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Quasi-geostrophic equations revisited: The case of the ocean

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Summary. — Oceanic circulation on the beta-plane demands the presence of (at least) two scales of the motion to describe completely the flow structure and evolution both in the central-eastern area of the basin and in the western boundary layer. In each of the two regions of the basin, well-established quasi-geostrophic vorticity equations govern the flow. However, since these two regions are physically connected, a single and general vorticity equation is requested to hold in the whole basin. In the present investigation, the inference of this last equation is analysed by comparing some methods found in the literature with another which is put forward by the authors.

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1. – Introduction

The present investigation aims to explore the fundation of two kinds of equations governing the quasi-geostrophic dynamics of the ocean, for a conserved fluid density and a constant density, respectively. Therefore, the subject does not concern models or solutions of these equations (even if we will consider special solutions on illustrative purposes) but mainly the method of their inference.

In the framework of the quasi-geostrophic approximation, the oceanic circulation on the beta-plane demands the consideration of (at least) two scales of the motion, one being referred schematically to the central and eastern areas of the basin and the other to the western boundary layer, in the westernmost side of the basin. By convention, we call basin scale dynamics that concerning the flow in the central and eastern areas and mesoscale dynamics that related to the flow evolution in the western side and to the eddies, although the eddy field lies outside our goal. Well-established quasi-geostrophic vorticity equations (depending on the considered density structure of the fluid) govern the flow in each of the two regions but the incessant transition of the fluid elements from

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one region to the other makes it unavoidable the resort to a single general equation, valid in the whole basin. This happens, for instance, in modelling Rossby wave packets or subtropical gyres if the effects produced by the western boundary layer are taken into account as well. Here, we will focus the attention to the construction of the quasi-geostrophic vorticity equation in its general form whose deduction, in our opinion, is not made always clear, as the following remarks show.

At times, for a flow with conserved fluid density, the mesoscale vorticity equation is assumed to be valid in the whole basin. This, since it can be formally transformed into the vorticity equation of the basin scale simply by a rescaling of its nondimensional parameters, is in contradiction however with the hypotheses on which the mesoscale equation is based. On the other hand, in the homogeneous model (constant fluid density), one should clearly distinguish the advective time of the mesoscale from the so-called Rossby wave period which yields the typical time at the basin scale. Only with this distinction in mind, not always recognized in the literature, a consistent general vorticity equation can be inferred for the homogeneous ocean.

Our point of view is that, irrespectively of the features of the density field, the vorticity equation referable to the complete fluid domain comes from suitable nonsingular transformations applied to all the variables, fields, operators and parameters of the starting mesoscale equation thus leading to a general vorticity equation, in a composite form, such that:

- All the transformed variables, fields, operators and parameters appearing in the general equation are expressed only through basin scale quantities and no term vanishes after the transformations.
- In the general equation the $O(1)$ terms are only those giving the Rossby waves equation, the Sverdrup balance and a combination of them while the remaining terms are much smaller than unity. We stress that an equation with terms of different orders of magnitude cannot be obtained by means of the perturbative method used, separately, to constitute the mesoscale and the basin scale quasi-geostrophic equations from the primitive equations.
- In the western boundary layer, relative vorticity and/or dissipation grow well beyond unity and thus further terms $O(1)$ are restored in the general equation.

2. – Summary of the nondimensional starting equations

We preliminarily summarize the nondimensional equations, with the perturbation pressure geostrophically scaled, from which the quasi-geostrophic equations are inferred. The horizontal mean current $\bar{u} = (u, v)$ referred to a certain beta-plane is governed by the equations

$$\varepsilon_T \frac{\partial u}{\partial t} + \varepsilon \bar{u} \cdot \bar{\nabla} u - (1 + \beta \varepsilon y) v = -\frac{\partial p}{\partial x} + \frac{E_\mathrm{h}}{2} \nabla_\mathrm{h}^2 u + \frac{E_\mathrm{v}}{2} \frac{\partial^2}{\partial z^2} u, \tag{1}$$

$$\varepsilon_T \frac{\partial v}{\partial t} + \varepsilon \bar{u} \cdot \bar{\nabla} v + (1 + \beta \varepsilon y) u = -\frac{\partial p}{\partial y} + \frac{E_\mathrm{h}}{2} \nabla_\mathrm{h}^2 v + \frac{E_\mathrm{v}}{2} \frac{\partial^2}{\partial z^2} v. \tag{2}$$

In (1) and (2), $\varepsilon_T \equiv 1/f_0 T$ is the temporal Rossby number, $\varepsilon \equiv U/f_0 L$ is the advective Rossby number, $\beta \equiv \beta_0 L^2/U$ is the *nondimensional* planetary vorticity gradient, $E_\mathrm{h} \equiv 2A_\mathrm{h}/f_0 L^2$ is the horizontal Ekman number and $E_\mathrm{v} \equiv 2A_\mathrm{v}/f_0 H^2$ is the vertical Ekman number. Each scale of the motion is fixed once the values of L, U and T are given, while the Coriolis parameter $f_0 = O(10^{-4})$ and the *dimensional* planetary vorticity gradient $\beta_0 = O(10^{-11})$ are constant. The Ekman numbers suffer the strong indeterminateness of the coefficients A_h and A_v which come from the parametrization of turbulence and therefore they cannot be singled out only by the values of L, U and T. The relations between these numbers and ε_T, ε, β will be specified in the following. The perturbation pressure and the perturbation density are related by the equation

$$\frac{\partial p}{\partial z} + \rho(\bar{x}, t) = 0. \tag{3}$$

The hypothesis of an isentropic and nondiffusive ocean, in which total density is conserved following the motion, yields the thermodynamic equation

$$w = \frac{\varepsilon}{Bu}\frac{1}{N^2}\left(\frac{\partial \rho}{\partial t} + \bar{u}\cdot\bar{\nabla}\rho\right), \tag{4}$$

where w is the vertical velocity and $Bu \equiv (HN_0/f_0L)^2$ is the Burger number. In turn, $N \equiv N_s/N_0$ is the nondimensional buoyancy frequency, N_s is its dimensional counterpart and $N_0 = O(5\cdot 10^{-3})$ is the typical value of N_s. Note also that, if the local and the advective terms are expected to give comparable contributions to the conservation of total density, the advective time scale, *i.e.* $T = L/U$, holds. In the absence of a conservation principle this is not necessarily true. For instance, in the homogeneous ocean where the density conservation is substituted by the hypothesis of a constant density and the thermodynamics is not necessary to complete the equations of motion, the typical time interval of the basin scale dynamics is $T = (\beta_0 L)^{-1}$.

3. – The quasi-geostrophic equation at the basin scale

At the basin scale the observed values of L, U and T lead to the relation

$$\varepsilon_T = \varepsilon \ll \beta\varepsilon < 1 \tag{5}$$

together with

$$\beta Bu = O(1). \tag{6}$$

It is also assumed

$$E_\mathrm{h} \ll \beta\varepsilon \qquad \text{and} \qquad E_\mathrm{v} \ll \beta\varepsilon \tag{7}$$

in order that the geostrophic equilibrium be dominant in the leading dynamic balance. Because of (5) and (7), the ordering parameter is $\beta\varepsilon$ and the vorticity equation takes the form

$$v_0 = \frac{\partial w_1}{\partial z}\,, \tag{8}$$

where v_0 is the zeroth-order term in the expansion of the meridional velocity in powers of $\beta\varepsilon$ while w_1 is the first-order term in the expansion of the vertical velocity in powers of $\beta\varepsilon$. According to the same expansion we obtain from this last equation the leading term of the thermodinamic equation which is

$$w_1 = \frac{1}{\beta Bu N^2}\left(\frac{\partial \rho_0}{\partial t} + \bar{u}_0 \cdot \bar{\nabla}\rho_0\right). \tag{9}$$

At this point, w_1 can be eliminated from (8) and (9) to yield an equation involving only the geostrophic current $\bar{u}_0$ and the perturbation density ρ_0. In terms of the perturbation pressure at the zeroth order in $\beta\varepsilon$, which is conventionally denoted by ψ (instead of p_0), we have $\bar{u}_0 = \hat{k} \times \bar{\nabla}\psi$ and, using (3), $\rho_0 = -\partial\psi/\partial z$. Hence the equation takes the form

$$\frac{\partial}{\partial t}\frac{\partial}{\partial z}\left(\frac{1}{BuN^2}\frac{\partial \psi}{\partial z}\right) + J\left(\psi, \frac{\partial}{\partial z}\left(\frac{1}{BuN^2}\frac{\partial \psi}{\partial z}\right)\right) + \beta\frac{\partial \psi}{\partial x} = 0, \tag{10}$$

where $\bar{u}_0 \cdot \bar{\nabla} = J(\psi, \cdot)$ is the advection operator expressed as a function of the Jacobian determinant. Equation (10) is the quasi-geostrophic equation at the basin scale for a stratified and isentropic fluid on the beta-plane.

We report two examples of unforced and forced solutions of (10) that point out their inadequacy to describe exhaustively the flow behaviour in a real ocean.

– In the simplified case $N = 1$ we set in accordance with (6) $\beta Bu = 1$, to find the wavelike solution of (10) inside the layer $-1 \le z \le 0$ given by

$$\psi = \cos(k\pi z)\exp\left[in\left(x + \frac{t}{k^2\pi^2}\right) + imy\right] \tag{11}$$

(integer k). Solution (11) represents a *westward translating* Rossby wave with the same phase (c_x) and group (c_{gx}) velocity in the zonal direction, *i.e.* $c_x = c_{gx} = -(k\pi)^{-2}$. As every ocean basin is bounded by a couple of meridional coasts, the unidirectional propagation of the wave packet in the zonal direction would imply an eastern source and a western sink of energy at the basin walls. Actually, such unphysical situation is avoided by smaller scale wavelike components not satisfying (10) and travelling eastward.

– The vertical integration of each term of the *steady* solution of (10) with the use of the vertical boundary conditions

$$w_1(z=-1) = 0 \qquad \text{and} \qquad w_1(z=0) = \hat{k} \cdot \bar{\nabla} \times \bar{\tau}$$

yields

$$\int_{-1}^{0} J\left(\psi, \frac{\partial}{\partial z}\left(\frac{1}{N^2}\frac{\partial \psi}{\partial z}\right)\right)\mathrm{d}z = -\hat{k} \cdot \bar{\nabla} \times \bar{\tau} \qquad \text{and} \qquad \int_{-1}^{0} \frac{\partial \psi}{\partial x}\mathrm{d}z = \frac{\partial \Phi}{\partial x}\,, \tag{12}$$

where the transport streamfunction

$$\Phi = \int_{-1}^{0} \psi\, \mathrm{d}z$$

has been introduced. From (10) and (12) the well-known Sverdrup balance

$$\frac{\partial \Phi}{\partial x} = \hat{k} \cdot \bar{\nabla} \times \bar{\tau} \tag{13}$$

follows. We stress that the typical length of the basin scale is the same as that of $\bar{\tau}$ and therefore $\hat{k} \cdot \bar{\nabla} \times \bar{\tau} = O(1)$. If we consider, for instance, the subtropical region where $\hat{k} \cdot \bar{\nabla} \times \bar{\tau} \leq 0$, we conclude that the meridional transport $\partial\Phi/\partial x$ is *southward everywhere* in the underlying basin, thus preventing the closure of the wind-driven circulation inside the basin itself by means of a suitable northward return flow (the Gulf Stream, say). Again, the basin scale dynamics governed by (13) must be supplemented by a smaller scale contribution in order to reproduce also the northward return flow.

4. – The quasi-geostrophic equation at the mesoscale

At the mesoscale the observed values of L, U and T imply the relations

$$\varepsilon_T = \varepsilon \ll 1, \qquad \beta = O(1) \tag{14}$$

together with

$$Bu = O(1). \tag{15}$$

It is also assumed

$$E_{\mathrm{h}} = O(\varepsilon) \quad \text{and} \quad E_{\mathrm{v}} = O(\varepsilon) \tag{16}$$

in order to have the possibility of including dissipation into the vorticity equation. Because of (14) and (16), the ordering parameter is ε and the vorticity equation, written by resorting to the previously introduced streamfunction and Jacobian, takes the form

$$\frac{\partial}{\partial t}\nabla^2\psi + J\big(\psi, \nabla^2\psi\big) + \beta\frac{\partial\psi}{\partial x} = \frac{\partial w_1}{\partial z} + \frac{E_{\mathrm{h}}}{2\varepsilon}\nabla^4\psi. \tag{17}$$

From (4) the thermodynamic equation

$$w_1 = -\frac{1}{BuN^2}\left(\frac{\partial}{\partial t}\frac{\partial\psi}{\partial z} + J\left(\psi, \frac{\partial\psi}{\partial z}\right)\right) \tag{18}$$

follows. By eliminating the vertical velocity w_1 from (17) and (18) we eventually obtain the quasi-geostrophic equation at the mesoscale on the beta-plane in the form

$$\frac{\partial}{\partial t}\left(\nabla^2\psi + \frac{\partial}{\partial z}\left(\frac{1}{BuN^2}\frac{\partial\psi}{\partial z}\right)\right) + J\left(\psi, \nabla^2\psi + \frac{\partial}{\partial z}\left(\frac{1}{BuN^2}\frac{\partial\psi}{\partial z}\right)\right) + \beta\frac{\partial\psi}{\partial x} = \frac{E_{\mathrm{h}}}{2\varepsilon}\nabla^4\psi. \tag{19}$$

It is worthwhile reconsidering the examples of the previous section to see how and why the above limitations in the flow propagation at the basin scale are overcome at the mesoscale.

– If $N = 1$, $E_{\mathrm{h}} = 0$ and, in accordance with (14) and (15), we set $\beta = Bu = 1$, the wavelike solution of (19) inside the layer $-1 \le z \le 0$ is found to be

$$\psi = \cos(k\pi z)\exp\left[i\left(nx + my + \frac{n}{n^2 + m^2 + k^2\pi^2}t\right)\right] \tag{20}$$

(integer k). Solution (20) represents a Rossby wave with westward phase velocity $c_x = -1/(n^2 + m^2 + k^2\pi^2)$ and zonal group velocity

$$c_{gx} = \frac{n^2 - \left(m^2 + k^2\pi^2\right)}{\left(n^2 + m^2 + k^2\pi^2\right)^2}\,. \tag{21}$$

Unlike (11), (20) admits both westward ($n^2 < m^2 + k^2\pi^2$) and eastward ($n^2 > m^2 + k^2\pi^2$) propagation of wave packets. The presence of relative vorticity in (19) produces in (21) the square wave numbers n^2 and m^2 which allow such bidirectional propagation.

– The vertical integration of each term of the *steady* solution of (19), in which we disregard relative vorticity in favour of that thermal on simplicity grounds (we have no formal arguments to justify otherwise this step) and set $w_1(z = -1) = 0$, yields

$$\frac{\partial \Phi}{\partial x} = \frac{L_{\mathrm{a}} U_{\mathrm{Sv}}}{LU}\hat{k} \cdot \bar{\nabla} \times \bar{\tau} + \frac{E_{\mathrm{h}}}{2\varepsilon\beta}\nabla^4\Phi. \tag{22}$$

In (22) we do not take into account bottom friction arising from the bentic Ekman layer (we will consider this kind of dissipation in sects. **8** and **9**). In the same equation L_{a} is the "atmospheric" length scale which is the same as that of the basin scale, while U_{Sv} is the typical velocity of the basin scale which can be identified with that given by the Sverdrup balance (13). It is reasonable to assume $L_{\mathrm{a}}U_{\mathrm{Sv}}/LU = O(1)$ but, since the typical length of the mesoscale is smaller than that characteristic of $\bar{\tau}$, we have $O(\hat{k}\cdot\bar{\nabla}\times\bar{\tau}) < 1$. Both the meridional transport and the dissipative term of (22) are $O(1)$. In other words, the slowly varying wind field is weakly detected by the gradient operator of the mesoscale and the related Ekman pumping velocity is depressed. Therefore the mesoscale dynamics on the beta plane is *a priori* consistent with a northward transport which is mainly controlled by dissipation, as (22) shows.

5. – Comments on the quasi-geostrophic equations

Comparison of (11) with (20) and (13) with (22) makes evident the opportuneness of handling with a unique quasi-geostrophic equation in dealing with models that involve the effect of lateral boundaries such as solid coastlines and/or latitude circles along which the wind-stress curl vanishes. This kind of equation should be able to take into account, at the same time, both the basin scale and the mesoscale quasi-geostrophic dynamics depending on the local growth or reduction of relative vorticity and/or dissipation. The need of a unique equation is as important as obvious in numerical modelling. Indeed, due to the resemblance between (10) and (19), this problem seems to be solvable simply by a rescaling of the parameters of (19) itself, as follows.

Equation (10) is equivalent to

$$\frac{\partial}{\partial t}\frac{\partial}{\partial z}\left(\frac{1}{\beta Bu N^2}\frac{\partial \psi}{\partial z}\right) + J\left(\psi, \frac{\partial}{\partial z}\left(\frac{1}{\beta Bu N^2}\frac{\partial \psi}{\partial z}\right)\right) + \frac{\partial \psi}{\partial x} = 0, \tag{23}$$

while (19) can be written as

$$
(24') \qquad \frac{\partial}{\partial t}\left(\frac{1}{\beta}\nabla^2\psi + \frac{\partial}{\partial z}\left(\frac{1}{\beta BuN^2}\frac{\partial\psi}{\partial z}\right)\right)+ + J\left(\psi, \frac{1}{\beta}\nabla^2\psi + \frac{\partial}{\partial z}\left(\frac{1}{\beta BuN^2}\frac{\partial\psi}{\partial z}\right)\right) + \frac{\partial\psi}{\partial x} = \frac{E_{\mathrm{h}}}{2\varepsilon\beta}\nabla^4\psi.
$$

Now, if

a$'$) we take in (24$'$) the limit for $\beta \to \infty$

b) while keeping $O(1)$ the product βBu,

then eq. (24$'$) can be *formally* transformed into (23). This approach is shortly quoted in Pedlosky [1], Section 6.19. The point is that (19), and hence (24$'$), has been inferred for $\beta = O(1)$. Thus, in our opinion, limit a$'$) is problematic. Moreover, even if hypothetically it were applied, point b) would not be consistent with (15).

Alternatively, we can write (19) as

$$
(24'') \qquad \frac{\partial}{\partial t}\left(Bu\nabla^2\psi+\frac{\partial}{\partial z}\left(\frac{1}{N^2}\frac{\partial\psi}{\partial z}\right)\right)+ + J\left(\psi, Bu\nabla^2\psi+\frac{\partial}{\partial z}\left(\frac{1}{N^2}\frac{\partial\psi}{\partial z}\right)\right) + \beta Bu\frac{\partial\psi}{\partial x} = \frac{E_{\mathrm{h}}}{2\varepsilon}\nabla^4\psi\,,
$$

whence, if

a$''$) we take in (24$''$) the limit for $Bu \to 0$

b) while keeping $O(1)$ the product βBu,

then also eq. (24$''$) can be *formally* transformed into (23). This approach is shortly quoted in Gill [2], Section 12.8. Here, the point is that (24$''$) has been inferred for $Bu = O(1)$. Thus, limit a$''$) seems to be problematic. Moreover, even if hypothetically it were applied, point b) would not be consistent with (14).

Since eq. (19) with suitable values of β, Bu and $E_{\mathrm{h}}/2\varepsilon$ is currently used in ocean modelling, the speculative problem arises about its inference in the framework of the basin scale. We stress that it cannot be obtained through scaling arguments (which, on the contrary, lead to (10)) nor, in our opinion, by using the above procedure based on points a$'$), a$''$) and b).

To overcome this conceptual impasse, we put forward the method described in next section.

6. – Inference of the composite quasi-geostrophic equation

First of all, in order to distinguish the quantities referred to the basin scale from those of the mesoscale, hereafter we will mark these last with the subscript "μ", while no subscript will be applied to the basin scale quantities. For instance, (19) is rewritten as

$$\frac{\partial}{\partial t_\mu}\left(\nabla_\mu^2\psi_\mu + \frac{\partial}{\partial z_\mu}\left(\frac{1}{Bu_\mu N_\mu^2}\frac{\partial\psi_\mu}{\partial z_\mu}\right)\right) + \\ + J_\mu\left(\psi_\mu, \nabla_\mu^2\psi_\mu + \frac{\partial}{\partial z_\mu}\left(\frac{1}{Bu_\mu N_\mu^2}\frac{\partial\psi_\mu}{\partial z_\mu}\right)\right) + \beta_\mu\frac{\partial\psi_\mu}{\partial x_\mu} = \frac{E_{\mathrm{h}\mu}}{2\varepsilon_\mu}\nabla_\mu^4\psi_\mu. \tag{25}$$

We start from (25) since it brings more information than (10) about relative vorticity and friction. To see how $\nabla_\mu^2\psi_\mu$ and $(E_{\mathrm{h}\mu}/2\varepsilon_\mu)\nabla_\mu^4\psi_\mu$ become vanishingly small in the transition to the basin scale, we investigate how (25) is looked at this last scale. To the purpose we state the transformation equations of all the quantities involved into (25). Once starred quantities are understood to be dimensional, given L_μ, U_μ and L, U we can write

$$(x_*, y_*) = L_\mu(x_\mu, y_\mu) = L(x, y) \tag{26}$$

and

$$\psi_* = U_\mu L_\mu \psi_\mu = UL\psi. \tag{27}$$

Setting in short

$$\frac{L_\mu}{L} = a \qquad \text{and} \qquad \frac{U_\mu}{U} = b, \tag{28}$$

we obtain from (26) and (28)

$$\frac{\partial}{\partial x_\mu} = a\frac{\partial}{\partial x}, \qquad \bar{\nabla}_\mu = a\bar{\nabla}, \qquad J_\mu = a^2 J, \qquad \nabla_\mu^2 = a^2\nabla^2, \qquad \nabla_\mu^4 = a^4\nabla^4. \tag{29}$$

Because of the advective time scales, (28) yield

$$\frac{\partial}{\partial t_\mu} = \frac{a}{b}\frac{\partial}{\partial t}\,. \tag{30}$$

From (27) and (28) it follows that

$$\psi_\mu = \frac{1}{ab}\psi. \tag{31}$$

Moreover, one easily finds that

$$\beta_\mu = \frac{a^2}{b}\beta \qquad \text{and} \qquad \frac{E_{\mathrm{h}\mu}}{2\varepsilon_\mu} = \frac{1}{ab}\frac{E_{\mathrm{h}}}{2\varepsilon}\,. \tag{32}$$

About the vertical coordinate, we put *a priori* $z_* = H_\mu z_\mu = Hz$ and therefore

$$\frac{\partial}{\partial z_\mu} = \frac{H_\mu}{H}\frac{\partial}{\partial z}\,. \tag{33}$$

Note that the nondimensional buoyancy frequency is scale independent. In fact

$$N(z_\mu) = \frac{N_s(H_\mu z_\mu)}{N_0} = \frac{N_s(Hz)}{N_0} = N(z). \tag{34}$$

We find also the following transformation law of the Burger number:

$$Bu_\mu = \left(\frac{H_\mu}{aH}\right)^2 Bu, \tag{35}$$

so, from (26), (28), (34) and (35) we obtain the transformation law

$$\frac{\partial}{\partial z_\mu}\left(\frac{1}{Bu_\mu N_\mu^2}\frac{\partial \psi_\mu}{\partial z_\mu}\right) = \frac{a}{b}\frac{\partial}{\partial z}\left(\frac{1}{BuN^2}\frac{\partial \psi}{\partial z}\right) \tag{36}$$

whatever the ratio H_μ/H may be. Indeed, observational evidence of basin scale gyres suggests that $H_\mu = H$. This assumption will be used in next sections.

Once transformations (26)-(36) are substituted into (25), this last becomes

$$\frac{\partial}{\partial t}\left(\nabla^2\psi+\frac{\partial}{\partial z}\left(\frac{1}{BuN^2}\frac{\partial\psi}{\partial z}\right)\right)+J\left(\psi,\nabla^2\psi+\frac{\partial}{\partial z}\left(\frac{1}{BuN^2}\frac{\partial\psi}{\partial z}\right)\right)+\beta\frac{\partial\psi}{\partial x}=\frac{E_{\mathrm{h}}}{2\varepsilon}\nabla^4\psi. \tag{37}$$

We wish to underline the following:

- The coincidence of (37) with (19) is only formal since (37) holds at the basin scale while (19) refers to the mesoscale.
- Unlike (19) which is inferred by scaling (1)-(4) under conditions (14)-(16), eq. (37) relies on transformations (26)-(36) applied to (19).
- Because of (32) and (35), the parameters Bu, β and $E_{\mathrm{h}}/2\varepsilon$ have different orders of magnitude in (19) and in (37).

With the aid of the notation $1/\beta = (\delta_I/L)^2(\ll 1)$ and $E_{\mathrm{h}}/2\beta\varepsilon = (\delta_M/L)^3(\ll 1)$ we write the "composite" quasi-geostrophic equation (37) in the final form

$$\begin{aligned}\frac{\partial}{\partial t}\left(\left(\frac{\delta_I}{L}\right)^2\nabla^2\psi + \frac{\partial}{\partial z}\left(\frac{1}{N^2}\frac{\partial\psi}{\partial z}\right)\right) + \\ +J\left(\psi, \left(\frac{\delta_I}{L}\right)^2\nabla^2\psi + \frac{\partial}{\partial z}\left(\frac{1}{N^2}\frac{\partial\psi}{\partial z}\right)\right) + \frac{\partial\psi}{\partial x} = \left(\frac{\delta_M}{L}\right)^3\nabla^4\psi.\end{aligned} \tag{38}$$

The length δ_I is the inertial boundary layer width, while δ_M is the dissipative width in the presence of lateral dissipation of relative vorticity. In the framework of the boundary layer theory it would be easy to check that $L_\mu = \max\{\delta_I, \delta_M\}$.

In the ocean interior $(\delta_I/L)^2$ and $(\delta_M/L)^3$ are much smaller than unity, so the $O(1)$ balance in (38) gives, recalling also (6), eq. (10). In the ocean regions where $\nabla^2\psi$ and/or $\nabla^4\psi$ grow well beyond unity (for instance in the western boundary layer), $L_\mu = L$ and therefore the terms $(\delta_I/L)^2\nabla^2\psi$ and/or $(\delta_M/L)^3\nabla^4\psi$ become $O(1)$. In these regions eq. (38), because of (14) and (15), reduces to (19).

7. – Applications of the composite quasi-geostrophic equation

In the present section, we reconsider the previous examples from the point of view of (38).

– In the first case, under the hypotheses $(\delta_M/L)^3 = 0$, $(\delta_I/L)^2 > 0$ and the usual assumption $N = 1$, the wavelike solution into the layer $0 \le z \le -1$ has the form

$$\psi = \cos(k\pi z)\exp\left[i\left(nx + my + nt\left(\left(\frac{\delta_I}{L}\right)^2(n^2+m^2) + k^2\pi^2\right)^{-1}\right)\right]$$

(integer k). It represents a Rossby wave with westward phase velocity

$$c_x = -\left[\left(\frac{\delta_I}{L}\right)^2(n^2+m^2) + k^2\pi^2\right]^{-1}$$

and zonal group velocity

$$c_{gx} = \frac{(\delta_I/L)^2 n^2 - \left[(\delta_I/L)^2 m^2 + k^2\pi^2\right]}{\left[(\delta_I/L)^2(n^2+m^2) + k^2\pi^2\right]^2}\,. \tag{39}$$

We see that a positive, although small, value of $(\delta_I/L)^2$ allows the eastward propagation of the wave packets, as (20) does but with zonal wave numbers high enough, *i.e.* such that

$$n > \left[m^2 + \left(\frac{L}{\delta_I}\right)^2 k^2\pi^2\right]^{1/2};$$

however, in the limit of a vanishingly small value of $(\delta_I/L)^2$, the group velocity (39) coincides with that of (11).

– In the second example, we consider the so-called Munk's model of the integrated transport, that is the steady version of (38) with friction retained ($(\delta_M/L)^3 > 0$) and wind-stress ($\bar\tau$) forcing but with $(\delta_I/L)^2 = 0$. The governing equation is therefore

$$J\left(\psi, \frac{\partial}{\partial z}\left(\frac{1}{N^2}\frac{\partial\psi}{\partial z}\right)\right) + \frac{\partial\psi}{\partial x} = \left(\frac{\delta_M}{L}\right)^3\nabla^4\psi \tag{40}$$

with the vertical boundary conditions

$$w_1(z=-1) = 0 \qquad \text{and} \qquad w_1(z=0) = \hat{k}\cdot\bar{\nabla}\times\bar\tau. \tag{41}$$

The governing equation of the transport streamfunction is obtained by vertically integrating (40) on the fluid thickness and using boundary conditions (41). Because of (12) and the trivial equation $\int_{-1}^{0}\nabla^4\psi\,\mathrm{d}z = \nabla^4\Phi$ we have

$$\frac{\partial\Phi}{\partial x} = \hat{k}\cdot\bar{\nabla}\times\bar\tau + \left(\frac{\delta_M}{L}\right)^3\nabla^4\Phi. \tag{42}$$

Once all the horizontal boundary conditions (no mass flux and dynamic) are imposed, eq. (42) can be solved by standard methods thus yielding the transport field. Due to the smallness of $(\delta_M/L)^3$, in the interior (42) is nothing but the Sverdrup balance (13). Equation (42) referred to the mesoscale with the use of (29), (31) and (32) becomes

$$\frac{\partial \Phi_\mu}{\partial x_\mu} = \frac{1}{ab}\hat{k} \cdot \bar{\nabla}_\mu \times \bar{\tau} + \frac{E_{\mathrm{h}\mu}}{2\beta_\mu \varepsilon_\mu} \tag{43}$$

and we can easily verify, resorting to (28), that (43) coincides with (22). Note that $\hat{k} \cdot \bar{\nabla}_\mu \times \bar{\tau} = O(L_\mu/L) < 1$, so the dominant balance of (43) is

$$\frac{\partial \Phi_\mu}{\partial x_\mu} \approx \frac{E_{\mathrm{h}\mu}}{2\beta_\mu \varepsilon_\mu}\,. \tag{44}$$

Unlike (42), (44) shows that at the mesoscale the northward transport is admissible and it is controlled by dissipation rather than the wind forcing as in (42).

8. – The quasi-geostrophic equations for the homogeneous ocean

The homogeneous ocean is characterized by a constant (rather than conserved) fluid density which, in turn, implies a depth-independent perturbation pressure and hence a depth-independent geostrophic current. This hypothesis yields a decoupling of the dynamics from the thermodynamics whence relevant consequences follow. As before, our aim is the inference of the "composite" quasi-geostrophic equation and, to this purpose, we begin by considering first the basin scale dynamics.

Without the constraint of density conservation, and with reference to (1) and (2), the time scale comes from the request that the Eulerian derivative of the horizontal current enters into the vorticity equation at the first order in $\beta\varepsilon$, that is to say for $\varepsilon_T = \beta\varepsilon$. Therefore the relation

$$\varepsilon \ll \varepsilon_T = \beta\varepsilon < 1 \tag{45}$$

holds instead of (5), while (7) is still retained. Relation (45) implies the time scale

$$T = (\beta_0 L)^{-1} \tag{46}$$

in accordance with the anticipation done at the end of sect. **2**. Note that time interval (46) is much shorter than advective, by a factor 10^{-3}–10^{-4} in the scale under investigation. By using (46), the vorticity equation takes the form

$$\frac{\partial}{\partial t}\nabla^2\psi + \frac{\partial \psi}{\partial x} = \frac{\partial w_1}{\partial z}\,. \tag{47}$$

Comparison of (47) with (8), which can be written also as $\partial\psi/\partial x = \partial w_1/\partial z$, shows that the time rate of change of relative vorticity becomes *explicit* in (47) because of (46) while, in the case of a stratified ocean, the time evolution is governed by the thermodynamics according to (9) and not by the vorticity balance (8) which is *implicitly* time dependent. Since the streamfunction of (47) is depth independent, the vertical integration of (47) itself on the geostrophic layer (of unitary thickness) is mathematically trivial but it depends on the physical system taken into account.

For a steady circulating wind-driven ocean we have $w_1(z = -1) = 0$ and $w_1(z = 0) = \hat{k} \cdot \bar{\nabla} \times \bar{\tau}$ and the vertical integration of (47) gives the Sverdrup balance for the homogenous ocean in the form

$$\frac{\partial \psi}{\partial x} = \hat{k} \cdot \bar{\nabla} \times \bar{\tau}. \tag{48}$$

One should compare (48) with (13) recalling that, in the homogenous ocean, $\int_{-1}^{0} \psi \, \mathrm{d}z = \psi$.

Wavelike solutions can be obtained from (47) once $w_1(z = 0)$ has been specified at the free surface of the fluid. We know that

$$w_1(z = 0) = F \frac{\partial \psi}{\partial t}, \tag{49}$$

with the Froude number $F = f_0^2 L^2 / gH$. Vertical integration of (47) with (49) yields the equation

$$\frac{\partial}{\partial t}\left(\nabla^2 \psi - F\psi\right) + \frac{\partial \psi}{\partial x} = 0 \tag{50}$$

whose solutions are extensively investigated in the literature. Here we stress only that, due to the relative vorticity appearing in (50), the Rossby waves satisfying this equation admit wave packets propagating both westward and eastward, depending on the zonal wave number.

We wish to point out the following remark. We have seen that the presence of relative vorticity is a consequence of scaling (46) which, in general, is not consistent with a hypothetical conservation principle coupled to the vorticity equation of the homogeneous ocean. Strictly speaking, in the homogeneous ocean where the (dimensional) geostrophic layer extends from z_{0^*} up to $z_{0^*} + H$, the conservation principle

$$\frac{D}{Dt_*} \frac{z_* - z_{0^*}}{H} = 0 \tag{51}$$

holds, whence the vertical velocity at the top of the layer turns out to be

$$w_*(z_* = z_{0^*} + H) = \left(\frac{\partial}{\partial t_*} + \bar{u}_* \cdot \bar{\nabla}_*\right) H. \tag{52}$$

If the Eulerian derivative and the advection on the r.h.s. of (52) were comparable, the advective time scale would be established in contradiction with (46). This contradiction, however, is only apparent. In fact, the nondimensional version of (52) relies on the positions

$$w_*(z_* = z_{0^*} + H) = \frac{UH}{L}\frac{\beta_0 L}{f_0} w_1(z=0), \tag{53}$$
$$\frac{\partial}{\partial t_*} + \bar{u}_* \cdot \bar{\nabla}_* = \beta_0 L \frac{\partial}{\partial t} + \frac{U}{L}\bar{u}_0 \cdot \bar{\nabla},$$
$$H = H_0 + \frac{f_0 UL}{g}\psi,$$

where $\beta_0 L/f_0 = \beta\varepsilon$, $H = H_0 + \eta_*$ with the free-surface elevation η_* in geostrophic equilibrium with $\bar{u}_0$ so that $\eta_* = (f_0 UL/g)\psi$. Using (53) into (52), we obtain the nondimensional vertical velocity of the free surface in the form

$$w_1(z=0) = \frac{f_0^2 L^2}{gH}\frac{\partial \psi}{\partial t} + \frac{f_0^2 U}{gH\beta_0}\bar{u}_0 \cdot \bar{\nabla}\psi, \tag{54}$$

but the very definition of ψ implies that the dot product $\bar{u}_0 \cdot \bar{\nabla}\psi$ is identically vanishing. Hence (54) is nothing but (50). To summarize, the conservation principle (51) is consistent with the time scale (46) since at the geostrophic level of approximation its material derivative is simply an Eulerian derivative thus yielding the kinematic boundary condition (50).

Second, we consider the mesoscale. At the mesoscale, relations (14) and (16) are left unchanged and the vertical integration of the vorticity equation (17) on the geostrophic layer of unitary thickness immediately gives

$$\frac{\partial}{\partial t_\mu}\nabla_\mu^2\psi_\mu + J_\mu\Big(\psi_\mu, \nabla_\mu^2\psi_\mu\Big) + \beta_\mu\frac{\partial \psi_\mu}{\partial x_\mu} = w_{1\mu}(z=0) - \frac{\sqrt{E_\mathrm{v}}}{2\varepsilon_\mu}\nabla_\mu^2\psi_\mu + \frac{E_{\mathrm{h}\mu}}{2\varepsilon_\mu}\nabla_\mu^4\psi_\mu, \tag{55}$$

where the vertical velocity above the bentic Ekman layer $w_{1\mu}(z=-1) = (\sqrt{E_\mathrm{v}}/2\varepsilon_\mu)\nabla_\mu^2\psi_\mu$ has been introduced into (55). All the terms appearing into (55) are $O(1)$.

For a wind-driven ocean, the vertical velocity at the top of the geostrophic layer can be written as

$$w_{1\mu}(z=0) = \frac{UL}{U_\mu L_\mu}\beta_\mu \hat{k} \cdot \bar{\nabla}_\mu \times \bar{\tau}, \tag{56}$$

while free, undamped solutions are obtained if

$$w_{1\mu}(z=0) = F_\mu \frac{\partial \psi_\mu}{\partial t_\mu} \tag{57}$$

with $E_\mathrm{v} = E_{\mathrm{h}\mu} = 0$. In (57), F_μ is the Froude number evaluated at the mesoscale.

9. – The composite quasi-geostrophic equation for the homogeneous ocean

To establish the "composite" quasi-geostrophic equation of the homogeneous ocean starting from (55) and including (48), (50), we adopt the same method as that described in sect. **6** and preliminarily find the transformation rules of $\partial/\partial t_\mu$, $w_{1\mu}$ and $(\sqrt{E_\mathrm{v}}/2\varepsilon_\mu)\nabla_\mu^2\psi_\mu$.

Because of the equality $t_* = (L_\mu/U_\mu)t_\mu = t/\beta_0 L$, we obtain $\partial/\partial t_\mu = (\beta_0 L_\mu L/U_\mu)(\partial/\partial t)$, that is to say the transformation rule

$$\frac{\partial}{\partial t_\mu} = \beta \frac{a}{b}\frac{\partial}{\partial t}\,. \tag{58}$$

Moreover, by equating the leading terms of the dimensional vertical velocity w_* looked at the considered scales, we obtain $w_* = (UH/L)\beta\varepsilon w_1 = (U_\mu H/L_\mu)\varepsilon_\mu w_{1\mu}$ whence the transformation rule

$$w_{1\mu} = U\beta_0\left(\frac{L_\mu}{U_\mu}\right)^2, \quad w_1 = \left(\frac{a}{b}\right)^2 \beta w_1. \tag{59}$$

Finally, it is trivial to check that

$$\frac{\sqrt{E_\text{v}}}{2\varepsilon_\mu}\nabla_\mu^2\psi_\mu = \left(\frac{a}{b}\right)^2 \frac{\sqrt{E_\text{v}}}{2\varepsilon}\nabla^2\psi. \tag{60}$$

At this point we have all the ingredients to establish the desired equation by applying (29), (31), (32), (58), (59) and (60) to (55). The result is

$$\beta\frac{\partial}{\partial t}\nabla^2\psi + J\big(\psi, \nabla^2\psi\big) + \beta\frac{\partial\psi}{\partial x} = \beta w_1(z=0) - \frac{\sqrt{E_\text{v}}}{2\varepsilon}\nabla^2\psi + \frac{E_\text{h}}{2\varepsilon}\nabla^4\psi. \tag{61}$$

After the division of each term of (61) by β and the introduction of the frictional boundary layer width δ_S by means of the position $\sqrt{E_\text{v}}/2\beta\varepsilon = \delta_S/L$, we obtain the final form

$$\frac{\partial}{\partial t}\nabla^2\psi + \left(\frac{\delta_I}{L}\right)^2 J\big(\psi, \nabla^2\psi\big) + \frac{\partial\psi}{\partial x} = w_1(z=0) - \frac{\delta_S}{L}\nabla^2\psi + \left(\frac{\delta_M}{L}\right)^3 \nabla^4\psi \tag{62}$$

in which $w_1(z=0)$ depends on the considered dynamics: in the wind-driven circulation it is given by (41) while in the inertial circulation it is given by (49). In the first case and under the assumption of a steady circulation, the smallness of all the ratios δ/L at the basin scale implies that (62) reduces to (48). In the second case and under the assumption of an undamped circulation, (62) reduces to (50) in the limit of vanishingly small $(\delta_I/L)^2$. On the contrary, if the Jacobian is retained into (62), we obtain the nonlinear equations of the interacting Rossby waves

$$\frac{\partial}{\partial t}\big(\nabla^2\psi - F\psi\big) + \left(\frac{\delta_I}{L}\right)^2 J\big(\psi, \nabla^2\psi\big) + \frac{\partial\psi}{\partial x} = 0. \tag{63}$$

Equation (62) is so widely used in marine dynamics that, in our opinion, every further comment about it would be certainly non-essential, but a final remark is in order.

Sometimes, in the literature the basic role played by (58) in the inference of (62) is not fully acknowledged. For instance, relation (58) is actually used both in Pedlosky [1], Section (3.26) to obtain an equation equivalent to (63) and in Pedlosky [3], Section 2.3. On the contrary, in Pedlosky [1], Section 5.2 the author resorts unexpectedly to the advective time scale in the formulation of the homogeneous model. In this case eq. (30)

holds instead of (58) and, if one applies (30) into (55), say in the case of the wind-driven circulation, then the equation

$$\left(\frac{\delta_I}{L}\right)^2\left(\frac{\partial}{\partial t}\nabla^2\psi + J\left(\psi,\nabla^2\psi\right)\right) + \frac{\partial\psi}{\partial x} = \hat{k}\cdot\bar{\nabla}\times\bar{\tau} - \frac{\delta_S}{L}\nabla^2\psi + \left(\frac{\delta_M}{L}\right)^3\nabla^4\psi \tag{64}$$

follows. Also Hendershott [4] introduces the advective time scale in the homogeneous model (eq. (1.4.10) of his paper) and, consequently, he finds an equation of the kind (64), that is eq. (2.1.1) of his paper. However, investigating the behaviour of an impulsively started midlatitude flow, this author resorts to the *linear and time-dependent* equation (3.2.1) of his paper that, according to our view point, cannot be inferred from an equation like (64). Indeed, eq. (3.2.1) of Hendershott [4] is nothing but the dimensional version of (50).

* * *

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