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On the solution of Long's equation over terrain

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Summary. — Several authors that investigated the numerical solutions of Long's equation over terrain found that the solution depends weakly on the nonlinear terms in this equation. The objective of this paper is to provide analytical proof of this statement in the context of gravity waves over topography. Furthermore we show that under mild restrictions the equation can be transformed to a Lienard-type equation and identify the "slow variable" that controls the nonlinear oscillations in this equation. Using the phase averaging method we derive also an approximate formula for the attenuation of the stream function perturbation with height. This result is generically related to the nonlinear terms in Long's equation.

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1. – Introduction

Long's equation [1-4] models the flow of stratified incompressible fluid (in the Boussinesq approximation) in two dimensions over terrain. The solutions of this equation in various settings were studied by [5-7] and several other authors [8-13]. (An extensive list of references appears in [14, 15].)

Previous studies of the solutions of Long's equation in two dimensions [6, 7] used this equation to calculate steady lee waves in unbounded domain over terrain. The numerical solution of this equation was investigated extensively in the limit where the parameters β and μ which appear in this equation are identically zero. In this (singular) limit the nonlinear terms and one of the leading second-order derivatives in the equation drop out and the equation reduces to that of a linear harmonic oscillator over two-dimensional domain. However they observed that these approximations do not hold when wave breaking is present in the solution. Peltier and Clark [4] simulated mountain lee waves using anelastic approximation and found that the physical assumptions used to derive Long's equation are no longer justified when "breaking waves" are present.

The objective of this paper is to study closely the impact that the approximations $\beta = 0$, $\mu = 0$ might have on the solution of Long's equation. Toward this end we first employ a set of (analytic) transformations which replace the nonlinearities in the first-order derivatives by a nonlinear term in the stream function. This leads to different limiting approximations of Long's equation and yields constraints on the numerical observations made by previous authors about the impact of the nonlinear terms. Furthermore we identify and derive analytical expressions for the attenuation to the solution of this equation in one and two dimensions due to the presence of the nonlinear terms. This attenuation was not taken into account by other authors [16, 17] who tried to detect experimentally gravity waves in the stratosphere and this led to various difficulties in these studies.

The plan of the paper is as follows: In sect. **2** we present a short review of the derivation of Long's equation and the solution of its linearized version. In sect. **3** we present some transformations that simplify this equation. In sect. **4**. we investigate the effect of the nonlinear terms on the solution in one and two dimensions. In the one dimension we let $\mu = 0$, while in two dimensions the full equation is considered. In both cases we use the method of phase averaging [18, 19] to derive the approximate form of the perturbation from the base state and the attenuation of this perturbation (and hence the amplitude of the gravity wave) due to the nonlinear terms in the equation. We end up in sect. **5**. with some considerations about the numerical solution of the transformed equation and some conclusions.

2. – Long's equation—A short review

In two dimensions (x, z) the flow of a steady inviscid and incompressible stratified fluid (in the Boussinesq approximation) is modeled by the following equations:

$$u_x + w_z = 0\,, \tag{2.1}$$

$$u\rho_x + w\rho_z = 0\,, \tag{2.2}$$

$$\rho(uu_x + wu_z) = -p_x\,, \tag{2.3}$$

$$\rho(uw_x + ww_z) = -p_z - \rho g\,, \tag{2.4}$$

where subscripts indicate differentiation with respect to the indicated variable, $\mathbf{u} = (u, w)$ is the fluid velocity, ρ is its density p is the pressure and g is the acceleration of gravity.

We can non-dimensionalize these equations by introducing

$$\bar{x} = \frac{x}{L}, \quad \bar{z} = \frac{N_0}{U_0}z, \quad \bar{u} = \frac{u}{U_0}, \quad \bar{w} = \frac{LN_0}{U_0^2}w, \tag{2.5}$$
$$\bar{\rho} = \frac{\rho}{\rho_0}, \quad \bar{p} = \frac{N_0}{gU_0\rho_0}p,$$

where L represents a characteristic length, and U_0, ρ_0 represent respectively the free stream velocity and density. N_0 is the characteristic Brunt-Vaisala frequency

$$N_0^2 = -\frac{g}{\rho_0}\frac{\mathrm{d}\rho_0}{\mathrm{d}z}. \tag{2.6}$$

In these new variables eqs. (2.1)-(2.4) take the following form (for brevity we drop the bars):

$$u_x + w_z = 0\,, \tag{2.7}$$

$$u\rho_x + w\rho_z = 0\,, \tag{2.8}$$

$$\beta\rho(uu_x + wu_z) = -p_z\,, \tag{2.9}$$

$$\beta\rho(uw_x + ww_z) = -\mu^{-2}(p_z + \rho)\,, \tag{2.10}$$

where

$$\beta = \frac{N_0 U_0}{g}\,, \tag{2.11}$$

$$\mu = \frac{U_0}{N_0 L}\,. \tag{2.12}$$

β is the Boussinesq parameter [20] which controls stratification effects (assuming $U_0 \neq 0$) and μ is the long-wave parameter which controls dispersive effects (or the deviation from the hydrostatic approximation). In the limit $\mu = 0$ the hydrostatic approximation is fully satisfied [13].

In view of eq. (2.7) we can introduce a stream function ψ so that

$$u = \psi_z, \quad w = -\psi_x\,. \tag{2.13}$$

After a long (and intricate) algebra one can show that $\rho = \rho(\psi)$ and derive the following equation for ψ [20]

$$\psi_{zz} + \mu^2\psi_{xx} - N^2(\psi)\left[z + \frac{\beta}{2}(\psi_z^2 + \mu^2\psi_x^2)\right] = G(\psi)\,, \tag{2.14}$$

where

$$N^2(\psi) = -\frac{\rho_\psi}{\beta\rho} \tag{2.15}$$

is the nondimensional Brunt-Vaisala frequency and $G(\psi)$ is some unknown function which can be determined by making proper assumptions on the upstream disturbance (see [14]). Equation (2.15) is referred to as Long's equation. If we let

$$\psi(\infty, z) = z\,, \tag{2.16}$$

then

$$G(\psi) = -N^2(\psi)\left(\psi + \frac{\beta}{2}\right) \tag{2.17}$$

and eq. (2.14) becomes

$$\psi_{zz} + \mu^2\psi_{xx} - N^2(\psi)[z - \psi + \frac{\beta}{2}(\psi_z^2 + \mu^2\psi_x^2 - 1)] = 0\,. \tag{2.18}$$

In the following we restrict our attention to this form of Long's equation.

For a flow in an unbounded domain over topography with shape $f(x)$ and maximum height H the following boundary conditions are imposed on ψ:

$$\psi(-\infty, z) = z\,, \tag{2.19}$$

$$\psi(x, \epsilon f(x)) = \text{const}, \quad \epsilon = \frac{HN_0}{U_0}\,, \tag{2.20}$$

where the constant in eq.(2.20) is (usually) set to zero. As to the boundary condition on $\psi(\infty, z)$ we observe that Long's equation contains no dissipation terms and therefore only radiation boundary conditions can be imposed in this limit. Similarly at $z = \infty$ it is customary to impose (following [7]) radiation boundary conditions or to let

$$u(x, \infty) = \psi_z(x, \infty) = 1. \tag{2.21}$$

For the perturbation from the base flow

$$\eta = \psi - z\,, \tag{2.22}$$

eq. (2.18) becomes

$$\eta_{zz} - \alpha^2\eta_z^2 + \mu^2(\eta_{xx} - \alpha^2\eta_x^2) - N^2(\eta)(\beta\eta_z - \eta) = 0\,, \tag{2.23}$$

where

$$\alpha^2 = \frac{N^2(\psi)\beta}{2}\,. \tag{2.24}$$

In the limit $\beta = 0$, $\mu = 0$ and $N(\psi)$ is a constant N over the domain, eq. (2.23) reduces to a linear equation

$$\eta_{zz} + N^2\eta = 0\,. \tag{2.25}$$

We observe that the limit $\beta = 0$ can be obtained either by letting $U_0 \to 0$ or $N_0 \to 0$. In the following we assume that this limit is obtained as $U_0 \to 0$ (so that stratification persists in this limit). The general solution of eq. (2.25) is

$$\eta(x, z) = p(x)\cos(Nz) + q(x)\sin(Nz)\,, \tag{2.26}$$

where the functions $p(x), q(x)$ have to be chosen so that the the boundary conditions derived from eqs. (2.19), (2.20) and the radiation boundary conditions (or eq. (2.21)) are satisfied. These lead in general to an integral equation for $p(x)$ and $q(x)$.

$$q(x)\cos(\epsilon N f(x)) + H[q(x)]\sin(\epsilon N f(x)) = -\epsilon f(x)\,, \tag{2.27}$$

where $H[q(x)]$ is the Hilbert transform of $q(x)$. This equation has to be solved numerically [6, 7].

It is clear from the form of the general solution given by eq. (2.26) that it represents a wave propagating in the z-direction and the properties of this wave (under varied physical conditions) were investigated by the authors which were mentioned in sect. **1**. It should be observed however that eq. (2.25) is a "singular limit" of Long's equation as one of the leading second-order derivatives drops when $\mu = 0$ and the nonlinear terms drop when $\beta = 0$. Under these circumstances it is uncertain that the solutions of "limit equation" relates continuously to the solutions of the original equation. It is imperative therefore to investigate other forms of eq. (2.18) (or equivalently eq. (2.23)) and explore the impact of these approximations on the solution.

3. – Transformations on Long's equation

Long's equation in the form (2.18) contains second-order derivatives in x, z and quadratic terms in the first-order derivatives. In this form it is a challenge to solve the full equation even numerically. In this section we apply on this equation a sequence of transformations which mitigate some of these difficulties. A similar treatment applies to eq. (2.23).

To begin with, we define $\bar{z} = z - \beta/2$, $\bar{x} = x/\mu$ and substitute in eq. (2.18) we obtain after dropping the bars that

$$(\psi_{zz} - \alpha^2\psi_z^2) + (\psi_{xx} - \alpha^2\psi_x^2) - N^2(\psi)(z - \psi) = 0. \tag{3.1}$$

We now let $g(\psi)$ be an invertible smooth solution of

$$g''(\psi) + \alpha^2(\psi)g'(\psi) = 0. \tag{3.2}$$

Introduce the transformation $\phi = g(\psi)$ and observe that

$$\psi_{zz} + \frac{g''(\psi)}{g'(\psi)}(\psi_z)^2 = \frac{\phi_{zz}}{g'(\psi)}. \tag{3.3}$$

Since $\psi = g^{-1}(\phi) = h(\phi)$ we can use eqs. (3.2), (3.3) to rewrite eq. (3.1) as

$$\nabla^2\phi - g'(h(\phi))N^2(h(\phi))(z - h(\phi)) = 0. \tag{3.4}$$

Using eq. (3.2) and the definition of α^2 (eq. (2.24)) this simplifies further to

$$\nabla^2\phi + \frac{2g''(h(\phi))}{\beta}(z - h(\phi)) = 0. \tag{3.5}$$

We give few examples to demonestrate how this transformation applies in some practical applications.

I. N = const. This assumption was made in most practical applications of Long's equation. In this case a solution of eq. (3.2) is

$$\phi = g(\psi) = e^{-\alpha^2\psi}, \quad \alpha \neq 0, \tag{3.6}$$

$$\psi = h(\phi) = -\frac{1}{\alpha^2}\ln(\phi) \tag{3.7}$$

and Long's equation takes the form

$$\nabla^2\phi + N^2\phi(\alpha^2 z + \ln\phi) = 0\,. \tag{3.8}$$

This can be written more succinctly as

$$\nabla^2\phi + N^2\phi\ln(\phi e^{\alpha^2 z}) = 0\,. \tag{3.9}$$

II. $N = 1/\psi$. From eq. (3.2) we obtain that $g(\psi)$ can be chosen as

$$\phi = g(\psi) = c\ln\psi, \quad \psi = e^{\phi/c} \tag{3.10}$$

and Long's equation takes the form

$$\nabla^2\phi - \frac{2}{\beta}e^{-2\phi/c}(z - e^{\phi/c}) = 0\,. \tag{3.11}$$

Similar transformations can be applied to the perturbation equation (2.23). In fact, if we let $\xi = g(\eta)$ satisfy an equation similar to eq. (3.2), then eq. (2.23) (after the transformation $\bar{x} = x/\mu$) takes the form

$$\nabla^2\xi + 2g''(h(\xi))\left[\frac{\xi_z}{g'} - \frac{h(\xi)}{\beta}\right] = 0\,, \tag{3.12}$$

where $\eta = g^{-1}(\xi) = h(\xi)$. In particular when N is a constant ξ can chosen as

$$\xi = e^{-\alpha^2\eta} \tag{3.13}$$

and eq. (2.23) becomes

$$\nabla^2\xi - 2\alpha^2\xi_z + N^2\xi\ln\xi = 0\,. \tag{3.14}$$

Although eqs. (3.5), (3.12) are nonlinear the nonlinearity has been shifted to terms containing only the unknown function. In this form these equations are better suited for the derivation of analytic insights some additional transformations and approximations as well as numerical solutions. Furthermore although the transformations derived above might seem to be "straightforward" we believe (after exhaustive literature search) that they did not appear in the literature before (see *e.g*, [14]).

We observe that the transformations in this section were applied to "the restricted form of Long's equation", *viz.* eq. (2.18). However the same transformations can be used also to simplify the more general form (2.14).

4. – The effect of the nonlinearity on the solutions

In this section we consider analytically the impact of of the nonlinear terms in eqs. (3.6) and (3.8) on the solutions of these equations. We prove that at least in two instances one can reduce these equations to Lienard-type equations which describe nonlinear oscillations. We further show that in these cases, the impact of the nonlinear terms can be described by a "slow variable" which affects only the amplitude of the oscillations. (Throughout this section we assume that N is constant.)

To begin with, we consider eq. (2.23) in the limiting case $\mu = 0$.

$$\eta'' - \alpha^2(\eta')^2 - 2\alpha^2\eta' + N^2\eta = 0\,. \tag{4.1}$$

This equation belongs to the class of Rayleigh equations [18]. To proceed, we differentiate this equation with respect to z and introduce $q = \eta'$. We obtain

$$q'' - 2\alpha^2(q+1)q' + N^2 q = 0\,. \tag{4.2}$$

Equation (4.2) is a Lienard-type equation which describes nonlinear oscillations. Using the method of phase averaging [18, 19] one can show that the two leading terms in the asymptotic solution of this equation as $\alpha \to 0$ can be written in the form

$$q(z) = A(\nu)\cos(Nz + B(\nu))\,, \tag{4.3}$$

where $\nu = \alpha^2 z$ and $A(\nu)$, $B(\nu)$ have to to satisfy (see [18], p. 299).

$$A_\nu = \frac{A}{\pi}\int_0^{2\pi}(A\cos t + 1)\sin^2 t\,\mathrm{d}t\,, \tag{4.4}$$

$$B_\nu = \frac{1}{\pi}\int_0^{2\pi}(A\cos t + 1)\sin t\cos t\,\mathrm{d}t\,. \tag{4.5}$$

In these equations the subscript ν denotes differentiation with respect to this variable. Since we assume that $\beta \ll 1$ and hence $\alpha \ll N$, it follows that the variation of $A(\nu)$, $B(\nu)$ over one cycle with frequency N is negligible (ν is a "slow" variable in these equations). Therefore, although in principle A inside the integral of eq. (4.4) should be considered as a function of the phase t, we can treat it as a constant inside the integral of eqs. (4.4), (4.5). (In fact this is the essence of the method of phase averaging, otherwise eqs. (4.4), (4.5) will be very difficult to solve.)

A simple computation then yields $A = A_0 e^{\alpha^2 z}$, $B = \text{const}$. Since ν can be recognized as a "slow variable" solution (4.3) is essentially periodic and the modifications due to the nonlinear terms are minimal. This is confirmed further by the numerical solution of eq. (4.1) with $\alpha^2 = 5.\times 10^{-3}$ and $\eta(0) = 0$ (see fig. 1).

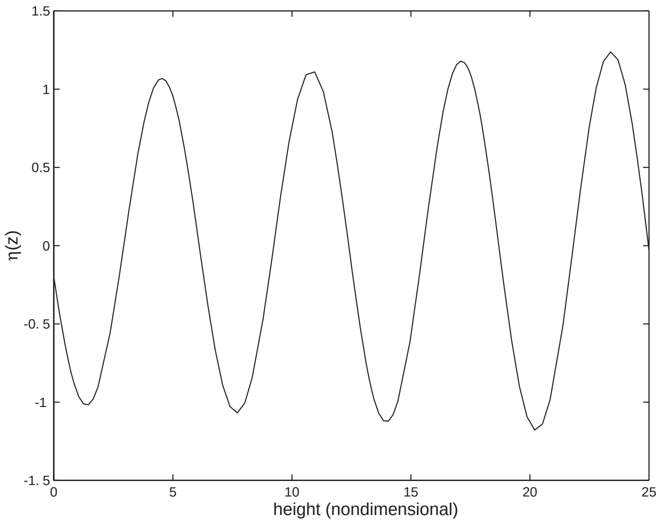

Fig. 1. – A plot of $\eta(z)$ which was computed numerically using eq. (3.14). This plot demonstrates the almost periodic behavior of this function. The parameters used are $\alpha^2 = 0.002$, $N = 1$.

To summarize: We showed that the solution for $q(z)$ can be approximated by

$$q(z) = A_0 e^{\alpha^2 z} \cos(Nz + B)\,, \tag{4.6}$$

where A_0, B are constants. Hence the approximate solution for the perturbation $\eta(z)$ is

$$\eta(z) = C_1 e^{\alpha^2 z}[N \sin(Nz + B) + \alpha^2 \cos(Nz + B)] + C_2\,, \tag{4.7}$$

where $C_1 = \frac{A_0}{\alpha^4 + N^2}$ and C_2 is a constant of integration. This result shows that the linear theory of gravity waves which is based on eq. (2.25) is valid only when the modulating factor $e^{\alpha^2 z}$ is close to 1, *i.e.* $|\alpha^2 z| \ll 1$. This result provides a rigorous analytical proof, and constraints, on the numerical observations made in [9,6] that the nonlinear terms in Long's equation have little effect on its solution. However it clear also that this factor has to be taken into account if z is allowed to vary over a long range, *e.g.*, when one considers gravity waves (generated by topography) in the stratosphere.

We further note that eq. (3.5) with $\mu = 0$ can be reduced to a first-order equation by introducing $y = \phi e^{\alpha^2 z}$ and $w = \phi'_z/\phi$, as the independent and dependent variables respectively (this requires that relation between y and z is invertible). This yields, assuming that $(\alpha^2 + w) \neq 0$,

$$\frac{\mathrm{d}w}{\mathrm{d}y} = -\frac{N^2 \ln y + w^2}{y(\alpha^2 + w)}\,. \tag{4.8}$$

To consider the case $\mu \neq 0$, we observe that eq. (3.8) (in an unbounded domain) is invariant with respect to translations in x and z. It is natural therefore to consider solutions of the form $\xi(x, z) = \xi(\chi)$ where $\chi = (kx+mz)$ and k,m are arbitrary constants.

Substitution of this form in eq. (3.8) leads to

$$(m^2+\mu^2k^2)\xi''-2\alpha^2 m\xi'+N^2\xi\ln\xi=0\,, \tag{4.9}$$

where primes denote differentiation with respect to χ. This demonstrates that eq. (4.9) admits "wave like structures" as solutions (if we think about z as playing the role of "time" in the solution).

Further insights about the nature of the solutions of eq. (4.9) can be obtained by introducing $\sigma=\frac{2\alpha^2 m}{m^2+\mu^2k^2}$, $\gamma^2=\frac{N^2}{m^2+\mu^2k^2}$ and $s=\sigma\chi$. Equation (4.9) becomes

$$\xi''-\xi'+\frac{\gamma^2}{\sigma^2}\xi\ln\xi=0 \tag{4.10}$$

(where primes denote differentiation with respect to s). By introducing $u=\xi'/\xi$ eq. (4.10) takes the form

$$u''+(2u-1)u'+\frac{\gamma^2}{\sigma^2}u=0 \tag{4.11}$$

which is once again a Lienard-type equation which describes nonlinear oscillations and can be treated exactly as we treated eq. (4.2).

However, to provide a uniform treatment for the one- and two-dimensional cases we substitute $\eta(x,z)=\xi(\chi)$ in eq. (2.23), differentiate the resulting equation with respect to χ, and then substitute $q(\chi)=\xi'$ (where primes now denote differentiation with respect to χ) to obtain

$$q''-2\alpha^2\left[q+\frac{\sigma}{(2\alpha^2)}\right]q'+\gamma^2 q=0\,. \tag{4.12}$$

The same procedure used to derive an approximate solution for eq. (4.2) can be used now to obtain an approximate solution of this equation. This leads to

$$q(\chi)=A_0e^{\sigma\chi/2}\cos(\gamma\chi+B)\,. \tag{4.13}$$

Hence $\eta(\chi)$ is given (approximately) by

$$\eta(\chi)=C_1e^{\frac{\sigma\chi}{2}}\left[\gamma\sin(\gamma\chi+B)+\frac{\sigma}{2}\cos(\gamma\chi+B)\right]+C_2\,, \tag{4.14}$$

where $C_1=4A_0/(\sigma^2+4\gamma^2)$ and $C2$ is a constant of integration. This can be rewritten more succinctly in the form

$$\eta(\chi)=Ce^{\sigma\chi/2}\cos(\gamma\chi+\phi)+C_2\,, \tag{4.15}$$

where $\phi=B+\tan^{-1}(\frac{-2\gamma}{\sigma})$.

Equations (4.7) and (4.15) provide an approximation for the attenuation (due to stratification) of the perturbation η. These results are possible only when one considers the nonlinear terms in Long's equation.

It should be observed that these solutions cannot satisfy the boundary conditions on η that can be derived from eqs. (2.19), (2.20). However since Long's equation contains no dissipation terms the practical interest in this type of solutions is primarily in a region outside the boundary layer but close to the topography that generates them. (Thus if dissipation is taken into account the perturbation will "die out" as x or z go to ∞ or $-\infty$.)

Another way to gauge the impact of the nonlinear terms on the solutions of Long's equation is provided by a comparison of a "linearized" form of eq. (3.14) and eq. (2.23) with $\beta = 0$ which is usually used for the detection of gravity wave in the statosphere [16, 21, 17, 15]. To carry out this comparison we will assume that $\xi \ln \xi$ in eq. (3.14) can be approximated by ξ (that is $\ln \xi \approx 1$). The resulting linear equation

$$\nabla^2 \xi - 2\alpha^2 \xi_z + N^2 \xi = 0 \tag{4.16}$$

retains it dependence on α (and μ) and offers insights about gravity waves when this parameter is not set to zero. On the other hand, eq. (2.18) (with $\bar{x} = x/\mu$) simplifies to

$$\nabla^2 \eta + N^2 \eta = 0\,. \tag{4.17}$$

An "elementary solution" of eq. (4.17) is of the form

$$\eta = \cos(mx + kz) \tag{4.18}$$

with $m^2 + k^2 = N^2$. On the other hand, an "elementary solution" of eq. (4.16) is of the form

$$\xi = e^{\alpha^2 z} \cos(mx + kz) \tag{4.19}$$

with $m^2 + k^2 + \alpha^4 = N^2$. The constraint $m^2 + k^2 = N^2$ led to a host of problems in the detection gravity waves which persist even today [16, 21, 17]. The modified relation $m^2 + k^2 + \alpha^4 = N^2$ might resolve some of this problems. (Observe that although this relation has been derived for ξ it must hold true for η also.)

5. – Conclusions

In this paper we applied several transformations to Long's equation in order to simplify its form. Using this new form we were able to prove analytically that at least in two instances the equation reduces to a Lienard-type equation. In these instances, the nonlinear terms in Long's equation (in the limit $\alpha \to 0$) convert the scalars that appear in the solution of the linearized equation (2.25) to functions of the slow variable $\alpha^2 z$. This confirms previous numerical observations made in the past by other authors. Using this form of the equation we were able to derive also new results regarding the attenuation of the perturbation η with height in one and two dimensions.

The transformed equation is still nonlinear but the nonlinearity has been shifted from the derivatives to terms containing the unknown function. In this form the equation is highly amenable to semi-analytic treatment, various approximations and numerical simulations. In fact it is easy to implement Newton's iteration scheme to solve Long's equation in the form (3.8). To this end we define (from eq. (3.8))

$$F(\xi) = \xi_{zz} + \mu^2 \xi_{xx} + N^2(\xi \ln \xi - \beta \xi_z) \tag{5.1}$$

and take the Frechet derivative of F, *i.e.* compute

$$F(\xi_0 + \delta) \cong F(\xi_0) + L(\xi_0)\delta + O(\delta^2)\,, \tag{5.2}$$

where $L(\xi_0)$ is a linear operator. A short computation yields

$$L(\xi_0) = \frac{\partial^2}{\partial z^2} + \mu^2 \frac{\partial^2}{\partial x^2} + N^2 \left[(1 + \ln \xi_0) - \beta \frac{\partial}{\partial z} \right]. \tag{5.3}$$

To devise Newton's iteration scheme to solve eq. (3.8) we now let $F(\xi_0 + \delta) = 0$ in eq. (5.2) and obtain

$$L(\xi_m)\xi_{m+1} = N^2 \xi_m\,, \tag{5.4}$$

where the index m denotes the iteration number.

We shall not pursue these numerical simulations further as these were carried out extensively in the past by many authors. Simulations of Long's equation near the hydrostatic limit (viz. β small but not zero) will be considered in a separate publication.

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