^2 n^{-5/3}\,, \tag{8}$$

where $\psi_\varepsilon = \varepsilon h/w_*^3$ is the nondimensional molecular dissipation rate function and $\alpha \cong 1.52$ [14, 23].

By setting in eq. (5) $\beta \cong 0.55U/\sigma_{\mathrm{I}}$ [24] and computing σ_{I} from the integration from n_{I} to infinity of eq. (8), the following expression for KTV is obtained:

$$\nu_{\mathrm{T}}=0.1\left(\frac{\psi_\varepsilon}{h}\right)^{1/3}\left(\frac{U}{n_{\mathrm{I}}}\right)^{4/3}w_*\,. \tag{9}$$

Now, by defining an arbitrary but fixed wave number k_{I}, Heisenberg divided the energy spectrum into a small-scale part comprising velocity fluctuations with wave numbers k' larger than k_{I}, including negligibility of correlation between these different Fourier elements of the spectrum. Dimensional analysis then yields [8]

$$\nu_{\mathrm{T}} = \int\limits_{k'=k_{\mathrm{I}}}^{\infty} C_{\mathrm{H}} \sqrt{\frac{\mathbf{S}\,(k')}{k'^3}} \mathrm{d}k' \,, \tag{10}$$

where C_{H} is Heisenberg dimensionless, spectral transfer constant and $\mathbf{S}\,(k')$ is the 3D turbulence energy spectrum in the inertial subrange [25], with the following form:

$$\mathbf{S}\,(k') = \alpha \varepsilon^{2/3} k'^{-5/3} \,. \tag{11}$$

Assuming that the small-scale turbulence (inertial subrange) should act on the large-scale turbulence like an additional eddy-viscosity we re-insert eq. (11) into eq. (10) where $C_{\mathrm{H}} \cong 0.47$ [14, 23] and obtain

$$\nu_{\mathrm{T}} = 0.44 \varepsilon^{1/3} k_{\mathrm{I}}^{-4/3} \tag{12}$$

that, with $k_{\mathrm{I}} = \frac{2\pi n_{\mathrm{I}}}{U}$, yields

$$\nu_{\mathrm{T}} = 0.038 \left(\frac{\psi_\varepsilon}{h}\right)^{1/3} \left(\frac{U}{n_{\mathrm{I}}}\right)^{4/3} w_* \,. \tag{13}$$

Finally the comparison of eq. (9), obtained from Taylor theory, with eq. (13), derived from Heisenberg model, shows that both equations give the same physical information about the 3D KTV. It is important to notice that the only difference between eqs. (9) and (13) lies in their numerical coefficients. On the basis of the above comparison, eq. (5) can be considered as a model to estimate KTV.

4. – Derivation of a one-dimensional KTV and K_z in the RL

As the 3D EDS for an isotropic and homogeneous turbulent flow can be related to the 1D EDS, we can calculate KTV and K_z in the RL. Our starting point is that in the first stage the turbulence structure of the RL is the same as that of the previously existing CBL. This means that the turbulence energy spectra are assumed to have the same form as they had in the recently decayed mixed layer.

The vertical turbulence energy spectrum in the inertial subrange, derived by Kaimal *et al.* [22] on the basis of direct observation, can be written as

$$S_w\,(n) = 0.36 \left(\frac{\kappa U \psi_\varepsilon}{h}\right)^{2/3} w_*^2 n^{-5/3} \,, \tag{14}$$

where $\kappa = 0.4$ is the von Karman constant. In convective conditions $\psi_\varepsilon \approx 0.65$ [26, 27]

By setting $\beta_w = 0.55U/\sigma_w$ [24], computing σ_w from the integration from n_{I} to infinity of eq. (14), the following expression for KTV is obtained (from eq. (5)):

$$\nu_{\mathrm{T}} = 0.067 w_* \left(\frac{U}{n_{\mathrm{I}}}\right)^{4/3} \left(\frac{\kappa \psi_\varepsilon}{h}\right)^{1/3} \,. \tag{15}$$

We would like to stress that differently from eq. (15), which is derived from 1D vertical turbulent energy spectrum, eq. (13) describes a KTV associated to the 3D turbulent energy spectrum.

Concerning eq. (6), let us remember that, according to the convective similarity theory [27, 28], the Eulerian velocity spectra under unstable conditions can be expressed by the following relationships:

$$S_{w,0}(n) = \frac{0.38\frac{z}{U}\left(\frac{\psi_\varepsilon z}{h}\right)^{2/3} w_*^2}{(f_m^*)_w^{5/3}\left[1+\frac{1.5nz}{U(f_m^*)_w}\right]^{5/3}}, \tag{16}$$

where $(f_m^*)_w$ is the reduced frequency of the convective spectral peak. Since the spectral peak for the vertical component can be approximated [22, 26] by $(\lambda_m)_w = a_w h q_w$, the following relationships hold:

$$(f_m^*)_w = \frac{z}{(\lambda_m)_w} = \frac{z}{a_w h q_w}, \tag{17a}$$

$$n_\mathrm{e} = \frac{U}{a_w h q_w} \tag{17b}$$

and, consequently, eq. (16) becomes

$$S_{w,0}(n) = \frac{0.38\frac{h}{U}(a_w q_w)^{5/3}\psi_\varepsilon^{2/3} w_*^2}{\left(1+\frac{1.5nha_w q_w}{U}\right)^{5/3}}. \tag{18}$$

We remind that, according to Caughey and Palmer [26] and Kaimal and Finnigan [27], for the vertical turbulent wind component $a_w = 1.8$ end $q_w = 1-\exp[-4z/h]-0.0003\exp[8z/h]$, q_w is thus related to the vertical profile of more energetic eddies and accounts for the level of turbulence at each height.

To derive explicit relationships to be used in practical applications, it is necessary to have a good experimental estimation of n_I to be inserted in eq. (15). Following Kaimal *et al.* [22], who found that the onset of the inertial subrange in the CBL occurs at a wavelength $\lambda_w \approx 0.1h$, we assume in this work $n_\mathrm{I} \approx 10U/h$. It is of interest to point out that, using n_e values estimated by Kaimal *et al.* [22] in a well-mixed layer and by, NB and Sorbjan [6] in large eddy simulations, the ratio $n_\mathrm{I}/n_\mathrm{e}$ ranges from 15 to 20. Consequently the condition $n_\mathrm{e} \ll n_\mathrm{I}$ is justified and the inertial subrange can be considered statistically independent of the subrange energy-containing eddies. With this assumption eq. (15) becomes

$$\nu_\mathrm{T} = 1.98 \times 10^{-3} h w_*. \tag{19}$$

Considering typical CBL values ($w_* = 2\,\mathrm{ms}^{-1}$ and $h = 1500\,\mathrm{m}$), there results $\nu_\mathrm{T} = 6.0\ \mathrm{m}^2\mathrm{s}^{-1}$ for the w component. It is worth noting that this value verifies the above condition $\nu_\mathrm{T} \gg \nu$. It is important to point out that the decaying vertical energy spectrum, calculated from eqs. (6) and (18), for different evolution times shows that the largest eddies decay slower than the smaller ones. In fact, our result agrees with Sorbjan's who found that the position of the spectral peak is shifted towards the largest eddies.

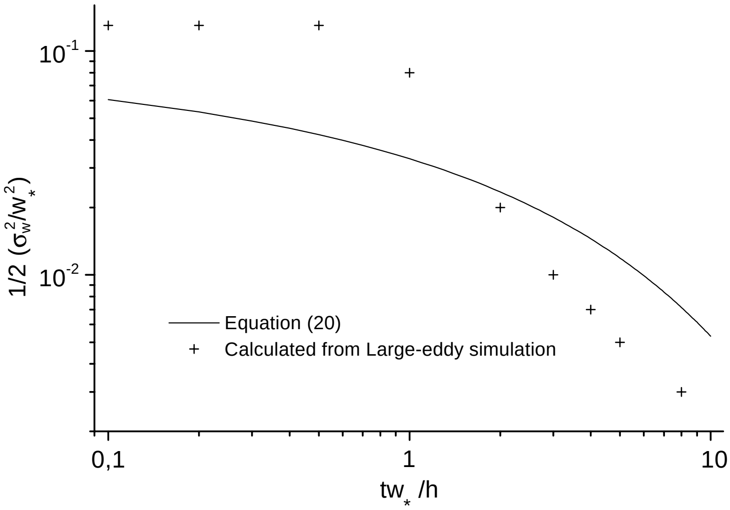

Fig. 1. – The temporal trend of the vertical velocity variance averaged across the CBL and normalized by w_*^2. The solid line is calculated from eq. (20), whereas crosses represent results obtained by NB from LES model.

By integrating eq. (6) (valid here for the 1D turbulent energy spectrum) from n_{e} to infinity yields

$$\sigma_w^2\,(z,t) = 0.76 q_w^{5/3} w_*^2 \int\limits_{(1.8q_w)^{-1}}^{\infty} \frac{\exp\left[-0.16 f_h^2 \frac{w_* t}{h}\right]}{\left(1+2.7 q_w f_h\right)^{5/3}}\,\mathrm{d}f_h\,, \tag{20}$$

where $f_h = nh/U$.

Figure 1 shows the vertical velocity variance averaged across the boundary layer and normalized by w_*^2, as a function of tw_*/h. Looking at fig. 1, we notice that for $tw_*/h < 1$ there is a good correspondence between the shape of the decaying vertical variance calculated from our model (eq. (20)) with those simulated by LES (NB points). For $tw_*/h > 1$ the LES data decrease more rapidly than the values estimated from eq. (20). In the case of LES data, this vertical variance decays as a function of time according to the power law t^{-2}. On the other hand, the vertical variance calculated from eq. (20) decays as $t^{-1.3}$. We note that this last exponent lies in the range usually observed for the decay of turbulent energy in the case of isotropic turbulence. This different decaying exponent can be explained considering the energy distribution among the velocity components. The energy in the vertical component is distributed between the lateral and longitudinal velocity. However our model was constructed for an isotropic three-dimensional turbulent flux subject to the energy conservation principle and as a consequence it cannot forecast and quantify the loss of energy associated to the vertical velocity component.

By considering eq. ($7a$) and eq. (20) and accounting for $n_{\mathrm{e}} = U(1.8 q_w h)^{-1}$, the

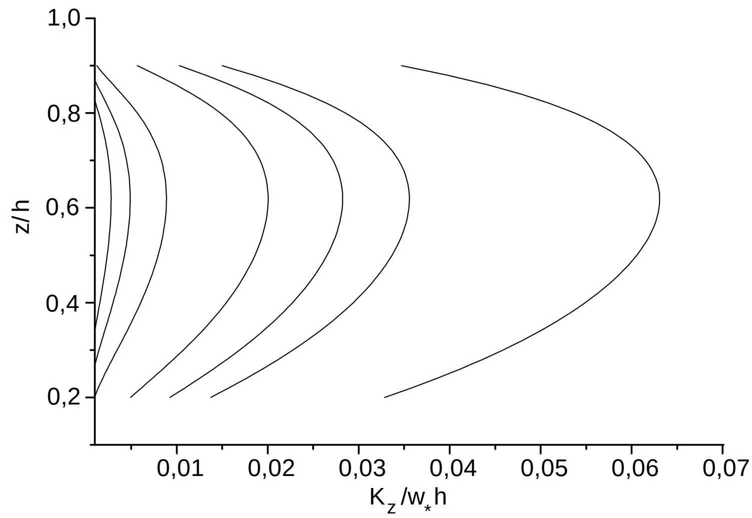

Fig. 2. – Temporal evolution of the vertical profiles of normalized RL eddy diffusivity as a function of dimensionless height. The profiles are evaluated at seven different times, $t = 0$, 1/2, 1, 2, 5, 8 and 11 h.

following expression for the decaying vertical eddy diffusivity results:

$$\frac{K_z\,(z,t)}{w_* h} = 0.15 q_w^{11/6} \left[\int\limits_{(1.8 q_w)^{-1}}^{\infty} \frac{\exp\left[-0.16 f_h^2 w_* t/h\right]}{\left(1 + 2.7 f_h q_w\right)^{5/3}}\, \mathrm{d} f_h \right]^{1/2} . \tag{21}$$

The vertical profile of RL eddy diffusivity (as given by eq. (21)), obtained by inserting the above-mentioned typical CBL values, and normalized by dividing by (hw_*), *vs.* z/h, is illustrated in fig. 2. Seven profiles, evaluated at different times ($t = 0$, 1/2, 1, 2, 5, 8 and 11 h), are shown in a linear-linear scale. In this graph, curves begin at $z/h = 0.2$ in order to exclude the lower part of the PBL, in which the nocturnal ground-based stable layer is likely to develop. The K_z temporal evolution shown in this figure is in a good qualitative agreement with the corresponding evolution presented by Freedman and Bornstein [3]. In particular, also in our case, it appears that RL eddy diffusivity is higher than that typical of the SBL by about an order of magnitude.

Lacking, as anticipated in the introduction, conclusive observations of the turbulence characteristics in the RL, we may propose the following comparison. By inserting eq. (17*b*), and $\beta_w = 0.55 U/\sigma_w$ into eq. (7*a*),

$$K_z\,(z,t) = 0.16 h q_w \sigma_w \tag{22}$$

is obtained; eq. (22) expresses the K_z vertical profile in the RL. If the $\sigma_w\,(z,t)$ values are known, K_z vertical profile computed by means of eq. (22) can be compared with those obtained by our model (eq. (21)). For this comparison we make use of the $\sigma_w\,(z,t)$ calculated by NB from LES data and reported in their fig. 6 (referring to their numerical experiment 10). The following characteristic parameters have been used: ($w_* = 2.3\ \mathrm{ms}^{-1}$

TABLE I. – *Vertical eddy diffusivities calculated for different dimensionless time from eqs.* (21) *and* (22).

	$tw_*/h = 0.7$		$tw_*/h = 1.5$		$tw_*/h = 2.2$	
z/h	Eq. (21)	Eq. (22)	Eq. (21)	Eq. (22)	Eq. (21)	Eq. (22)
0.25	81	154	66	90	59	63
0.4	121	216	105	130	95	90
0.5	137	234	119	140	108	98
0.6	144	241	124	140	113	97
0.7	138	226	118	130	107	92
0.8	115	187	97	101	88	72

and $h = 1350$ m, from NB). The K_z values were computed for the dimensionless times $tw_*h^{-1} = 0.7$, 1.5 and 2.2. The results are shown in table I. An inspection to this table suggests that the vertical eddy diffusivity calculated by the present model (eq. (21)) adequately describes those obtained from eq. (22) in which the NB standard deviations $\sigma_w\,(z,t)$ were used.

The above considerations mean, in particular, that effluents, either released directly into the RL during nighttime from elevated sources or emitted during daytime and trapped in the RL because of the time evolution of the PBL structure, may be effectively diluted.

In this section the following assumptions were assumed:

a) isotropy and homogeneity of turbulent flow;

b) the initial vertical energy spectrum is considered to have the same form as it had in the CBL;

c) the eddies in the inertial subrange are considered statistically independent of the energy-containing eddies subrange;

d) the characteristic frequencies n_{I} and n_{e} in a CBL are experimentally determinated.

5. – An algebraic approximation for the vertical eddy diffusivity in the RL

Based on the NB study (their eq. (11)), the complex integral for the vertical eddy diffusivity expressed by eq. (21) can be approximated by a simple algebraic formula presented in terms of the normalized time range and written here as

$$\frac{K_z(z,t)}{w_*h} = 0.16q_w\left(C_1 + C_2\sqrt[m]{\frac{w_*t}{h}}\right). \tag{23}$$

For $0 \leqslant w_*t/h \leqslant 24$ we have

$$C_1 = \left(\frac{\sigma_w}{w_*}\right)_0, \qquad C_2 = \frac{\left(\frac{\sigma_w}{w_*}\right)_{24} - \left(\frac{\sigma_w}{w_*}\right)_0}{\sqrt[m]{24}} \qquad \text{and} \quad m = 4\,;$$

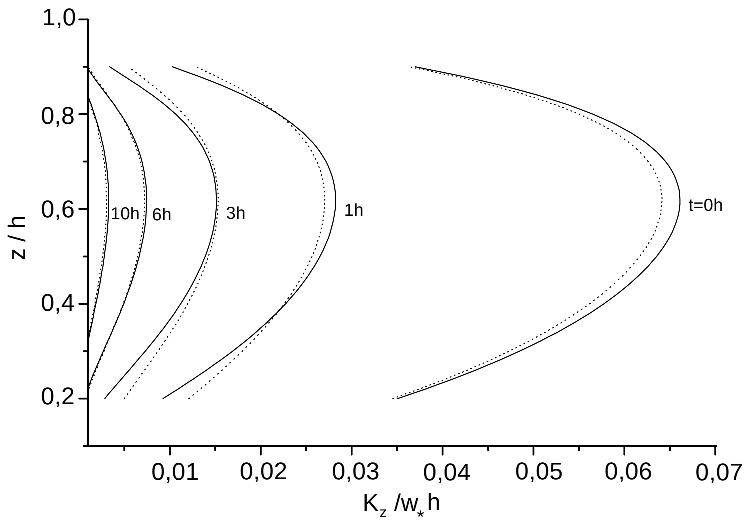

Fig. 3. – Vertical eddy diffusivity calculated from eqs. (21) (integral, solid line) and (23) (algebraic, dot line). The profiles are evaluated at five different times ($t = 0$, 1, 3, 6 and 10 h).

$$\left(\frac{\sigma_w}{w_*}\right)_0 = 0.48 q_w^{1/3}$$

and

$$\left(\frac{\sigma_w}{w_*}\right)_{24} = -0.0096 - 0.056\frac{z}{h} + 1.0813\left(\frac{z}{h}\right)^2 - 0.6995\left(\frac{z}{h}\right)^3 - \\ -5.8958\left(\frac{z}{h}\right)^4 + 14.6222\left(\frac{z}{h}\right)^5 + 13.5\left(\frac{z}{h}\right)^6 + 4.4246\left(\frac{z}{h}\right)^7 ;$$

appearing in the C_1 and C_2 coefficients are the nondimensional vertical turbulent velocity standard deviations calculated at, respectively, nondimensional times $w_* t/h = 0$ and $w_* t/h = 24$.

On the other hand, for $24 \leqslant w_* t/h \leqslant 48$, we have

$$C_1 = \left(\frac{\sigma_w}{w_*}\right)_{24} - C_2 \sqrt[m]{24}\,, \qquad C_2 = \frac{\left(\frac{\sigma_w}{w_*}\right)_{48} - \left(\frac{\sigma_w}{w_*}\right)_{24}}{\sqrt[m]{48} - \sqrt[m]{24}}\,, \quad \text{with} \quad m = 10$$

and

$$\left(\frac{\sigma_w}{w_*}\right)_{48} = -0.0033 + 0.1161\frac{z}{h} - 1.5722\left(\frac{z}{h}\right)^2 + 9.3963\left(\frac{z}{h}\right)^3 - 25.757\left(\frac{z}{h}\right)^4 + \\ +37.0279\left(\frac{z}{h}\right)^5 - 27.4259\left(\frac{z}{h}\right)^6 + 8.2247\left(\frac{z}{h}\right)^7 .$$

In fig. 3 the graph of K_z/w_*h given by integral (21) and algebraic (23) formulations for five different times ($t = 0$, 1, 3, 6, 10 h) *vs.* nondimensional height, is depicted, values $w_* = 2$ ms^{-1} and $h = 1500$ m were considered in these computations.

The comparison of these figures shows a very good agreement between the two expressions (eqs. (21) and (23)) for the major part of the RL indicating that the suggestion of using an algebraic formulation (eq. (23)) as a surrogate of eq. (21) for vertical eddy diffusivity in the turbulent RL is valid.

6. – Conclusions

A general semi-empirical method to derive eddy diffusivities in a decaying turbulence is proposed. The theoretical framework is the classical statistical diffusion theory, the dynamical equation for the energy spectrum and Heisenberg's elementary decaying turbulence theory [9]. The main point in Heisenberg's derivation is the hypothesis of a virtual turbulent viscosity, called kinematic turbulence viscosity, which is invoked to explain the mechanism of inertial transfer of energy from large to small eddies. This KTV is assumed to represent the friction produced by the smaller eddies and acting on the larger eddies. According to Batchelor [16], a net separation (statistical independence) between small and large eddies is the condition for this KTV to be physically correct. Employing principal frequencies, that characterise the dimensions of the more energetic eddies in different energy spectral subranges and are statistically independent, an expression for KTV can be obtained.

Concerning this point, the principal assumption of the present paper is the existence of a frequency n_{I}, located in the inertial subrange, which is associated to the size of the eddies that provide the main contribution to inertial energy flux. Based on this assumption, theoretical expressions (eqs. (5) and (7a)) for K_z (eddy diffusivity in decaying turbulence) and KTV are derived.

As an application of the derived eddy diffusivity in decaying turbulence, the problem of the decay of convective turbulence in the RL, the nearly adiabatic remnant of the daytime boundary layer, is addressed. The following relationship, $n_{\mathrm{I}} \approx (10U)/h$, based on the work done by Kaimal *et al.* [22], was assumed. It is worth noting that this assumption guarantees, at least in a first approximation, the statistical independence between n_{I} and the frequency n_{e}, associated to the subrange energy-containing eddies. Equations (15), (21) are the main results.

Equation (21) has no analytical solution. Consequently, in order to obtain a relationship that could be used in practical applications, eq. (23) is proposed. This is an algebraic approximated formula that demonstrated to give results very similar to those of eq. (21).

The nature of the subject of the present paper is not easily suited for a direct check between experiment and model. However we compared our temporal decaying eddy diffusivity evolution with the results obtained by Freedman and Bornstein [3] in a numerical simulation of Wangara Day 33 case. The agreement was found to be qualitatively good.

It is our opinion that the expressions for the eddy diffusivity derived in this paper may be used in mesoscale transport and dispersion simulations.

REFERENCES

[1] Stull R. B., *An Introduction to Boundary Layer Meteorology* (KIuwer Academic Publishers, Boston) 1988.

[2] Kim J. and Mahrt L., *Tellus A*, **44** (1992) 381.
[3] Freedman F. K. and Bornstein R. D., *Study of the spatial and temporal structure of turbulence in the nocturnal residual layer*, in *Proceedings of 22nd NATO/CCMS International Technical Meeting on Air Pollution Modelling and its Application, Clermont-Ferrand, France, June 2-6, 1997* (1997).
[4] Schayes G., Thunis P. and Bornstein R. D., *J. Appl. Meteorol.*, **35** (1996) 1815.
[5] Nieuwstadt F. T. M. and Brost R. A., *J. Atmos. Sci.*, **43** (1986) 532.
[6] Sorbjan Z., *Boundary-Layer Meteorol.*, **82** (1997) 501.
[7] Desiato F., Anfossi, Trini Castelli S. and Ferrero E., *Atmos. Environ.*, **32** (1998) 4157.
[8] Hinze J. O., *Turbulence* (Mc Graw Hill) 1975.
[9] Stanisic M. M., *The Mathematical Theory of Turbulence* (Springer Verlag, Berlin and New-York) 1988.
[10] Chandraseikhar F. R. S., *Proc. R. Soc. London, Ser. A*, **200** (1949) 20.
[11] Batchelor G. K., *Proc. R. Soc. London, Ser. A*, **195** (1949) 513.
[12] Frisch U., *Turbulence* (Cambridge University Press) 1995.
[13] Kaimal J. C., Wyngaard J. C., Izumi Y. and Cote' O. R., *Q. J. R. Meteorol. Soc.*, **98** (1972) 563.
[14] Muschinski A. and Roth R., *Contributions to Atmospheric Physics*, **66/4** (1993) 335.
[15] Ferziger J. H., *Large-eddy simulation: An introduction and perspective, Lecture 2*, in *New Tools in Turbulence*, edited by O. Matais and J. Ferziger (Les edition de Physique, Springer) 1997.
[16] Batchelor G. K., *The theory of homogeneous turbulence*, *Cambridge Monograph of Mechanics and Applied Mathematics* (Cambridge University Press) 1953.
[17] Rose H. A. and Sulem P. L., *J. Phys. (Paris)*, **T39** (1978) 441.
[18] Mann J., *J. Fluid Mech.*, **273** (1994) 141.
[19] Wilson K. D., *Boundary-Layer Meteorol.*, **85** (1997) 35.
[20] Hanna S. R., *J. Appl. Meteorol.*, **20** (1981) 242.
[21] Weil J. C., *Stochastic modeling of dispersion in the convective boundary layer*, in *Air Pollution Modeling and its Application VII*, edited by H.V. Dop (Plenum Press) 1989.
[22] Kaimal J. C., Wyngaard J. C., Haugen D. A., Cote' O. R., Izumi Y., Caughey S. J. and Readings C. J., *J. Atmos. Sci.*, **33** (1976) 2152.
[23] Corrsin S., *J. Atmos. Sci.*, **20** (1963) 115.
[24] Degrazia G. A. and Anfossi D., *Atmos. Environm.*, **32** (1998) 3611.
[25] Kolmogorov A. N., *Dokl. Akad. Nauk SSSR*, **30** (1941) 9.
[26] Caughey S. J. and Palmer S. G., *Q. J. R. Meteorol. Soc.*, **105** (1979) 811.
[27] Kaimal J. C. and Finnigan J. J., *Atmospheric Boundary Layer Flows* (Oxford University Press) 1994.
[28] Degrazia G. A., Moraes O. L. L., and Oliveira A. P., *J. Appl. Meteorol.*, **35** (1996) 974.