The virial theorem and the atmosphere

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Summary. — In our atmosphere, most of the energy resides as internal energy, U, and gravitational energy P, and the proportionality $U/P = c_V/R = 5/2$ is maintained in an air column provided there is hydrostatic equilibrium. In this paper we show that this result is a consequence of the virial theorem.

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1. – Introduction

As mentioned in the summary, in a planetary atmosphere most of the energy resides as thermodynamic internal energy, U, and gravitational potential energy, P, and in a hydrostatic column, the ratio $U/P = c_V/R$ [1] is maintained no matter what the vertical profile of the temperature may be in the air column. R and c_V are the gas constant and specific heat at constant volume, respectively, of the molecules existing in the atmosphere. In the case of the atmosphere of the earth, as most of the molecules are diatomic, $c_V = (5/2)R$. Qualitatively speaking, this constant ratio means that if the internal energy of an air column is increased by some heating process, the air column must expand vertically, thereby increasing its gravitational potential energy. These ideas are at the basis of the important concept of available potential energy.

To check this ratio, we use the equation of hydrostatic equilibrium and the gas law:

$$\mathrm{d}p = -g\rho\,\mathrm{d}z\,,$$
$$p = \rho R T\,. \tag{1}$$

Then, the internal energy of a vertical air column of section unity extending from the surface up to the end of the atmosphere, *i.e.* up to a height where pressure and density are nil is

$$U = c_V \int_0^\infty T\rho\,\mathrm{d}z = (c_V/R) \int_0^\infty p(z)\,\mathrm{d}z\,, \tag{2}$$

and the gravitational potential of the energy column is

$$(3) \qquad P = \int\limits_0^\infty zg\rho\, \mathrm{d}z = -\int\limits_0^\infty z\mathrm{d}p\,.$$

Integrating by parts and introducing the gas law

$$(4) \qquad P = R\int\limits_0^\infty \rho T\mathrm{d}z = \int\limits_0^\infty p(z)\mathrm{d}z\,.$$

This shows the constant proportionality $U/P = c_V/R$.

It is important to realise that in this calculation of U and P, we have adopted as the state of minimum energy, or level of reference, that where $T(z) = 0$ for the whole column, and consequently all the gas molecules are deposited at ground level, that is $U_0 = P_0 = 0$.

The Virial Theorem (VT) when applied to a bound classical many-particle system says [2] that the total kinetic energy, E_{kin}, is given by

$$(5) \qquad 2E_{\mathrm{kin}} = I = -\left\langle \sum_i \vec{r}_i \cdot \vec{F}_i \right\rangle,$$

where I is known as the virial of the system; $\vec{F}_i$ denotes the force acting on the i-th particle whose position is defined by $\vec{r}_i$, and the brackets denote a time or spatial average. This theorem is a fundamental result deduced by invoking only the equations of motion and the assumption that all the coordinates remain finite. Depending on the nature of the problem under consideration, the virial I is a sum of terms corresponding to the different types of forces acting on the particles of the system. The VT is easily generalized to quantum and relativistic physics.

Very likely the two areas in physics where the application of the VT has been most fruitful are: i) the equation of state of non-ideal gases, where this theorem indicates the systematic procedure [3] to compute the successive corrections to the equation of state of the ideal gas, and ii) astrophysics, where the application of this theorem provides a qualitative understanding of the evolution of a star [4].

In this paper, we deduce and discuss the implication of the VT for a planetary atmosphere. In sect. **2** we identify the relevant forces acting in the atmosphere and deduce the meaning of eq. (5) for this system. In sect. **3**, we perform the same job for a vertical column of air, under the simplifying hypothesis of dealing with a planar atmosphere. As expected, the two results are equivalent. Finally, in sect. **4** we recognise the VT implication for the atmosphere as the reason for the proportionality between U and P, commented on in this section.

2. – The virial theorem and the atmosphere

In the case of the atmosphere, the individual constituting particles are the molecules, and the forces acting upon them are of three types. The first type is the long-range gravitational force, which is vertically directed towards the centre of the planet. The

second type is the contact repulsive force exerted by the planetary surface, which is directed vertically outwards. The third type corresponds to the force exerted by the rest of the molecules, by collisions. These inter-molecule short-range effects certainly exist but in the first instance are not so important. Note that the atmospheric gas behaves like an ideal gas.

For these reasons, the virial of the atmosphere, I, will contain just two terms,

$$(6) \qquad\qquad I = I_{\mathrm{g}} + I_{\mathrm{s}} \,,$$

which correspond to the gravitational attraction and to the contact effect with the surface, respectively.

Locating our coordinate system in the centre of the planet and using polar spherical coordinates, we find

$$(7) \qquad\qquad I_{\mathrm{g}} = -\left\langle -\sum_i r_i \frac{GMm}{r_i^2} \right\rangle = GMm \left\langle \sum_i \frac{1}{r_i} \right\rangle = -V_{\mathrm{g}} \,,$$

where m denotes the mean molecular mass in the atmosphere, M is the planetary mass, and V_{g} the total gravitational energy of the atmosphere. Now, if a and p_0 denote the planetary radius and the value of the atmospheric pressure at surface, we obtain

$$(8) \qquad\qquad I_{\mathrm{s}} = -\left\langle \sum_i a \, f_{c,\,i} \right\rangle = -a(4\pi a^2)\, p_0 = -4\pi a^3 p_0.$$

This relation comes from the fact that the temporal average of the contact forces, f_{c}, exerted by the surface on the molecules is equal and opposite to the force exerted by the molecules on the ground, and this is p_0 times the area of the planet's surface.

Thus the VT implies

$$(9) \qquad\qquad 2E_{\mathrm{kin}} + V_{\mathrm{g}} + 4\pi\, a^3 p_0 = 0 \,.$$

Henceforth we will assume that all magnitudes of the gas such as the pressure, p, temperature, T, and the mass density, ρ, depend only on the radial coordinate, r.

Considering the atmospheric gas as a continuum, V_{g} is expressed as

$$(10) \qquad V_{\mathrm{g}} = -\int_0^{2\pi}\!\!\int_0^{\pi}\!\!\int_a^{\infty} \frac{GM}{r}\rho(r)\,\mathrm{d}\vec{r} = -\int_0^{2\pi}\!\!\int_0^{\pi}\!\!\int_a^{\infty} (rg(r))\rho(r)\,\mathrm{d}\vec{r},$$

where $g(r) = (GM/r^2)$ stands for the gravity acceleration, and $\mathrm{d}\vec{r} = r^2\mathrm{d}r\mathrm{d}\Omega$ ($\mathrm{d}\Omega = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$) is the differential of volume. Using the condition of hydrostatic equilibrium

$$(11) \qquad\qquad \mathrm{d}p(r) = -\rho(r)g(r)\mathrm{d}r \,,$$

and integrating by parts, we obtain

$$(12) \qquad\qquad V_{\mathrm{g}} = -4\pi\, a^3 p_0 - 3\int_0^{2\pi}\!\!\int_0^{\pi}\!\!\int_a^{\infty} p(r)\,\mathrm{d}\vec{r}.$$

And thus eq. (9) leads to

$$(13) \qquad E_{\text{kin}} = \frac{3}{2} \int\limits_0^{2\pi} \int\limits_0^{\pi} \int\limits_a^{\infty} p(r)\,\mathrm{d}\vec{r},$$

which is the VT implication for the spherical hydrostatic atmosphere. This result agrees with the kinetic theory [3].

3. – The virial theorem in a planar atmosphere

The planar approximation is a good one for the atmosphere because the scale height, H, for the vertical decrease in the air density is around 10 km, which is very small compared to the planetary radius ($a \gg H$). Thus, we can suppose that the ground surface is represented by the plane $z = 0$, and the air density is a decreasing function of $z > 0$. It is implicitly assumed, as in sect. **2**, that beyond a high enough value of z, the gas density vanishes. For such a planar gaseous distribution, we now deduce the consequences of eq. (5). Specifically we will focus on the air contained in a vertical cylinder whose axis coincides with the OZ axis, and whose base is a circle of surface unity lying on the ground. Thus, all the energy terms calculated in this section are per surface unity. We will take the origin of coordinates in the centre of the circle.

For this system, the term I_{s} accounting for the surface contact effect is nil because the height of the ground is $z = 0$. With respect to the gravitational contribution to the virial, I_{g}, we have

$$(14) \qquad I_{\text{g}} = -\left\langle \sum_i -z_i m g_i \right\rangle.$$

Passing to the continuum, and assuming a constant value of the gravity acceleration we find

$$(15) \qquad I_{\text{g}} = \int\limits_0^{\infty} z g_0 \rho(z)\mathrm{d}z \,,$$

which coincides with the familiar form of the potential energy, P, of an arbitrary mass distribution ρ, near the Earth's surface if one takes the $z = 0$ plane as the level of reference. Thus,

$$(16) \qquad I_{\text{g}} = P \,.$$

Using the hydrostatic condition, $\mathrm{d}p(z) = -g_0 \rho(z)\mathrm{d}z$ we obtain

$$(17) \qquad I_{\text{g}} = -\int\limits_0^{\infty} z\mathrm{d}p \,,$$

and integrating by parts

$$(18) \qquad I_{\mathrm{g}} = \int_0^\infty p(z)\mathrm{d}z \, .$$

Thus, supposing that $I = I_{\mathrm{g}}$ we obtain

$$(19) \qquad 2E_{\mathrm{kin}} = \int_0^\infty p(z)\mathrm{d}z \, ,$$

which disagrees with eq. (13) by a factor 3 and contradicts the kinetic theory. This contradiction indicates that some dynamical effect, in the virial I, is missing in the calculation. This is exactly the case; the pressure effect exerted by the molecules lying just on the other side of the cylindrical border that defines our subsystem has not been taken into account. For this reason, in this case of the planar atmosphere, the correct virial is

$$(20) \qquad I = I_{\mathrm{g}} + I_{\mathrm{b}} \, ,$$

where I_{g} is the calculated gravitational effect and I_{b} comes from the above-mentioned lateral border effect on the subsystem. I_{b} amounts to

$$(21) \qquad I_{\mathrm{b}} = \int_0^\infty bp(z)(2\,\pi\,b\,\mathrm{d}z) \, ,$$

where b represents the radius of the base of the cylinder. As the surface of the base is 1, we have $\pi\,b^2 = 1$, and thus

$$(22) \qquad I_{\mathrm{b}} = 2\int_0^\infty p(z)\mathrm{d}z \, .$$

Adding eq. (18) and eq. (21) we find the correct formula for the VT when it is applied to a planar atmosphere:

$$(23) \qquad 2E_{\mathrm{kin}} = 3\int_0^\infty p(z)\mathrm{d}z \, .$$

Thus, the previous contradiction with eq. (13) has disappeared.

4. – Discussion and conclusion

We have obtained

$$(24) \qquad E_{\mathrm{kin}} = \frac{3}{2}\int_0^\infty p(z)\mathrm{d}z \, ,$$

as the VT prediction for the atmospheric column. As the kinetic energy of the molecules is not a thermodynamic function, we use the principle of equipartition of energy to pass from E_{kin} to U, that is, to the internal energy. We will denote by ν the number of active degrees of freedom in the gas, *i.e.* $\nu = 3$ for the translational modes of monoatomic molecules, $\nu = 5$ for the 3 translational modes plus 2 rotational modes of diatomic molecules, etc. In the atmosphere of the Earth, the N_2 and O_2 molecules are dominant and therefore $\nu = 5$. Thus, invoking the energy equipartition, from the proportion

$$(25) \qquad \frac{U}{\nu} = \frac{E_{\text{kin}}}{3} \,,$$

eq. (24) adopts the form

$$(26) \qquad U = \frac{\nu}{2} \int\limits_0^\infty p(z)\mathrm{d}z = \frac{\nu}{2}P \,.$$

As in a perfect gas $R(\nu/2) = c_V$, eq. (26) is equivalent to

$$(27) \qquad \frac{U}{P} = \frac{c_V}{R} \,,$$

which is the ratio checked in sect. **1**.

Therefore, we conclude that the result of the VT in the atmosphere is in itself the proportionality between U and P. In this sense, note that the VT theorem leads to eq. (27) without the necessity of adopting any particular level of reference for the energies of the column, *i.e.* there is no ambiguity in the deduction.

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