IL NUOVO CIMENTO Vol. 27 C, N. 1 Gennaio-Febbraio 2004
DOI 10.1393/ncc/i2003-10016-6

Nonlinear stability of the Sverdrup flow against mesoscale disturbances

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(ricevuto il 6 Febbraio 2003; revisionato il 6 Aprile 2004; approvato il 13 Aprile 2004)

Summary. — In the framework of the homogeneous model of ocean circulation, we prove the nonlinear asymptotic stability, in the energy norm, of the Sverdrup flow against mesoscale disturbances. Stability holds in the same parameter condition that the Sverdrup theory itself requires for validity.

PACS `92.10.Fj` – Dynamics of the upper ocean.
PACS `47.20.-k` – Hydrodynamic stability.

1. – Introduction

The Sverdrup flow, that is the steady flow satisfying the Sverdrup balance together with the boundary condition of no-mass flux at the eastern coast of the ocean, is one of the basic constituents of the basin-scale circulation. The validity of the Sverdrup balance has been extensively investigated both from the theoretical and the experimental point of view and here we will assume it without any further discussion. A detailed review is found in [1]. The Sverdrup flow undergoes uninterruptly to the interaction with the background unsteady current field at the mesoscale that plays the role of a disturbance within the present context. More precisely "*It is now well known that the mid-ocean flow is almost everywhere dominated by so-called synoptic or meso-scale eddies, rotating about nearly vertical axes and extending throughout the water column. A typical mid-ocean horizontal scale is 100 km and a time scale ia 100 days: these meso-scale eddies have swirl speed of order 10 cm/s which are usually considerably greater than the long term average flow*" [2]. In spite of these perturbations, as a matter of fact the Sverdrup flow exhibits a persistent character in space and in time and such situation leads us to conjecture its dynamic stability. We stress that, quite in general, the possible conclusion in favour of the dynamic stability of any flow does not rely simply on observations or numerical simulations having, of necessity, a finite duration. On the contrary, we need

(*) E-mail: fulvio.crisciani@ts.ismar.cnr.it a formal proof to exclude the presence of slowly growing instabilities that could elude any evidence extended only to a finite time interval. The Sverdrup flow is, typically, a nonparallel one and, according to J. Pedlosky, "*Compared to our understanding of the instability of parallel flows the instability of curvy, nonparallel flow is small". Moreover "Nonparallel flows... may bacome unstable in ways which are novel in the context of the traditional theory pertaining to parallel flow*s" [3]. These considerations clearly underline the relevance of the stability of the Sverdrup flow which could be hardly inferred in an obvious way from pre-existing principles. From the spectral point of view, in general the viscous dissipation of enstrophy augments the transfer of energy to larger scales and it locks up even more tightly the energy at lower wave numbers. May be that such a mechanism could give another basis to explain the stability of the Sverdrup flow, but we will not follow this line.

Two questions arise in the stability investigation:

– Since the streamlines of the Sverdrup flow are neither parallel nor closed on themselves but rather impinge on the western boundary, they cannot support alone any circulating fluid, so we have to define preliminarly what kind of basic state we consider and where it can be identified with that of Sverdrup.

– In dealing with vorticity dynamics, we must take into account the presence of two different scales of motion, one pertinent to the basic state, with a typical horizontal length scale L and a typical horizontal velocity U and the other relative to the mesoscale disturbances, having a characteristic horizontal scale $L_\mu \ll L$ and a characteristic horizontal velocity $U_\mu \gg U$.

About the first point, we assume that the Sverdrup flow is governed by the basin-scale dynamics in the linear regimen and that, physically, it forms only at a suitable distance from the western boundary layer. Therefore, we identify the Sverdrup flow with those obtained in the framework of the linear models (where the streamlines are closed) but restrict the stability analysis inside a region where the streamlines of the linear models do coincide among themselves and with those of the Sverdrup transport. About the second point, we introduce a set of transformation equations among coordinates, fields and parameters which allow us to pass from one scale of motion to the other and, in particular, to describe how the Sverdrup flow is "seen" by the mesoscale.

The conclusion is that the same parameter condition that the Sverdrup theory itself requires for validity yields also the nonlinear asymptotic stability, in the energy norm, of the Sverdrup flow against mesoscale disturbances.

We stress that, unlike the case of the Sverdrup flow, instability mechanisms in the ocean interior actually do exist and they play a fundamental role in baroclinic systems. For instance, in the framework of the model of baroclinic wind-driven circulation of Young and Rhines (reported for instance in [3]), numerical experiments show that "*The equivalence of the condition for closed geostrophic contours and the criterion for instability satisfyingly identifies the mechanism for the production of motion with the necessary condition for the existence of that motion. When the motion is allowed, the advent of baroclinic instability is capable of producing it*" [3].

2. – Basin-scale dynamics

We anticipate that nondimensional quantities are hereafter understood, unless different specifications are claimed.

We denote with D the whole fluid domain and consider the following vorticity equation:

$$J\left(\psi, \nabla^2\psi\right) + \beta\frac{\partial\psi}{\partial x} = \beta w + F\left(\psi\right)\,, \tag{2.1}$$

where $\beta = \beta_0 L^2/U = O\left(10^3\right)$ is the nondimensional planetary vorticity gradient, β_0 ($= O\left(10^{-11}\right)$ in S.I. units) is the dimensional one, $w = \hat{k} \cdot \vec{\nabla} \times \vec{\tau} = O\left(1\right)$ is the Ekman pumping velocity at the top of the geostrophic layer, $\overset{\omega}{\tau}$ is the wind stress and $F\left(\psi\right)$ is the standard parametrization of the turbulence given by

$$F\left(\psi\right) = -\frac{\sqrt{E_{\mathrm{v}}}}{2\varepsilon}\nabla^2\psi + \frac{E_{\mathrm{h}}}{2\varepsilon}\nabla^4\psi \tag{2.2}$$

where E_{v} and E_{h} are the vertical and the horizontal Ekman numbers, respectively, while ε is the Rossby number. It is well known that (2.1) implies the dissipation integral $\beta \oint\limits_{\partial D} \vec{\tau} \cdot \hat{t}\mathrm{d}s + \int\limits_D F\left(\psi\right)\mathrm{d}x\mathrm{d}y = 0$ whence

$$\frac{1}{\beta}\int\limits_D F\left(\psi\right)\mathrm{d}x\mathrm{d}y = O\left(1\right). \tag{2.3}$$

Consistently with (2.3) we assume also

$$\frac{1}{\beta}F\left(\psi\right) = O\left(1\right). \tag{2.4}$$

Due to the smallness of $\beta^{-1} = O\left(10^{-3}\right)$, we expand the streamfunction ψ in powers of β^{-1}, *i.e.*

$$\psi = \psi_0 + \frac{1}{\beta}\psi_1 + O\left(\beta^{-2}\right). \tag{2.5}$$

Then, substitution of (2.5) into (2.1) together with assumption (2.4) leads to the leading order the linear vorticity equation

$$\frac{\partial\psi_0}{\partial x} = w + \frac{1}{\beta}F\left(\psi_0\right). \tag{2.6}$$

All the linear solutions of the homogeneous model come from (2.6) with proper boundary conditions along ∂D but, outside the western boundary layer, the term $(1/\beta)F\left(\psi_0\right)$ plays no role while the remaining balance, *i.e.* $\partial\psi_0/\partial x = w$, yields just the Sverdrup flow. In other words, the streamlines of all the linear models (which are closed because of the term $(1/\beta)F\left(\psi_0\right)$) coincide with those of the Sverdrup flow in a certain region outside the western boundary layer. Therefore we will investigate the stability of the Sverdrup flow in a restricted domain $D_0 \subset D$ where the western boundary layer is excluded. Last but not least, we see that the Sverdrup streamfunction in D_0 is nothing but the zeroth-order term in $1/\beta$ of expansion (2.5).

3. – Mesoscale dynamics

We denote with ϕ the mesoscale disturbance and with ψ_μ the streamfunction ψ of (2.5) when it is "seen" by the mesoscale. Whenever it is necessary, we use the subscript μ to distinguish the quantities referred to the mesoscale from those referred to the basin-scale. Then, the evolution equation for $\phi + \psi_\mu$ at the mesoscale is

$$\frac{\partial}{\partial t}\nabla_\mu^2\phi + J_\mu\left(\phi+\psi_\mu, \nabla_\mu^2\phi+\nabla_\mu^2\psi_\mu\right) + \beta_\mu\frac{\partial\phi}{\partial x_\mu} + \beta_\mu\frac{\partial\psi_\mu}{\partial x_\mu} = w_\mu + F_\mu\left(\phi+\psi_\mu\right)\,, \tag{3.1}$$

where $\beta_\mu = O\left(1\right)$ and, according to (2.2),

$$F_\mu\left(\phi+\psi_\mu\right) = -\frac{\sqrt{E_\mathrm{v}}}{2\varepsilon_\mu}\nabla_\mu^2\left(\phi+\psi_\mu\right) + \frac{E_{\mathrm{h}\mu}}{2\varepsilon_\mu}\nabla_\mu^4\left(\phi+\psi_\mu\right). \tag{3.2}$$

In order to localize ϕ inside D_0, we demand the no-mass flux boundary condition

$$\phi = 0 \qquad \forall\left(x,y\right)\in\partial D_0 \tag{3.3}$$

and, if, in (3.2), $E_{\mathrm{h}\mu} > 0$, also dynamic boundary conditions consistent with the circuit integral

$$\oint_{\partial D_0} \nabla_\mu^2\phi\ \vec{\nabla}_\mu\ \phi\cdot\hat{n}\mathrm{d}s = 0. \tag{3.4}$$

Setting, in short, $a = UL/U_\mu L_\mu$ and $b = L_\mu/L$, we point out the following transformation rules which hold between couples of nondimensional quantities which are related to the same dimensional counterpart (see Appendix A):

$$\begin{aligned}&\frac{\partial}{\partial x_\mu} = b\frac{\partial}{\partial x}, \frac{\partial}{\partial y_\mu} = b\frac{\partial}{\partial y}, \psi_\mu = a\psi, \beta_\mu = ab^3\beta,\\ &w_\mu = a^2b^4\beta w, F_\mu\left(\psi_\mu\right) = a^2b^4F\left(\psi\right).\end{aligned} \tag{3.5}$$

Substitution of (3.5) into (3.1) yields

$$\begin{aligned}&\frac{\partial}{\partial t}\nabla_\mu^2\phi + J_\mu\left(\phi, \nabla_\mu^2\phi+\nabla_\mu^2\psi_\mu+\beta_\mu y_\mu\right) + J_\mu\left(\psi_\mu, \nabla_\mu^2\phi\right) +\\ &\quad + a^2b^4J\left(\psi,\nabla^2\psi\right) + a^2b^4\beta\frac{\partial\psi}{\partial x} = a^2b^4\beta w + a^2b^4F\left(\psi\right) + F_\mu\left(\phi\right)\,.\end{aligned} \tag{3.6}$$

Because of (2.1), (3.6) simplifies into

$$\frac{\partial}{\partial t}\nabla_\mu^2\phi + J_\mu\left(\phi, \nabla_\mu^2\phi+\nabla_\mu^2\psi_\mu+\beta_\mu y_\mu\right) + J_\mu\left(\psi_\mu, \nabla_\mu^2\phi\right) = F_\mu\left(\phi\right). \tag{3.7}$$

Note that (3.7) does not rely on special values of a and b. The term $J_\mu\left(\psi_\mu, \nabla_\mu^2\phi\right)$ takes into account the interaction between the basic state ψ_μ and the generic disturbance ϕ and it is the key ingredient of the subsequent stability analysis.

4. – Stability analysis

Stability in a given norm, say N, is assured if, for every perturbation ϕ, we prove that

$$\frac{\mathrm{d}}{\mathrm{d}t} N(\phi) \leqslant 0. \tag{4.1}$$

If, besides (4.1), relation

$$\lim_{t\to\infty} N(\phi) = 0 \tag{4.2}$$

is valid, then the basic state is asymptotically stable in the same norm.

With (4.1) in mind, we multiply each term of (3.7) by ϕ and integrate the products on D_0 with the aid of (3.3). The result is

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{D_0} \left|\vec{\nabla}_\mu\, \phi\right|^2 \mathrm{d}x_\mu \mathrm{d}y_\mu + \int_{D_0} \psi_\mu J_\mu\left(\phi, \nabla_\mu^2 \phi\right) \mathrm{d}x_\mu \mathrm{d}y_\mu = -\int_{D_0} \phi F_\mu(\phi)\, \mathrm{d}x_\mu \mathrm{d}y_\mu, \tag{4.3}$$

and, in view of (4.3), it is quite obvious to introduce the energy norm defined by

$$N(\phi) \equiv \left\|\vec{\nabla}_\mu\, \phi\right\| = \left\{\int_{D_0} \left|\vec{\nabla}_\mu\, \phi\right|^2\right\}^{1/2}. \tag{4.4}$$

The inequality (see Appendix B)

$$-\int_{D_0} \phi F_\mu(\phi)\, \mathrm{d}x_\mu \mathrm{d}y_\mu \leqslant -C\left\|\vec{\nabla}_\mu\, \phi\right\|^2, \tag{4.5}$$

where C is a positive constant, allows us to bound from above the time derivative of the norm (4.4) square appearing in (4.3) as follows:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\vec{\nabla}_\mu\, \phi\right\|^2 \leqslant -C\left\|\vec{\nabla}_\mu\, \phi\right\|^2 - \int_{D_0} \psi_\mu J_\mu\left(\phi, \nabla_\mu^2 \phi\right) \mathrm{d}x_\mu \mathrm{d}y_\mu. \tag{4.6}$$

With reference to the last term of (4.6), Schwarz inequality yields

$$-\int_{D_0} \psi_\mu J_\mu\left(\phi, \nabla_\mu^2 \phi\right) \mathrm{d}x_\mu \mathrm{d}y_\mu \leqslant \left\{\int_{D_0} \psi_\mu^2 \mathrm{d}x_\mu \mathrm{d}y_\mu\right\}^{1/2} \left\{\int_{D_0} J_\mu^2\left(\phi, \nabla_\mu^2 \phi\right) \mathrm{d}x_\mu \mathrm{d}y_\mu\right\}^{1/2}. \tag{4.7}$$

Recalling that $\psi_\mu = a\psi$, in order to explore the stability problem of the dynamical regimen in which the Sverdrup balance is valid, that is to say for $\beta \to \infty$, we write the integral $\left\{\int\limits_{D_0} \psi_\mu^2 \mathrm{d}x_\mu \mathrm{d}y_\mu\right\}^{1/2}$ appearing in (4.7) as

$$\left\{\int\limits_{D_0} \psi_\mu^2 \mathrm{d}x_\mu \mathrm{d}y_\mu\right\}^{1/2} = a\left\{\int\limits_{D_0} \psi^2 \mathrm{d}x\mathrm{d}y\right\}^{1/2}. \tag{4.8}$$

By means of the inequality (see Appendix A)

$$a < \frac{1}{\beta}\frac{1}{\varepsilon_\mu^3\beta_\mu^2}\,, \tag{4.9}$$

we easily bound from above (4.8), and hence also the r.h.s. of (4.6), to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\vec{\nabla}_\mu\,\phi\right\|^2 < -C\left\|\vec{\nabla}_\mu\,\phi\right\|^2 + \frac{1}{\beta}\left\{\int\limits_{D_0}\left[\psi_0 + \frac{1}{\beta}\psi_1 + O\left(\beta^{-2}\right)\right]\mathrm{d}x\mathrm{d}y\right\}^{1/2} \times \\ \times \frac{1}{\varepsilon_\mu^3\beta_\mu^2}\left\{\int\limits_{D_0} J_\mu^2\left(\phi, \nabla_\mu^2\phi\right)\mathrm{d}x_\mu\mathrm{d}y_\mu\right\}^{1/2}, \tag{4.10}$$

where use has been made of (2.5). The quantity

$$\frac{1}{\beta}\left\{\int\limits_{D_0}\left[\psi_0 + \frac{1}{\beta}\psi_1 + O\left(\beta^{-2}\right)\right]\mathrm{d}x\mathrm{d}y\right\}^{1/2} \tag{4.11}$$

includes only basin-scale constituents and goes to zero for $\beta \to \infty$. Therefore, in the same dynamical regimen in which the Sverdrup balance is valid, (4.10) simplifies into

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\vec{\nabla}_\mu\,\phi\right\|^2 < -C\left\|\vec{\nabla}_\mu\,\phi\right\|^2. \tag{4.12}$$

Inequality (4.12) satisfies (4.1) and thus the stability of the Sverdrup flow ψ_0 in the energy norm (4.4) is proved. Moreover, time integration of (4.12) gives

$$\left\|\vec{\nabla}_\mu\,\phi\,(t)\right\| < \exp\left[-Ct\right]\left\|\vec{\nabla}_\mu\,\phi\,(0)\right\| \tag{4.13}$$

and (4.13) satisfies condition (4.2) for the asymptotic stability of ψ_0 in the same norm.

5. – Concluding remarks

– The inference of the stability of the Sverdrup flow expounded in last section relies basically on the parallel limits

$$\lim_{\beta\to\infty}\psi = \psi_0 \qquad \text{and} \qquad \lim_{\beta\to\infty}\psi_\mu = 0\,.$$

We have seen that the first limit selects the Sverdrup streamfunction as the basic state while the second one implies the stability of the selected basic state. The different values of the limits above are explained by the different scales of the motion ascribed to the basic state and to the perturbations. Besides, the idealization inherent in the expression $\beta \to \infty$ is widely justified from the quantitative point of view: for instance, if (in S.I. units) $L = O\left(10^6\right)$, $U = O\left(5 \cdot 10^{-3}\right)$ and $L_\mu = O\left(10^5\right)$, $U_\mu = O\left(10^{-1}\right)$, then $\beta/\beta_\mu = O\left(2 \cdot 10^3\right)$.

– We wish to stress that the dissipation of vorticity at the mesoscale plays a minor role in the stability proof. In fact (4.12) shows that, in accordance with (4.1), $\frac{\mathrm{d}}{\mathrm{d}t}\left\|\vec{\nabla}_\mu\, \phi\right\| < 0$ no matter how small the constant C is, that is to say (see Appendix B) how small the quantities $\frac{\sqrt{E_\mathrm{v}}}{2\varepsilon_\mu}$ and $\frac{E_{\mathrm{h}\mu}}{2\varepsilon_\mu}$ are.

– One could wonder why the stability in the enstrophy norm $\left\|\nabla^2_\mu \phi\right\|$ is not considered here, even if multiplication of (3.7) by $\nabla^2_\mu \phi$ and the subsequent integration on D_0 trivially yields, as first term, $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla^2_\mu \phi\right\|^2$. Actually, the problematic aspect lies in the integral $I \equiv \int\limits_{D_0} \nabla^2_\mu \phi \frac{\partial \phi}{\partial x_\mu} \mathrm{d}x_\mu \mathrm{d}y_\mu$ coming from the beta-term. In fact, we have the following alternative:

1) To apply Schwarz inequality and inequality (B.3) of Appendix B, whence

$$I \leqslant \left\|\vec{\nabla}_\mu\, \phi\right\| \left\|\nabla^2_\mu \phi\right\| \leqslant \frac{1}{K}\left\|\nabla^2_\mu \phi\right\|^2. \tag{5.1}$$

2) To apply the divergence and the Green theorems, whence

$$I = \frac{1}{2}\oint\limits_{\partial D_0} \left|\vec{\nabla}_\mu\, \phi\right|^2 \mathrm{d}y_\mu. \tag{5.2}$$

In case (5.1), $\frac{1}{K}\left\|\nabla^2_\mu \phi\right\|^2$ is opposed to the dissipative terms with the possibility, for small K, to have a dissipation with the "wrong" sign.

In case (5.2), if no-slip boundary conditions are taken along the meridional boundaries, then $I = 0$. On the other hand, horizontal diffusion of relative vorticity arises the term $\oint\limits_{\partial D_0} \nabla^2_\mu \phi\, \vec{\nabla}_\mu \left(\nabla^2_\mu \phi\right) \cdot \hat{n} \mathrm{d}s - \left\|\vec{\nabla}_\mu \left(\nabla^2_\mu \phi\right)\right\|^2$ which can be bounded by $-K^2\left\|\nabla^2_\mu \phi\right\|^2$ by using the Wirtinger-Poincaré inequality [4] only if free-slip conditions are taken along the boundary. This last condition is clearly incompatible with that of no-slip.

Appendix A.

Starred quantities are dimensional. Recall also that $L = L_\mu/b$ and $U = abU_\mu$.
Consider the horizontal coordinates

$$\left.\begin{array}{l} (x_*, y_*) = L\,(x, y) \\ (x_*, y_*) = L_\mu\,(x_\mu, y_\mu) \end{array}\right\} \Rightarrow (x_\mu, y_\mu) = \frac{L}{L_\mu}\,(x, y)\,.$$

This last implies

$$\left(\frac{\partial}{\partial x_\mu}, \frac{\partial}{\partial y_\mu}\right) = b\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right), \tag{A.1}$$

whence $J_\mu = b^2 J$, $\nabla_\mu^n = b^n \nabla^n, n = 2, 4$.

Analogously for the perturbation pressure

$$\left.\begin{array}{l}\psi_* = UL\psi \\ \psi_* = U_\mu L_\mu \psi_\mu\end{array}\right\} \Rightarrow \psi_\mu = \frac{UL}{U_\mu L_\mu}\psi,$$

that is to say

$$\psi_\mu = a\psi. \tag{A.2}$$

Rossby number:

$$\varepsilon = \frac{U}{f_0 L} = \frac{ab^2 U_\mu}{f_0 L_\mu} = ab^2 \varepsilon_\mu. \tag{A.3}$$

Horizontal Ekman number:

$$E_{\mathrm{h}} = \frac{2A_{\mathrm{h}*}}{f_0 L^2} = \frac{2A_{\mathrm{h}*} b^2}{f_0 L_\mu^2} = b^2 E_{\mathrm{h}\mu}. \tag{A.4}$$

Nondimensional planetary vorticity gradient:

$$\beta = \frac{\beta_0 L^2}{U} = \frac{\beta_0 L_\mu^2}{ab^3 U_\mu} = \frac{1}{ab^3}\beta_\mu. \tag{A.5}$$

Vertical velocity in the geostrophic layer:

$$\left.\begin{array}{l}w_* = \frac{UH}{L}\beta\varepsilon w \\ w_* = \frac{U_\mu H}{L_\mu}\varepsilon_\mu w_\mu\end{array}\right\} \Rightarrow w_\mu = \beta\frac{UL_\mu\varepsilon}{U_\mu L\varepsilon_\mu} w,$$

that is to say, using (A.3),

$$w_\mu = a^2 b^4 \beta w. \tag{A.6}$$

Dissipative term: using (A.3) and (A.4) we obtain

$$F_\mu = -\frac{\sqrt{E_{\mathrm{v}}}}{2\varepsilon_\mu}\nabla_\mu^2 + \frac{E_{\mathrm{h}\mu}}{2\varepsilon_\mu}\nabla_\mu^4 = ab^4\left(-\frac{\sqrt{E_{\mathrm{v}}}}{2\varepsilon}\nabla^2 + \frac{E_{\mathrm{h}}}{2\varepsilon}\nabla^4\right) = ab^4 F$$

and hence

$$F_\mu(\psi_\mu) = ab^4 F(a\psi) = a^2 b^4 F(\psi). \tag{A.7}$$

Equations (A.1), (A.2), (A.5), (A.6) and (A.7) are listed in (3.5). Moreover, from (A.3) and (A.5) we find

$$a = \frac{1}{\beta}\frac{(\varepsilon\beta)^3}{\varepsilon_\mu^3 \beta_\mu^2} \tag{A.8}$$

and

$$b = \frac{\varepsilon_\mu \beta_\mu}{\varepsilon \beta}. \tag{A.9}$$

Now we stress that the very validity of the beta-plane approximation demands $\varepsilon\beta < O(1)$ [3], so from (A.8) we infer the inequality $a < \frac{1}{\beta}\frac{1}{\varepsilon_\mu^3 \beta_\mu^2}$ which is (4.9). Note that, for the same reason, (A.9) yields $b > \varepsilon_\mu \beta_\mu$ and therefore the estimate $ab^3\beta = O(1)$ that comes from (A.5) implies that $\beta \to \infty \Leftrightarrow a \to 0$.

Appendix B.

To infer (4.5) we start from the direct evaluation of the integral on the l.h.s. of inequality above by using the definition of F_μ. We obtain

$$-\int_{D_0} \phi F_\mu(\phi)\,\mathrm{d}x_\mu \mathrm{d}y_\mu = -\frac{\sqrt{E_\mathrm{v}}}{2\varepsilon_\mu}\left\|\vec{\nabla}_\mu \phi\right\|^2 + \frac{E_{\mathrm{h}\mu}}{2\varepsilon_\mu}\oint_{\partial D_0} \nabla_\mu^2 \phi\, \vec{\nabla}_\mu \phi \cdot \hat{n}\mathrm{d}s - \frac{E_{\mathrm{h}\mu}}{2\varepsilon_\mu}\left\|\nabla_\mu^2 \phi\right\|^2. \tag{B.1}$$

Condition (3.4) states that the circuit integral of (B.1) is zero, so

$$-\int_{D_0} \phi F_\mu(\phi)\,\mathrm{d}x_\mu \mathrm{d}y_\mu = -\frac{\sqrt{E_\mathrm{v}}}{2\varepsilon_\mu}\left\|\vec{\nabla}_\mu \phi\right\|^2 - \frac{E_{\mathrm{h}\mu}}{2\varepsilon_\mu}\left\|\nabla_\mu^2 \phi\right\|^2. \tag{B.2}$$

Because of (3.3), the inequality

$$K^2\left\|\vec{\nabla}_\mu \phi\right\|^2 \leqslant \left\|\nabla_\mu^2 \phi\right\|^2 \tag{B.3}$$

can be established [4], where K is a constant depending on the shape and the size of D_0. In turn, from (B.3) we have trivially

$$-\left\|\nabla_\mu^2 \phi\right\|^2 \leqslant -K^2\left\|\vec{\nabla}_\mu \phi\right\|^2 \tag{B.4}$$

and thus the r.h.s. of (B.2) can be bounded as

$$-\int_{D_0} \phi F_\mu(\phi)\,\mathrm{d}x_\mu \mathrm{d}y_\mu \leqslant -\left(\frac{\sqrt{E_\mathrm{v}}}{2\varepsilon_\mu} + K^2\frac{E_{\mathrm{h}\mu}}{2\varepsilon_\mu}\right)\left\|\vec{\nabla}_\mu \phi\right\|^2. \tag{B.5}$$

Inequality (B.5) coincides with (4.5) if we set $C = \frac{\sqrt{E_\mathrm{v}}}{2\varepsilon_\mu} + K^2\frac{E_{\mathrm{h}\mu}}{2\varepsilon_\mu}$.

REFERENCES

[1] Pedlosky J., *Ocean Circulation Theory* (Springer-Verlag) 1996 (the Sverdrup theory is revisited in the first chapter).
[2] Allan R. Robinson (Editor), *Eddies in Marine Science* (Springer-Verlag) 1983.
[3] Pedlosky J., *Geophysical Fluid Dynamics* (Springer-Verlag) 1987 (Sections 6.2, 6.3 and 6.19).
[4] Straughan B., *The Energy Method, Stability and Nonlinear Convection* (Springer-Verlag) 1992 (inequality (B.3) and that of Wirtinger-Poincaré are found in Appendix 1 of the book).