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# On non-linear baroclinic adjustment with the stratosphere(*)

I. Bordi([1]), K. Fraedrich([2]), F. Lunkeit([2]) and A. Sutera([1])

([1]) *Dipartimento di Fisica, Università di Roma "La Sapienza" - I-00185 Rome, Italy*
([2]) *Meteorologisches Institut, Universität Hamburg - D-20146 Hamburg, Germany*

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**Summary.** — The effect of the stratosphere on the baroclinic adjustment of a non-linear Eady model is presented. The classical linear Eady model has been modified by including an additional layer (the stratosphere), Ekman dissipation at the bottom boundary and a Newtonian cooling at the surface and the tropopause, respectively; non-linearity is introduced by wave-mean flow interaction for a single eddy mode. Results for the rigid-lid case and for small troposphere/stratosphere stratification ratio are compared with those for the linear Eady model with Ekman dissipation at the surface. For these cases model solutions consist of a steady zonal correction and an eddy field with a travelling constant amplitude wave. The equilibrated field, as a function of small stratification ratio, shows that the minimum amplitude of the eddy component raises to a height close to the tropopause (its steering level), denoting that the wave solution becomes vertical evanescent. When realistic values for the static stability in the stratosphere are considered, the zonal correction is no more time independent and reveals a degree of chaotic behaviour, while the eddy field is fully chaotic. Effects of changes in the zonal wind vertical shear and a further decreasing static stability in the stratosphere are also analysed. Results suggest that the minimum amplitude is, in average, higher than the one computed for the classical rigid lid with Ekman dissipation at the surface. Thus, as in the linear Eady model, the stratosphere induces a stabilising effect on the baroclinic dynamics. Finally, the model solutions are compared with the time behaviour of a simplified General Circulation Model.

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## 1. – Introduction

The nature of the interaction between the troposphere and the stratosphere has been discussed in many papers. These studies were motivated by the first report on stratospheric warming [1] and the successful theoretical insight of Matsuno [2] on the physical

(*) The authors of this paper have agreed to not receive the proofs for correction.

 

causes of this phenomenon. Along the years there has been a great deal of ideas on this sort of interaction mostly revolving around the main results of Charney and Drazin [3]. In this work it was shown how tropospheric stationary planetary waves could penetrate into the stratosphere and therein be absorbed. The latter would be leading to a wave-mean flow interaction [4].

However, there is another case where these interactions may play a role, namely the baroclinic adjustment problem, where the equilibrium of an atmosphere subject to forcing and dissipation is sought. In principle, in a baroclinic atmosphere the full spectrum of the eddy field should be involved. There is, however, observational evidence that the meridional gradient of the zonal potential vorticity (hereafter, PV) allows the instability of only a few zonal modes. This scale selection suggests that the atmosphere may be in a state of nearly neutrality with respect to baroclinic instability processes. Therefore, it may be possible that a closure theory may be at hand. Proposals for this theory have been advanced. Most of them rely on Charney-Stern [5] theorem, which states a necessary condition for zonal mean flow instability with respect to eddy perturbations. Modifying the zonal PV meridional gradient of the basic state, which satisfies the theorem, leads to a neutral state where the eddy Reynolds stresses cannot modify the zonal mean flow. A shortcoming of such an approach is that the neutral state achieved is often characterised by a weak lower level temperature gradient. This feature is far off from the observed zonal mean flow, for which the largest meridional temperature gradient is often found at the lower levels. As suggested by Lindzen [6], a neutral state, which avoids these unrealistic features, exists and can be calculated in terms of tropopause height. This approach has been extended to include the stratospheric interactions [7] and the form-drag process [8].

In pursuing a theory of baroclinic adjustment of this type, we may look at other aspects of troposphere-stratosphere interaction, namely those of a finite amplitude baroclinic eddy on the zonal flow driven by forcing and dissipation, going beyond the Charney-Drazin [3] results.

Let us consider a single growing baroclinic wave. The associated heat transport will induce a zonal flow correction that, in turn, modifies the radiative-convective basic state. These corrections constrain the baroclinic disturbances to saturate to a finite amplitude state, in which both zonal mean disturbance and the eddy coexist. On the other hand, radiative-convective processes operate to restore radiative-convective equilibrium of the zonal field so that the wave may restart its growth, and so on. Questions are whether we are able to parameterise the associated eddy heat fluxes and if the stratosphere plays any role at all in this adjustment process.

This paper attempts to address this issue in the simplest model where such behaviour may occur, namely a Boussinesq atmosphere constrained to have zero PV meridional gradients in the interior, whereas they do exist at the lower boundary and at the tropopause. The assumptions above are plagued clearly, among other things, by the unrealistic choice of zero PV gradient in the stratospheric basic state; however, they make our problem manageable by using analytical means. As mentioned, in previous studies [7, 8] we have addressed the problem of how baroclinic instability of an Eady wave is affected by a stratosphere. We found that both the ratio of stratosphere-troposphere static stabilities and the basic state zonal wind vertical shear are the most sensitive parameters in determining the dispersion relationship. Overall, these studies have shown that the stratosphere reduces the growth rate of the baroclinic wave, as if the short wave cut-off of Eady waves is moved to smaller zonal wave numbers. As a consequence, it appears that, in these circumstances, a theory of near critical baroclinic adjustment, such as the one proposed by Stone [9] or Lindzen [6], would seem to be suitable for the eddy heat fluxes to be parameterised in terms of zonal mean flow, also when a finite amplitude model is considered. In the present paper we shall pursue this idea presenting a minimal model where zonal mean fields are related to the eddy stresses and to a simple parameterisation of diabatic cooling. Model solutions are then compared with the behaviour of a simplified General Circulation Model (GCM) output.

The method employed here follows the study of Hart [10,11] and Weng *et al.* [12] who pursued finite amplitude analysis in a parameter space close to the annulus experimental set-up. We will depart from their approach assuming both physical processes and related parameters closer to an atmosphere-like environment. We will assume that the zonal mean flow consists of two terms, one, time and space independent, satisfying the thermal wind equation everywhere in the domain. It simulates a (extra-tropical) Hadley circulation in thermal wind balance. The other component, the zonal flow correction, instead, depends on space and time. However, in the absence of the eddy heat fluxes, it will be exponentially relaxing to zero in time. With this set-up, we can study the baroclinic adjustment of our flow.

The paper is organised as follows. In sect. **2** the model physical assumptions and the associated equations of motion are presented. In sect. **3** we explore the behaviour of the model for the rigid-lid case, while in sect. **4** we present results of baroclinic adjustment for realistic values of the static stability in the stratosphere. In sect. **5** we will show evidence that our results may help in interpreting simple General Circulation Model runs. In the final section conclusions and the perspective for further work are outlined.

## 2. – Equations and model set-up

Outside the Hadley cell, the lowest-order balance is geostrophically constrained. Let the flow be confined in a mid-latitude channel of width $L$ and depth $H$. In avoiding the frontal collapse of disturbances on the basic state, occurring if an inviscid interior flow is considered (see [13] for non–quasi-geostrophic approach with internal dissipation), we use quasi-geostrophic dynamics for a stratified atmosphere to describe perturbations. We scale $(x, y)$ by $L$, $z$ by $H$, the horizontal wind components $(u, v)$ by the thermal wind $U$, the geostrophic stream function by $UL$ and $t$ by the advective time $L/U$.

For a Boussinesq atmosphere, the geostrophic constraint implies that the mass-wind field balance is approximately described by the non-dimensional zonally symmetric stream function:

$$\phi_{\mathrm{b}} = -(U_0 + \Lambda z)y, \tag{1}$$

where the subscript "b" refers to the basic state, $U_0$ is the zonal wind at the surface and $\Lambda = \partial U/\partial z$ the constant vertical wind shear.

The governing equations are the conservation of potential vorticity and the thermodynamic equation [14]:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} q &= \frac{\partial}{\partial z}\left(\frac{\Im}{S}\right), \\ \frac{\mathrm{d}}{\mathrm{d}t}\theta + wS &= \Im, \end{aligned} \tag{2}$$

where $q = \nabla_H^2 \phi + \beta y + \frac{\partial}{\partial z}(\frac{\theta}{S})$, $\theta = \partial\phi/\partial z$, $S = N^2H^2/f^2L^2$, with $f$ the Coriolis parameter and $\beta$ its meridional gradient, $N^2$ the Brunt-Väisälä frequency, $\nabla_H^2$ the horizontal

Laplacian operator, $w$ is the vertical velocity, $\phi$ is the total stream function, and $\Im$ denotes the diabatic heating rate. Notice that the same equations hold in the log-pressure coordinates (see the formulation by Holton [15]). In the following subsections we discuss the rigid-lid case before analysing the two-layer model.

**2**˙1. *Rigid-lid case.* – Now we consider the case where $w$ is zero both at the surface ($z = 0$) and at the tropopause ($z = H_{\mathrm{T}}$), *i.e.* the rigid-lid case.

Let the total stream function be

$$\phi = \phi_{\mathrm{b}} + \varphi(x, y, z, t) + \Phi(y, z, t),$$

with $\phi_{\mathrm{b}}$ the basic state, $\varphi$ represents the wave field related to the eddies and $\Phi$ the zonal mean flow correction. The Eady problem [16] implies that $q_\varphi = q_\Phi = 0$ (*i.e.* the potential vorticity of $\varphi$ and $\Phi$) everywhere unless $z = 0$ and $z = H_{\mathrm{T}}$. Such a statement requires zero meridional gradient of the potential vorticity of the basic flow and no $\beta$ effects. The horizontal domain is zonally periodic and, since the flow is limited in latitude by walls, the meridional velocity must vanish there. For the $x$-independent portion of the perturbation field ($\Phi$), for a quasi-geostrophic flow, the zonal momentum equation requires also that no zonally integrated acceleration can occur at the rigid $y$-boundaries (see the second equation of system (3)).

Suppose that $\Im$ in the interior flow is zonally homogeneous and constant with height. Then, it must be specified at the surface and at the tropopause. Thus, in the thermodynamic equation, $\Im$ may be parameterised in terms of $\theta$ as a Newtonian cooling which, in the absence of any perturbation, will restore the geostrophic balance described by (1). However, the relative rate of restoration should take into account both the nature of the emitting surfaces and their different heat balance requirements. Time constants of this relaxation process can be evaluated by Taylor series expansion of the Stefan-Boltzman law around the radiative-convective equilibrium temperature of the two surfaces. This leads to a modest variation in the time constant with height except at the lower boundary where the temperature field is strongly constrained by the heat capacity of the ocean. As a matter of fact, in the absence of landmasses, the role of the atmosphere in constraining the surface temperature is more related to the associated air-sea fluxes rather than to the radiative equilibrium. Thus, it is worthwhile to consider problems where the relaxation times are different at the ground and at $H_{\mathrm{T}}$. A signature of this difference may be inferred by the relatively small temperature variance at lower levels in the Southern Hemisphere mid-latitude winter when compared with the corresponding variance at the tropopause level (see [17], or NCEP/NCAR reanalysis maps of the mean zonal temperature). Therefore, in studying the baroclinic adjustment process for our simple system, we consider cases where the relaxation term at $z = 0$ is smaller than the one at $z = H_T$.

With the assumptions above and taking the total stream function just described, the zonal average of the quasi-geostrophic equations (2) yield the following equations for the zonal mean flow correction, where a constant $S$ and an Ekman pumping at the ground have been considered [12]:

$$\begin{aligned} &\partial_{zz}\Phi + S\partial_{yy}\Phi = 0, && \\ &\partial_{yt}\Phi = 0, && \text{at } y = 0, L_y, \\ &\partial_{tz}\Phi + \langle J(\varphi, \partial_z\varphi)\rangle + \delta_{\mathrm{E}}\partial_{yy}\Phi + 1/\tau_0\partial_z\Phi = 0, && \text{at } z = 0, \\ &\partial_{tz}\Phi + \langle J(\varphi, \partial_z\varphi)\rangle + 1/\tau_H\partial_z\Phi = 0, && \text{at } z = H_{\mathrm{T}}. \end{aligned} \tag{3}$$

Equations for the eddy dynamics are obtained by subtracting the zonal flow correction (3) from eqs. (2), providing

$$
(4)\quad
\begin{aligned}
&\partial_{zz}\varphi + S\nabla_H^2\varphi = 0, && \\
&\partial_x\varphi = 0, && \text{at } y = 0, L_y, \\
&\partial_{tz}\varphi + U\partial_{xz}\varphi - \Lambda\partial_x\varphi + J(\Phi, \partial_z\varphi) + J(\varphi, \partial_z\Phi) + \delta_{\mathrm{E}}\nabla_H^2\varphi + && \\
&\qquad + J(\varphi, \partial_z\varphi) - \langle J(\varphi, \partial_z\varphi)\rangle = 0, && \text{at } z = 0, \\
&\partial_{tz}\varphi + U\partial_{xz}\varphi - \Lambda\partial_x\varphi + J(\Phi, \partial_z\varphi) + J(\varphi, \partial_z\Phi) + && \\
&\qquad + J(\varphi, \partial_z\varphi) - \langle J(\varphi, \partial_z\varphi)\rangle = 0\,, && \text{at } z = H_{\mathrm{T}},
\end{aligned}
$$

where $U = U_0 + \Lambda z$, $\delta_{\mathrm{E}} = E^{1/2}S/2R_0$ is the Ekman dissipation parameter with $E$ and $R_0$ the Ekman and Rossby numbers; $J$ is the Jacobian, while $\langle\ldots\rangle$ denotes the average over the $x$-direction:

$$\langle\ldots\rangle = 1/L_x \int_0^{Lx} (\ldots)\mathrm{d}x.$$

Here $L_x$ and $L_y$ are the zonal and meridional scale of the fluid, while $\tau_0$ and $\tau_H$ are the relaxation times for the mean flow correction at the lower and upper boundary, respectively.

**2**˙2. *Two-layer model.* – Now the rigid-lid assumption is removed by imposing that the stream function and the vertical velocity are continuous at $H_{\mathrm{T}}$, while the wave and the mean flow correction fields vanish in the limit for $z$ going toward the outer space. Of course, it is an implicit assumption that the motion is not accompanied by a tropopause deformation [18]. The tropopause will remain fixed and independent of time and space. Additional effort is required to relax this assumption, since the thermodynamic equation at the tropopause should be matched on a space- and time-varying surface. However, the associated effect might require a departure from quasi-geostrophic theory, where $w\partial_z\theta \approx w\partial_z\theta_0$ (the subscript stands for the geostrophically balanced and time-independent value); therefore, vertical velocity continuity at the material surface leads to infinite vertical heat fluxes at the tropopause. On the other hand, we may think that the sharp change in the static stability at the tropopause occurs without vertical discontinuity in $\theta_0$ as long as its vertical variation is confined in a layer of a small thickness compared with the scale height of the perturbation. Notice that this assumption is usually considered when GCMs are used, since vertical derivatives are replaced by finite differences. Nevertheless, it is fair to say that this assumption may be a critical constraint on the choice of a meridional scale for the $x$-independent component of the perturbation field (which has the smallest scale height). The removal of a rigid tropopause may lead to interesting consequences, which is beyond the scope of this paper and will be discussed elsewhere. We feel, however, that this assumption is, by and large, of minor relevance when it is contrasted with the assumed linearity of the interior flow. The latter, in fact, inhibits the flow to adjust by developing interior potential vorticity gradients that will change the meridional scale of the eddies and, thereby, their stability.

Thus, within the assumptions above and denoting by the up-scripts $n = 1$, 2 troposphere and stratosphere, respectively, eqs. (3), (4) for the mean flow correction and eddy dynamics of the two-layer model become

$$\partial_{zz}\Phi^{(1)} + S^{(1)}\partial_{yy}\Phi^{(1)} = 0, \tag{5}$$

$$\partial_{zz}\Phi^{(2)} + S^{(2)}\partial_{yy}\Phi^{(2)} = 0, \tag{6}$$

$$\partial_{yt}\Phi^{(1)} = \partial_{yt}\Phi^{(2)} = 0, \quad \text{at } y = 0, L_y, \tag{7}$$

$$\partial_{tz}\Phi^{(1)} + \big\langle J\big(\varphi^{(1)}, \partial_z\varphi^{(1)}\big)\big\rangle + \delta_{\mathrm{E}}\partial_{yy}\Phi^{(1)} + 1/\tau_0\partial_z\Phi^{(1)} = 0, \quad \text{at } z = 0, \tag{8}$$

$$\begin{aligned}&\partial_{tz}\Phi^{(1)} + \big\langle J\big(\varphi^{(1)}, \partial_z\varphi^{(1)}\big)\big\rangle + 1/\tau_H\partial_z\Phi^{(1)} = \\ &\quad = \gamma\big[\partial_{tz}\Phi^{(2)} + \big\langle J\big(\varphi^{(2)}, \partial_z\varphi^{(2)}\big)\big\rangle + 1/\tau_H\partial_z\Phi^{(2)}\big] \quad \text{at } z = H_{\mathrm{T}},\end{aligned} \tag{9}$$

$$\partial_{zz}\varphi^{(1)} + S^{(1)}\nabla^2_H\varphi^{(1)} = 0, \tag{10}$$

$$\partial_{zz}\varphi^{(2)} + S^{(2)}\nabla^2_H\varphi^{(2)} = 0, \tag{11}$$

$$\partial_x\varphi^{(1)} = \partial_x\varphi^{(2)} = 0, \quad \text{at } y = 0, L_y, \tag{12}$$

$$\begin{aligned}&\partial_{tz}\varphi^{(1)} + U^{(1)}\partial_{xz}\varphi^{(1)} - \Lambda\partial_x\varphi^{(1)} + J\big(\Phi^{(1)}, \partial_z\varphi^{(1)}\big) + J\big(\varphi^{(1)}, \partial_z\Phi^{(1)}\big) + \\ &\quad + \delta_{\mathrm{E}}\nabla^2_H\varphi^{(1)} + J\big(\varphi^{(1)}, \partial_z\varphi^{(1)}\big) - \big\langle J\big(\varphi^{(1)}, \partial_z\varphi^{(1)}\big)\big\rangle = 0 \quad \text{at } z = 0,\end{aligned} \tag{13}$$

$$\begin{aligned}&\partial_{tz}\varphi^{(1)} + U^{(1)}\partial_{xz}\varphi^{(1)} - \Lambda\partial_x\varphi^{(1)} + J\big(\Phi^{(1)}, \partial_z\varphi^{(1)}\big) + \\ &\quad + J\big(\varphi^{(1)}, \partial_z\Phi^{(1)}\big) + J\big(\varphi^{(1)}, \partial_z\varphi^{(1)}\big) - \big\langle J\big(\varphi^{(1)}, \partial_z\varphi^{(1)}\big)\big\rangle = \\ &\quad = \gamma\big[\partial_{tz}\varphi^{(2)} + U^{(2)}\partial_{xz}\varphi^{(2)} - a_0\Lambda\partial_x\varphi^{(2)} + J\big(\Phi^{(2)}, \partial_z\varphi^{(2)}\big) + \\ &\quad + J\big(\varphi^{(2)}, \partial_z\Phi^{(2)}\big) + J\big(\varphi^{(2)}, \partial_z\varphi^{(2)}\big) - \big\langle J\big(\varphi^{(2)}, \partial_z\varphi^{(2)}\big)\big\rangle\big] \quad \text{at } z = H_{\mathrm{T}},\end{aligned} \tag{14}$$

$$\Phi^{(1)} = \Phi^{(2)} \text{ and } \varphi^{(1)} = \varphi^{(2)}, \quad \text{at } z = H_{\mathrm{T}}, \tag{15}$$

$$\Phi^{(2)} \to 0 \text{ and } \varphi^{(2)} \to 0, \quad \text{for } z \to \infty, \tag{16}$$

where

$$\begin{aligned}U^{(1)} &= U_0 + \Lambda z, && \text{for } z \le H_{\mathrm{T}},\\ U^{(2)} &= U^{(1)}(H_{\mathrm{T}}) + a_0\Lambda(z - H_{\mathrm{T}}), && \text{for } z > H_{\mathrm{T}}.\end{aligned}$$

In view of the horizontal boundary condition for $\varphi^{(n)}$, we may consider the following expansion:

$$\varphi^{(n)}(x,y,z,t) = \sum_{k,l}\varphi^{(n)}_{k,l}(z,t)e^{ikx}\sin(ly) + \text{c.c.}, \tag{17}$$

where $k$ and $l$ are the zonal and meridional wave numbers of the wave field, c.c. stands for the complex conjugate, while the $\varphi^{(n)}_{k,l}(z,t)$ satisfy the corresponding interior equations. Substituting (17) into (13) and (14), we obtain the equations for the time evolution of $\varphi^{(n)}_{k,l}(z,t)$ by projecting into the base functions $e^{ikx}\sin(ly)$. In the spirit of baroclinic adjustment theory [9], we may consider a single term of (17) by selecting a value for $k$ and $l$. Consequently, equations for the zonal mean correction (8), (9) become

$$\begin{aligned}&\partial_{tz}\Phi^{(1)} + 2ikl\varphi^{(1)}_{k,l}\partial_z\varphi^{(1)*}_{k,l}\sin(2ly) + \delta_{\mathrm{E}}\partial_{yy}\Phi^{(1)} + \frac{1}{\tau_0}\partial_z\Phi^{(1)} = 0 && \text{at } z = 0,\\ &\partial_{tz}\Phi^{(1)} + 2ikl\varphi^{(1)}_{k,l}\partial_z\varphi^{(1)*}_{k,l}\sin(2ly) + \frac{1}{\tau_H}\partial_z\Phi^{(1)} = \gamma\left[\partial_{tz}\Phi^{(2)} + \frac{1}{\tau_H}\partial_z\Phi^{(2)}\right] && \text{at } z = H_{\mathrm{T}},\end{aligned} \tag{18}$$

where the upscript star denotes the complex conjugate.

In view of the identity $\sin(2ly) = (2/l)\sum_{j=1}^{\infty} g_{2j-1}\cos((2j-1)ly)$, where $g_{2j-1} = 1/(1-(j-0.5)^2)$, we may solve (18) by setting

$$\Phi^{(n)}(y,z,t) = \sum_{lb} \Phi^{(n)}_{lb}(z,t)\cos(lby),$$

with $\Phi^{(n)}_{lb}(z,t)$ satisfying the interior equations by projection on the meridional structure.

Thus, zonal mean corrections may retain full dependence on space. For the sake of both clarity and focus of the paper, we truncate also the zonal correction field to a single mode over the $y$-direction. It must be noted that under the single mode condition, the thermodynamic equations for the eddy component at the two boundaries become linear in the eddy field, limiting the model to describe the wave-mean flow interaction, as in the classical theory of troposphere-stratosphere dynamics. Thus, the consequence of adopting a single modal solution is that to neglect terms in the eddy equations that are quadratic in the amplitude of perturbation. We point out also that the same equations are obtained by an asymptotic analysis near criticality as shown by Pedlosky [14] or Chou and Loesch [19] for the rigid-lid case and without Newtonian cooling. Thus, we can write

$$\text{(19)}\quad \Phi^{(1)}(y,z,t) = \left[A_{0,lb}(t)\sinh\left(\alpha^{(1)}{}_{0,lb}z\right) + B_{0,lb}(t)\cosh\left(\alpha^{(1)}{}_{0,lb}z\right)\right](2/L_y)^{1/2}\cos(lby),$$

$$\text{(20)}\quad \Phi^{(2)}(y,z,t) = \left[D_{0,lb}(t)\exp\left[-\alpha^{(2)}{}_{0,lb}z\right]\right](2/L_y)^{1/2}\cos(lby),$$

$$\text{(21)}\quad \varphi^{(1)}(x,y,z,t) = \left[A_{k,l}(t)\sinh\left(\alpha^{(1)}{}_{k,l}z\right) + B_{k,l}(t)\cosh\left(\alpha^{(1)}{}_{k,l}z\right)\right]\times$$
$$\times(2/L_y)^{1/2}\sin(ly)\exp[ik(x-ct)]/(L_x)^{1/2} +$$
$$+\,\text{c.c.},$$

$$\text{(22)}\quad \varphi^{(2)}(x,y,z,t) = \left[D_{k,l}(t)\exp\left[-\alpha^{(2)}{}_{k,l}z\right]\right]\times$$
$$\times(2/L_y)^{1/2}\sin(ly)\exp[ik(x-ct)]/(L_x)^{1/2} + \text{c.c.},$$

where $c$ is the phase speed of the wave, $\alpha^{(n)}{}_{k,l} = [(k^2+l^2)S^{(n)}]^{1/2}$, $\alpha^{(n)}{}_{0,lb} = lbS^{1/2}$, with $lb$ the meridional wave number of the mean flow correction. $A_{k,l}$, $B_{k,l}$ are complex functions of time, while $A_{0,lb}$, $B_{0,lb}$ are real functions. It must be noted that by using condition (15) we may eliminate the dependence on $D_{k,l}(t)$ from the equations.

We set $U = 10\,\text{m/s}$, $L = 10^6\,\text{m}$ and $H = 10^4\,\text{m}$ for the scaling of dimensional variables. This scaling leads to the bulk static stability parameter $S$ to be $O(1)$. We take the usually accepted value of the stratification ratio $\gamma = 1/4$, though we will change this ratio for performing sensitivity studies. We fix the other free parameters to Earth-like values as: $k = 2\pi s/L_x$, $l = \pi m/L_y$, $lb = \pi m_b/L_y$, $L_x = 2\pi r_a\cos(\theta_0)$, $L_y = r_a\Delta\theta$, with $r_a$ the non-dimensional Earth's radius, $\theta_0 = 45^{\text{o}}$, $\Delta\theta = 30^{\text{o}}$, $\Lambda = 3$, $U_0 = 0.5$, $\delta_{\text{E}} = 0.1$ and $m = m_b = 1$. The latter choice maximises the zonal correction at mid-channel, that is, far away from the lateral boundaries where, in the real atmosphere, possible interactions with the Hadley cell may occur (a feature excluded by this model). This leaves us with the following free parameters: $H_{\text{T}}$, $s$, $\gamma$, $\tau_0$, $\tau_H$, $a_0$, which form a six-dimensional space. Given the model assumptions, exhausting the study of the model behaviour in such a large parameter space might imply that we credit our model more than its physical contents warrant. Thus, we grossly sample the parameter space by considering the linear theory of Eady wave with a rigid lid at $H_{\mathrm{T}} = 0.8$ and by fixing the non-dimensional zonal wave number $s$ close to the most unstable wave ($s = 5$). Then, we change, for a few discrete values, $\gamma$ and $a_0$. About the relaxation terms, following the considerations made above ($\tau_0 < \tau_H$), we set $\tau_0 = 3$ time units and $\tau_H = 10$ time units as their reference values, a choice which has no particular relevance for the physics here analysed. With this model set-up we integrate the system in time by using the leapfrog scheme (see Appendix for details).

**2**˙3. *Some general deductions: Steady state and eddy parameterisation.* – The aim of the paper, as discussed in the introduction, is to investigate about the possibility to parameterise the eddy heat fluxes in terms of zonal mean flow also when a finite amplitude model, which takes into account troposphere-stratosphere interaction, is considered. Besides the restrictions here adopted and previously discussed, few general deductions can be obtained from the equations.

If a steady state occurs, the equations for the zonal mean perturbation yield the following relation (by subtracting (8) from (9)):

$$\bar{w}^{(1)}\big|_{z=0} + 1/\tau_0 \partial_z \Phi^{(1)}\big|_{z=0} - 1/\tau_H \partial_z \Phi^{(1)}\big|_{z=H} = -\gamma/\tau_H \partial_z \Phi^{(2)}\big|_{z=H}, \tag{23}$$

where $\bar{w}^{(1)}$ is the vertical velocity set by the Ekman layer at the bottom. This implies that the difference of the zonally symmetric perturbation radiative cooling at the two boundaries (minus a contribution related to some stratospheric effect if $\gamma \neq 0$) must balance the Ekman-induced vertical velocity at the ground. On the other hand, the horizontally averaged heat fluxes (divergences) maintain the balance of the radiative cooling of the zonal temperature correction at the tropopause (by adding (8) and (9)):

$$\left\langle J\left(\varphi^{(1)}, \partial_z \varphi^{(1)}\right)\right\rangle = -1/\tau_H \partial_z \Phi^{(1)}\big|_{z=H} + \gamma/\tau_H \partial_z \Phi^{(2)}\big|_{z=H}. \tag{24}$$

The existence of the latter balance requires the horizontally integrated eddy heat fluxes to be time independent, while the horizontal wind-temperature eddy fields maintain a constant phase difference. We like to stress that (24) represents an exact parameterisation of the eddy heat fluxes in terms of the zonal mean field for this model; that is, the model can be baroclinically adjusted. To fully satisfy the statements (23) and (24), it is required that the zonally integrated baroclinic conversion term balances the Ekman-induced vertical eddy heat fluxes at the ground. This requirement is readily obtained by considering the eddy field temperature equation, multiplying it by the eddy temperature and integrating over the domain; this is a statement of conservation of eddy available potential energy.

Thus, the steady state or equilibrium formulated above (see (24)), if it exists, consists of a travelling eddy wave with constant amplitude and a constant correlation between the eddy geopotential and its associated temperature field. The phase speed $c$ of the wave depends on the relative strength of the zonal mean vorticity gradient induced by the eddy heat fluxes at the two (vertical) boundaries. The dispersion relationship may be computed by inserting (21), (22) into (13), (14). Then, the phase speed can be obtained by assuming that the zonal mean correction is steady and known. Thus, for a constant-amplitude travelling wave, we have

$$c = \frac{\left(U^{(1)}_{H_\mathrm{T}} - \frac{\Lambda^{(1)}_{H_\mathrm{T}}}{\alpha^{(1)}_{k,l}} \tanh\left(\alpha^{(1)}_{k,l} H_\mathrm{T}\right)\right) + \gamma \left(\frac{\alpha^{(2)}_{k,l}}{\alpha^{(1)}_{k,l}} U^{(2)}_{H_\mathrm{T}} \tanh\left(\alpha^{(1)}_{k,l} H_\mathrm{T}\right) + \frac{\Lambda^{(2)}_{H_\mathrm{T}}}{\alpha^{(1)}_{k,l}} \tanh\left(\alpha^{(1)}_{k,l} H_\mathrm{T}\right)\right)}{1 + \gamma \frac{\alpha^{(2)}_{k,l}}{\alpha^{(1)}_{k,l}} \tanh\left(\alpha^{(1)}_{k,l} H_\mathrm{T}\right)}, \tag{25}$$

where the zonal wind and the zonal wind vertical shear are those for the basic state plus the corrections at the tropopause. By projecting into the mode $\sin(ly)$, they are (for $I_2$ see the Appendix):

$$U^{(1)}_{H_\mathrm{T}} = U^{(2)}_{H_\mathrm{T}} = U_0 + \Lambda H_\mathrm{T} + lbI_2 \left(A_{0,lb} \sinh\left(\alpha^{(1)}_{0,lb} H_\mathrm{T}\right) + B_{0,lb} \cosh\left(\alpha^{(1)}_{0,lb} H_\mathrm{T}\right)\right),$$
$$\Lambda^{(1)}_{H_\mathrm{T}} = \Lambda + \alpha^{(1)}_{0,lb} lbI_2 \left(A_{0,lb} \cosh\left(\alpha^{(1)}_{0,lb} H_\mathrm{T}\right) + B_{0,lb} \sinh\left(\alpha^{(1)}_{0,lb} H_\mathrm{T}\right)\right),$$
$$\Lambda^{(2)}_{H_\mathrm{T}} = a_0 \Lambda - \alpha^{(2)}_{0,lb} lbI_2 \left(A_{0,lb} \cosh\left(\alpha^{(1)}_{0,lb} H_\mathrm{T}\right) + B_{0,lb} \sinh\left(\alpha^{(1)}_{0,lb} H_\mathrm{T}\right)\right).$$

Note that for $\gamma \to 0$ we get the rigid lid.

For the rigid-lid case, we can estimate the phase difference between $A_{k,l}$ and $B_{k,l}$ as a function of the zonal flow correction. We can write

$$A_{k,l}(t) = |A_{k,l}(t)| e^{i\omega t}$$
$$B_{k,l}(t) = |B_{k,l}(t)| e^{i(\omega t + \Theta(t))},$$

where $\omega = kc$ is the constant frequency of both $A_{k,l}$ and $B_{k,l}$ and $\Theta$ the relative phase difference. By moving our frame of reference with the phase speed of the wave, the stationary state for the eddy dynamics under rigid-lid conditions gives

$$\tan(\Theta) = \frac{\delta_\mathrm{E}\left(k^2 + l^2\right)}{k \Lambda^{(1)}_{z=0}}, \tag{26}$$

with $\Lambda^{(1)}_{z=0}$ the total wind vertical shear at the ground. This means that the Ekman dissipation at the lower level provides the necessary phase difference between $A_{k,l}$ and $B_{k,l}$ for the heat transport to occur and such a phase is a constant.

In summarising, at first glance, it appears that we may have achieved our aim, the parameterisation of synoptic eddies with a closure (24), at least for a steady-state solution. The following numerical analysis of the stratification ratio and vertical wind shear effects will show more details.

## 3. – Numerical results: Small stratification ratio $\gamma$

Next we will analyse the time behaviour of the model. Again for sake of simplicity we consider a single wave and its associated zonal mode. The equations of motion are summarised in the Appendix. Let us consider $\gamma = 0$. In the absence of Newtonian cooling the time behaviour of the model is presented in fig. 1. As shown, after an initial exponential growth, the wave amplitude decays rapidly towards zero, leaving, however, a non-zero zonal mean correction. The solution strongly resembles the one described by Chou and Loesch [19] for the case where the full spatial structure of the zonal mean

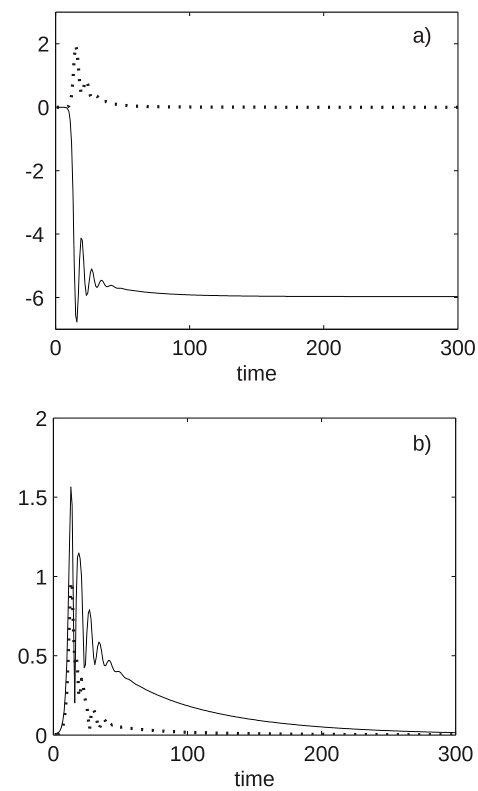


Fig. 1. – Model solution for $\gamma = 0$ and no Newtonian cooling: Time behaviour for the first 300 time units (corresponding to 300 days) of $A_{0,l}$ (solid line) and $B_{0,l}$ (dotted line) (a); b) time behaviour of the amplitude of $A_{k,l}$ (solid line) and $B_{k,l}$ (dotted line). The other parameters are as described in sect. **2**; units are dimensionless.

correction is retained. This is a positive check for our numerical model and its truncation assumption. Let us remark that the wave free solution found above has been analysed by Pedlosky [20]. The final zonal correction is zero at the lower layer, since in the presence of wave forcing the Ekman pumping will restore the initial state, while at the tropopause the zonal velocity will only be reduced. Paradoxically, the decay of the wave is connected to the absence of dissipation in the upper layer, which prevents the zonal correction to be dissipated there. However, for a baroclinic adjustment theory the relaxation of the zonal mean correction temperature field is sufficient to restore the balance between the wave production and the mean flow dissipation, so that, as we shall see in the steady state, the wave field is different from zero.

For small stratification ratio $\gamma$ we get the balance described in the previous section, *i.e.* the zonal correction is steady and the eddy field is a travelling constant amplitude wave. In table I we report actual values of $A_{0,l}$, $B_{0,l}$, $|A_{k,l}|$, $|B_{k,l}|$, $\Theta_{k,l}$, $c$ and the corresponding period $T$ for some selected values of $\gamma$. Note that the steady solution, as a function of $\gamma$, remains stable for any disturbance of the form (19)-(22) as long as $\gamma \leq 1/20$ where the wave bifurcates to a quasi-periodic solution. From there on, as we will see later on in the text, there is a route to chaos, which depends on the values of the parameters $\gamma$

TABLE I. – *Values of $A_{0,l}$, $B_{0,l}$, $|A_{k,l}|$, $|B_{k,l}|$, $\Theta_{k,l}$, $c$ and the corresponding period $T$ for stratification values $\gamma = 0$, 1/100 and 1/30. The other parameters are listed in sect.* **2**. *Units are dimensionless.*

| | $\gamma = 0$ | $\gamma = 1/100$ | $\gamma = 1/30$ |
|---|---|---|---|
| $A_{0,l}$ | −1.73 | −1.64 | −1.59 |
| $B_{0,l}$ | −1.96 | −1.67 | −1.47 |
| $\|A_{k,l}\|$ | 1.36 | 1.30 | 1.27 |
| $\|B_{k,l}\|$ | 0.89 | 0.96 | 1.01 |
| $\Theta_{k,l}$ | 0.11 | 0.11 | 0.11 |
| $c$ | −0.34 | 0.03 | 0.28 |
| $T$ | 16.66 | 187.68 | 19.88 |

and $a_0$. Thus, as expected, for small values of $\gamma$, the stratosphere does not have a strong impact on the nature of the equilibrium. However, two main differences distinguish the rigid-lid solution to those for small $\gamma$. The first one is the wave propagation direction, which reverses becoming eastward for small values of $\gamma$ (although these values are far from the ones proper for atmospheric-like conditions) and the other one is the position of the minimum wave amplitude. For the cases listed in table I, we show in fig. 2 the amplitude of the disturbance as a function of height. As shown by the figure, when the

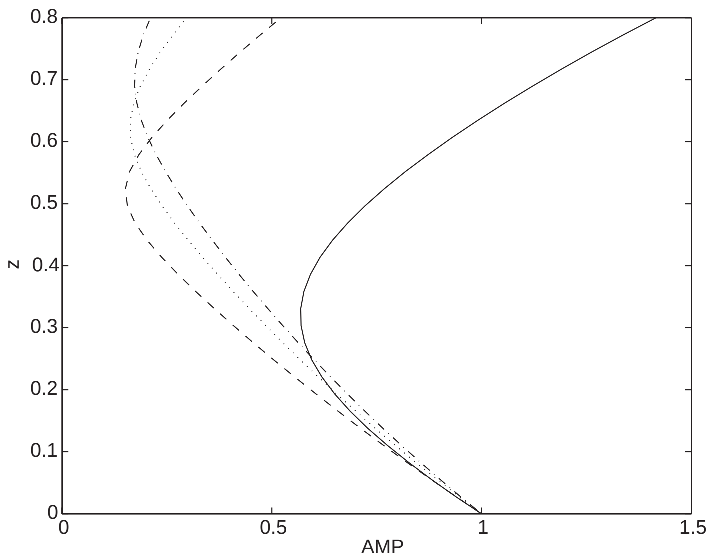


Fig. 2. – The amplitude of the disturbance as a function of height for linear Eady model with Ekman dissipation at the lower boundary (solid line); the model solution with $\gamma = 0$ (dashed line), with $\gamma = 1/100$ (dotted line) and for $\gamma = 1/30$ (dash-dotted line). The other parameters are as described in sect. **2**. Units are dimensionless.

Ekman dissipation is taken into account, the minimum wave amplitude of the Eady linear unstable mode is slightly below the mid height, *i.e.* its steering level. The equilibrated wave found here, instead, has its steering level nearby the tropopause (this can be easily deduced from (25)), while the minimum amplitude is located below this level. At the bifurcation the minimum amplitude approaches the tropopause, denoting that the wave solution becomes vertical evanescent, *i.e.* as if its associated effective Rossby height equals $H_{\mathrm{T}}$. Thus, for a vertically evanescent wave the heat fluxes do not allow to balance the radiative cooling at the tropopause as required by (24). It follows that the decay of the zonal mean correction, because the diabatic cooling, restores those conditions that allow the Eady wave to become enough unstable to draw available potential energy from the basic state. This is the fundamental baroclinic adjustment process operating in the system.

## 4. – Baroclinic adjustment: Realistic static stability ratio and zonal wind vertical shear, and the parameterisation problem

Relaxing the condition of the steady mean flow correction for small static stability ratio $\gamma$, the system demonstrates a rich behaviour, which allows dynamical insight into the stratosphere-troposphere interaction.

*A first example*: For standard atmospheric values, $\gamma = 1/4$ and $a_0 = 0$, used for the Eady model, the solution is shown in fig. 3 (hereafter, following the Appendix, we denote in the figures the zonal corrections with $x(1)$ and $x(2)$ and the amplitudes of eddy components with Amp$[x(3)]$ and Amp$[x(4)]$). A cursory look portrays a system behaving with a complexity that deserves attention. The zonal correction, not only is not steady, but also shows a degree of chaos. It appears that two main frequencies are detectable, the time sequence is widely spread around this quasi-periodic behaviour, making the motion highly unpredictable. Moreover, there are several regions where (see the widening of points density in the phase plane of the mean flow correction $x(1)$, $x(2)$) the trajectories tend to reside a longer time close the time averaged solution (denoted by a star in the figure), but none of them surrounds it. Also the amplitudes of the eddy component, $x(3)$ and $x(4)$, are chaotic and describe an erratic baroclinic cycle where the eddy heat flux attempts to balance radiative cooling. The motion may be described, indeed, as a vacillation of the minimum amplitude level where the effective Rossby height progressively rises stopping the available energy extraction from zonal mean reservoir, *i.e.* the process that in turn maintains the zonal correction against the dissipative effect of the zonal mean Ekman pumping and radiative cooling. Once this feeding ceases, the zonal correction tends to die out (because of the radiative cooling) establishing conditions for a rejuvenation of the baroclinic instability of the Eady wave.

*Zonal wind vertical shear in the stratosphere*: The effect of changes in the zonal wind vertical shear in the stratosphere are analysed next. We grossly sample the parameter space by choosing two values of $a_0$ (0.5 and $-0.5$). In the case $a_0 < 0$, we assume that the basic state temperature gradient reverses above the tropopause. Sections of the phase space are presented in fig. 4. In both cases the dynamics remains chaotic while the zonal correction motion appears to simplify by reinforcing the apparent periodic behaviour ($a_0 < 0$) or eliminating one of the two periods ($a_0 > 0$). It appears that this parameter tends to simplify the dynamics leading to a more predictable behaviour of the zonal mean correction. However, the use of (24) as a parameterization of the heat fluxes seems to be disfavoured, because there is no more a steady state for the zonal mean correction.

*Increasing static stability ratio*: A further increase of $\gamma$ presents the case for the

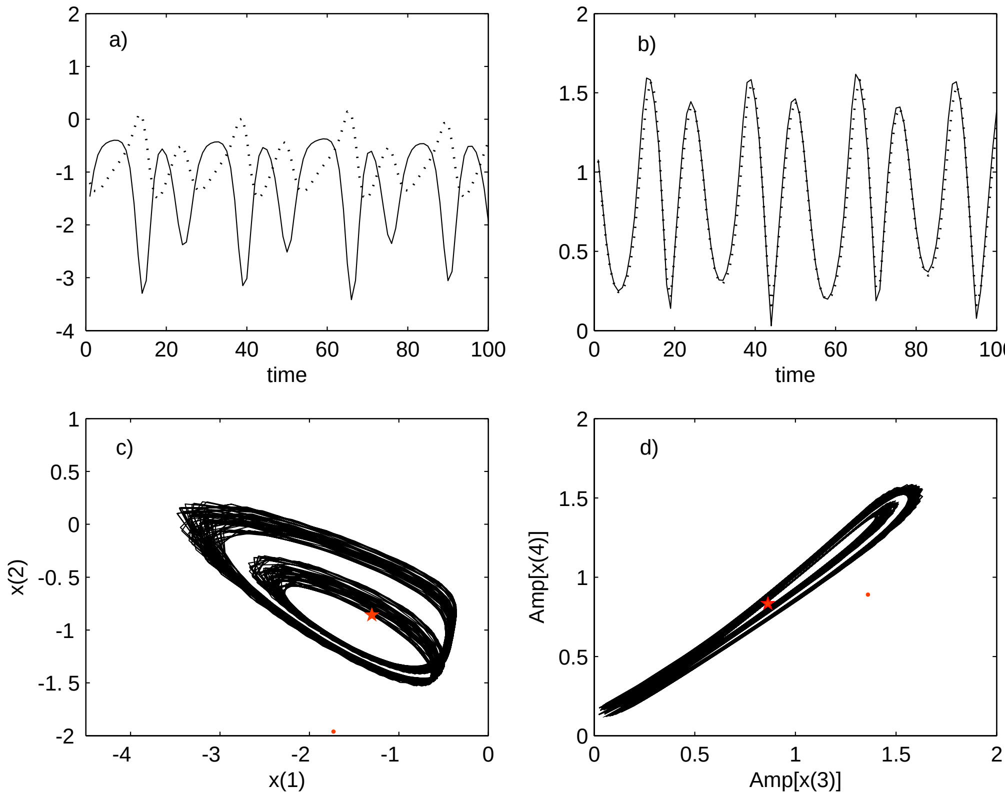


Fig. 3. – Model solutions for $\gamma = 1/4$ and $a_0 = 0$: a) Time behaviour for a sample of 100 time units of $x(1)$ (solid line) and $x(2)$ (dotted line); b) time behaviour of Amp[$x(3)$] (solid line) and Amp[$x(4)$] (dotted line); c) point density in the phase plane $x(1)$ *vs.* $x(2)$; d) point density in the phase plane Amp[$x(3)$] *vs.* Amp[$x(4)$]. Star denotes the time average solution, while filled circle the rigid-lid solution. The other parameters are as described in sect. **2**. Units are dimensionless.

possible dynamical effect of a declining ozone layer, leading to a less stable colder stratosphere [7]. In fig. 5 we present the relevant sections of the phase space for $\gamma = 1/2$ with the same three values of $a_0$ previously discussed. The overall picture seems to be close the one described in the above subsection except for the case $a_0 = -0.5$ where the zonal mean correction becomes fully chaotic. This solution is characterised by episodes during which the zonal mean correction goes to zero reaching radiative equilibrium (see fig. 6), which is unstable. There is an erratic nature in the duration of these episodes, which disrupts the main frequencies of the motion of zonal mean correction leading to an overall chaotic behaviour. Probably the same physics is present in other regions of the parameter space. It is more than just accidental to identify chaotic behaviour for increasing $\gamma$. For example, as reported elsewhere [7], a decrease of the static stability in the stratosphere (if all the rest remains unchanged) leads to a baroclinic less unstable atmosphere favourable for intermittant behaviour (end of sect. **3**).

*On the minimum amplitude*: The simple nature of the phase plane of the zonal mean correction conceals a complex dynamics distinguished by preferred regions of the phase space wherein an almost neutral (with respect to baroclinic instability) situation appears

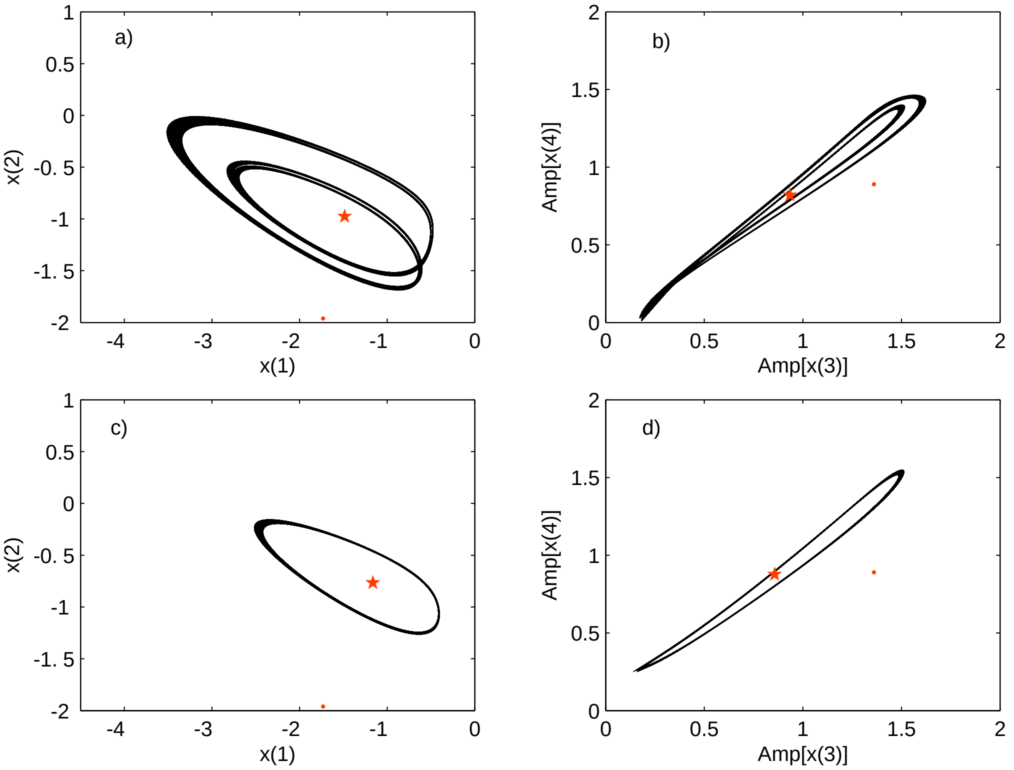


Fig. 4. – Model solutions for $\gamma = 1/4$ and $a_0 = -0.5$: a) points density in the phase plane $x(1)$ *vs.* $x(2)$, b) points density in the phase plane Amp$[x(3)]$ *vs.* Amp$[x(4)]$. Model solutions for $\gamma = 1/4$ and $a_0 = 0.5$: c) points density in the phase plane $x(1)$ *vs.* $x(2)$, d) points density in the phase plane Amp$[x(3)]$ *vs.* Amp$[x(4)]$. Star denotes the time average solution, while filled circle the rigid-lid solution. The other parameters are as described in sect. **2**. Units are dimensionless.

to be the most frequent case. We computed the probability density function of the height of the eddy minimum amplitude for $z < H_{\mathrm{T}}$. In fig. 7 the corresponding histograms for different values $\gamma$ and $a_0$ are shown. All panels suggest that the most recurrent position for the minimum amplitude level is at the tropopause. In particular, for $\gamma = 1/2$ and $a_0 = 0.5$ the position of the minimum amplitude varies between 0.4 and $H_{\mathrm{T}}$, while for $a_0 = 0$ or $a_0 = -0.5$ the height of the minimum amplitude sometimes is located below half the troposphere depth. This occurs when the zonal correction approaches zero and the atmosphere becomes again unstable with respect to baroclinic instability. A similar histogram is obtained for $\gamma = 1/4$ and $a_0 = -0.5$, while for $\gamma = 1/4$ and $a_0 = 0$ or 0.5 the minimum amplitude is confined at the tropopause (especially for $a_0 = 0.5$).

It must be noted that the resulting minimum amplitude, in average, is higher than the one computed for the classical rigid lid with Ekman dissipation at the surface (see fig. 2). This means that the stratosphere, also in the non-linear Eady model, has a finite amplitude stabilising effect with respect to baroclinic instability.

*On the eddy heat flux parameterization*: The rigid-lid solution appears to be greatly different from those solutions obtained for realistic $\gamma$-values. In these cases, the zonal mean correction has a much weaker time average despite that the time mean eddy heat

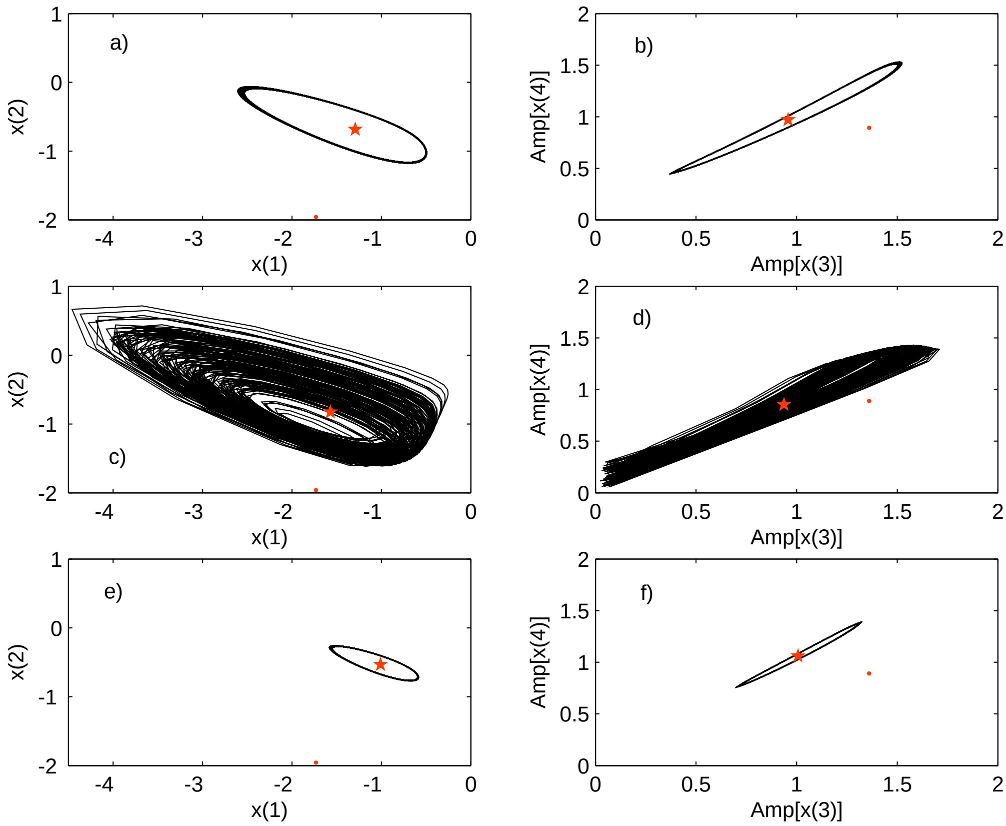


Fig. 5. – Model solutions for $\gamma = 1/2$ and $a_0 = 0$: a) points density in the phase plane $x(1)$ *vs.* $x(2)$, b) points density in the phase plane Amp$[x(3)]$ *vs.* Amp$[x(4)]$. Model solutions for $\gamma = 1/2$ and $a_0 = -0.5$: c) points density in the phase plane $x(1)$ *vs.* $x(2)$, d) points density in the phase plane Amp$[x(3)]$ *vs.* Amp$[x(4)]$. Model solutions for $\gamma = 1/2$ and $a_0 = 0.5$: e) points density in the phase plane $x(1)$ *vs.* $x(2)$, f) points density in the phase plane Amp$[x(3)]$ *vs.* Amp$[x(4)]$. Star denotes the time average solution, while filled circle the rigid-lid solution. The other parameters are as described in sect. **2**. Units are dimensionless.

TABLE II. – *Time mean heat fluxes, time mean zonal corrections* $\langle A_{0,l}\rangle_t$, $\langle B_{0,l}\rangle_t$ *and those computed from* (23), (24) *for different values of* $\gamma$ *and* $a_0$.

| | $\gamma = 0$ | $\gamma = 1/2$ $a_0 = 0$ | $\gamma = 1/2$ $a_0 = -0.5$ | $\gamma = 1/2$ $a_0 = 0.5$ |
|---|---|---|---|---|
| $\langle \text{Flux}\rangle_t$ | 0.48 | 0.45 | 0.54 | 0.35 |
| $\langle A_{0,l}\rangle_t$ | $-1.73$ | $-1.29$ | $-1.57$ | $-1.01$ |
| $\langle B_{0,l}\rangle_t$ | $-1.96$ | $-0.69$ | 0.83 | $-0.53$ |
| $\langle A_{0,l}\rangle_t$ from (23), (24) | $-1.76$ | $-0.92$ | $-1.10$ | $-0.71$ |
| $\langle B_{0,l}\rangle_t$ from (23), (24) | $-1.90$ | 0.59 | 0.71 | 0.46 |

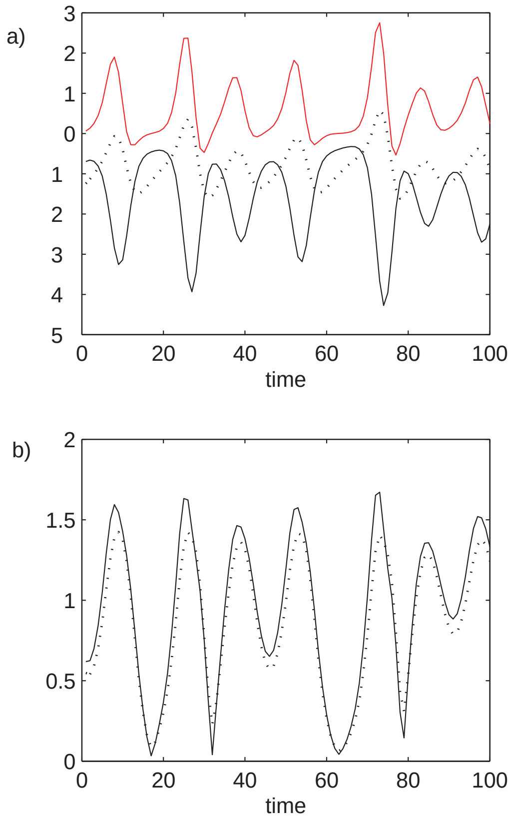


Fig. 6. – Model solutions for $\gamma = 1/2$ and $a_0 = -0.5$: a) Time behaviour of $x(1)$ (solid line), $x(2)$ (dotted line) and heat flux (red line); b) time behaviour of Amp[$x(3)$] (solid line) and Amp[$x(4)$] (dotted line). Units are dimensionless.

transport remains comparable with that for the rigid-lid case. Moreover, if we use the time mean fluxes to compute the corresponding zonal mean corrections by using balances (23) and (24), we find that the resulting corrections for the zonal mean are greatly apart from the actual values. This is illustrated in table II where the time mean heat fluxes, the time mean zonal corrections and those obtained from (23), (24) are listed for the cases $\gamma = 1/2$, $a_0 = 0$, $a_0 = -0.5$ and $a_0 = 0.5$.

In concluding, we note that, due to the erratic nature of the solutions, the heat flux parameterization (24) becomes a sterile expression of a steady-state balance and its usage may be worthless in any further application. Thus, the inclusion of the troposphere/stratosphere dynamics not only forces us to correct our view of the heat fluxes parameterization, but also leads us to the rather bleak outlook that the process may be not parameterized at all.

## 5. – Some GCM results

At a first glance, it seems that the analysis carried out is merely speculative and the results are far from realistic, especially for the set of constraints adopted (we used a highly

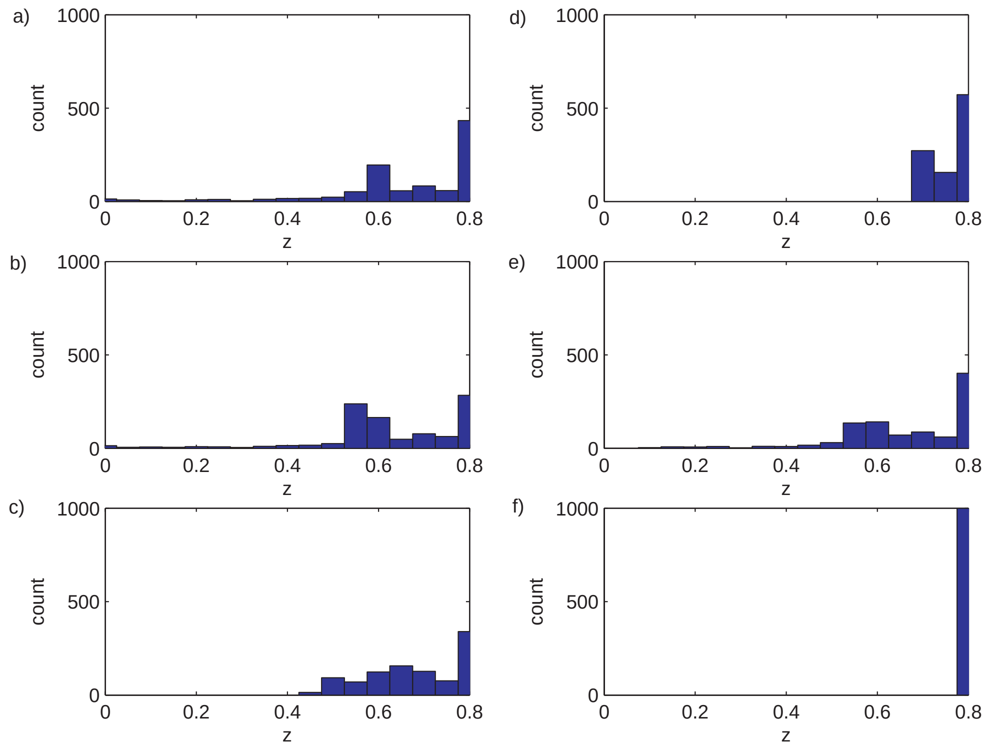


Fig. 7. – Histograms of the height of the minimum amplitude level for a) $\gamma = 1/4$ and $a_0 = 0$, b) $\gamma = 1/4$ and $a_0 = -0.5$, c) $\gamma = 1/4$ and $a_0 = 0.5$, d) $\gamma = 1/2$ and $a_0 = 0$, e) $\gamma = 1/2$ and $a_0 = -0.5$, f) $\gamma = 1/2$ and $a_0 = 0.5$. Units are dimensionless.

simplified model of the atmosphere). Our main result is that, in the parameter range here sampled, the effect of the stratosphere is that of inducing a vacillation of the zonal flow due to the eddy heat fluxes. To extrapolate our result to realistic conditions, we should investigate the observed zonal mean fluctuations in the troposphere and show evidence for a vacillation driven by eddy heat fluxes. Some hint of this behaviour was supported by McGuirk and Reiter [21], who identify a 24-day period vacillation for atmospheric energy parameters, but they left its origin and, more importantly, its connection with the stratosphere, unanswered. Although further progress along this study should be in order, we present evidence of such a vacillation in a simplified General Circulation Model, where it is induced by the eddy heat fluxes and apparently related to a finite static stability in the stratosphere.

The behaviour of the Eady wave-mean flow interaction is tested with a simplified GCM (PUMA, Portable University Model of the Atmosphere, Fraedrich *et al.* [22], available under `http://www.mi.uni-hamburg.de/puma`), which solves the primitive equations on sigma levels on a sphere with optional orography [23]. The diabatic and dissipative processes are represented by Rayleigh friction at the lowest level and Newtonian cooling. At each time step the model temperature is relaxed towards a prescribed restoration temperature field $T_{\mathrm{R}}$, which can be physically interpreted as a radiative-convective equilibrium temperature. The restoration temperature is identical in both hemispheres representing

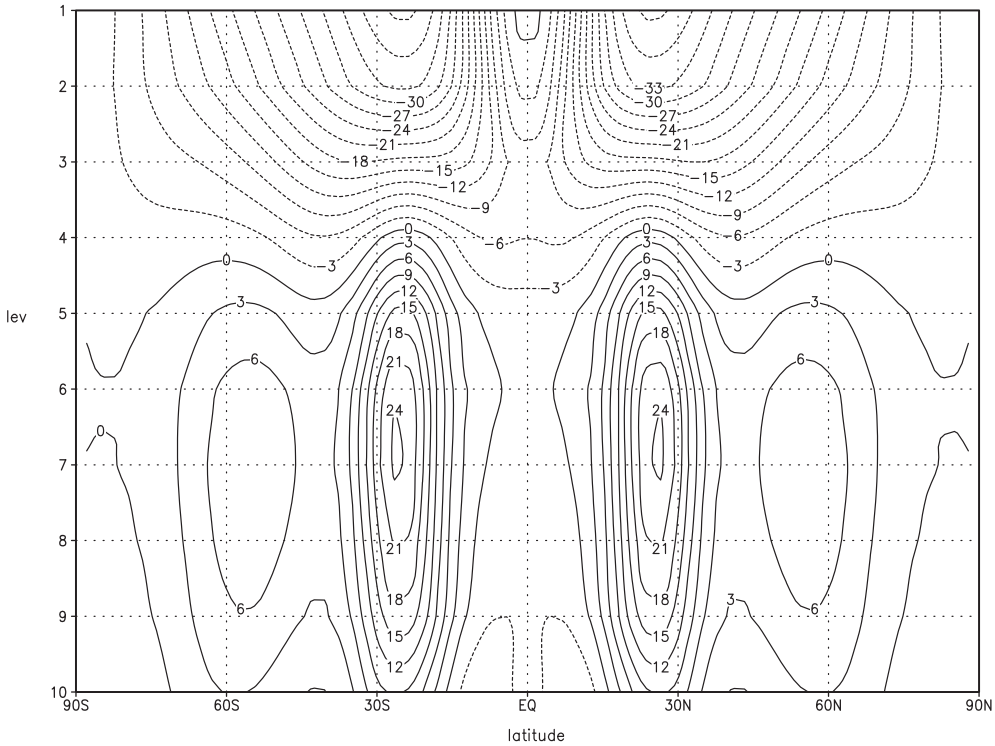


Fig. 8. – Time mean of the zonal wind as a function of latitude and model levels. Unit is $\mathrm{m\,s^{-1}}$.

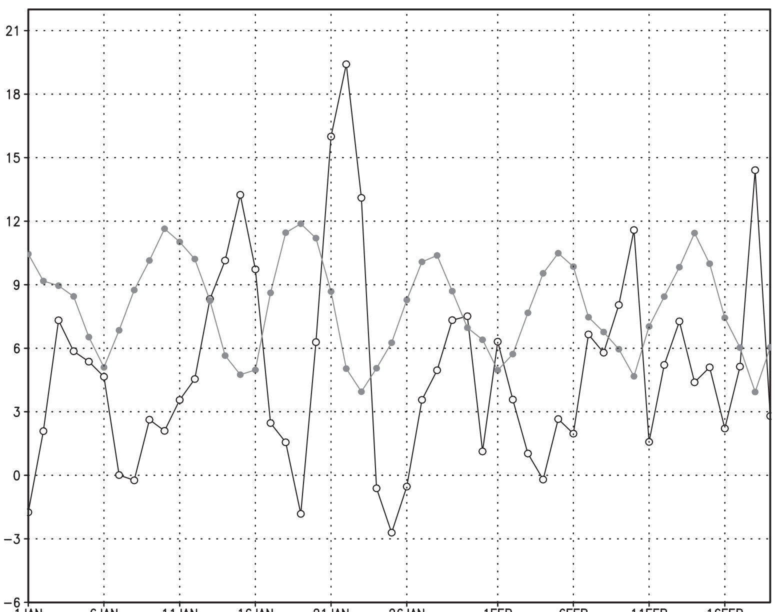


Fig. 9. – Time behaviour of the zonal wind (solid line with filled circles) and eddy heat flux (solid line with open circles) at 55 °N.

hypothetical equinox conditions. It consists of a meridional and vertical contribution as

$$T_{\mathrm{R}}(\varphi,\sigma) = \mathrm{f}(\sigma) T_{\mathrm{R}}(\varphi) + T_{\mathrm{R}}(\sigma)$$

with

$$\begin{aligned}
&T_{\mathrm{R}}(\varphi) = \Delta_{\mathrm{EP}}\left[1/3 - \sin(\varphi)^2\right],\\
&T_{\mathrm{R}}(\sigma) = T_{\mathrm{R}}(\sigma = 1) + \mathrm{lapse(height)},\\
&f(\sigma) = \sin\left(\frac{\pi(\sigma-\sigma_{\mathrm{T}})}{2(1-\sigma_{\mathrm{T}})}\right) \qquad \text{for } \sigma \geq \sigma_{\mathrm{T}},\\
&f(\sigma) = \sin\left(\frac{\pi(\sigma-\sigma_{\mathrm{T}})}{(\sigma_{\mathrm{T}}-\sigma_1)}\right) \qquad \text{for } \sigma < \sigma_{\mathrm{T}}\,,
\end{aligned}$$

where $\varphi$ is the latitude and $\sigma$ the sigma coordinate; $\sigma_{\mathrm{T}}$ denotes the tropopause level, $\sigma_1$ the uppermost level and $\Delta_{\mathrm{EP}}$ the equator to pole temperature difference. Above the tropopause, the equator-to-pole $T_{\mathrm{R}}$-difference may change sign and the temperature gradient may be reversed (this is accomplished by a negative $f(\sigma)$). In the tropopause the lapse rate is set to $6.5\,\mathrm{Kkm}^{-1}$, while in the stratosphere it is $-1\,\mathrm{Kkm}^{-1}$. For our analysis we use T42 horizontal resolution and 10 equally spaced sigma levels. With this set-up, in the absence of eddies, the model converges to the thermal wind balance everywhere with zero zonal mean wind at the ground. If we set $\sigma_{\mathrm{T}} = 0.55$ and we choose initial conditions such that a small non-symmetric perturbation is activated, the model achieves a state having the time mean zonal wind as displayed in fig. 8. It must be noted that a subtropical jet coexists with a mid latitude jet. Inspecting the time behaviour, we see that the subtropical jet (it is important to note that in the present case this jet is maintained by eddies alone, since there is no vertical diffusion of momentum) has minor fluctuations, while the mid-latitude jet undergoes a vacillation. In fig. 9 we show the time series of the zonal wind and the eddy meridional heat flux ($\langle v'T'\rangle$, with $v$ the meridional velocity) at $\sigma = 0.75$ at the same latitude 55N. The evident correlation between the two time series strongly supports a vacillation mechanism similar to the one discussed in the previous section on the Eady wave-mean flow interaction. Of course, the GCM experiments present other relevant features that would require further analysis. We will discuss them in a following paper.

As a final comment, we anticipate that when the tropopause level $\sigma_{\mathrm{T}}$ is decreased substantially, most of the features illustrated above are lost and the model behaviour appears to approach a state with only a single mid-latitude jet (almost stationary) accompanied by almost constant meridional heat fluxes. Since smaller $\sigma_{\mathrm{T}}$ means to approach the rigid-lid condition, we may conclude that the effect of the stratosphere in this simplified GCM appears to be of the same kind as discovered in our Eady wave-mean flow interaction model.

## 6. – Conclusions

The effect of the stratosphere on the Eady model with wave-mean flow interaction as the only non-linearity is analysed. We find that, at variance of the rigid lid, the stratosphere introduces a considerable amount of variability to the model dynamics. In particular, the traditional approach towards eddy parameterisation seems to be disfavoured because the large variability induced by the stratosphere.

Different conclusions may be drawn, if we consider a set of unstable and/or neutral Eady waves far from criticality interacting with the mean flow (*i.e.* for the case when the terms in the eddy field equations, which are quadratic in wave amplitude, cannot be neglected). In this case, presumably, even the rigid-lid case may be highly chaotic and the adjustment process needs to be addressed differently. Probably some progresses may be achieved by using tools of statistical mechanics, as Eady already envisioned in the conclusion of his celebrated paper.

Finally, a careful analysis of results obtained with a simplified GCM shows surprising similarities for the time behaviour of the zonal mean wind and meridional eddy heat transport. This suggests that a simple non-linear Eady-type model coupled to the stratosphere is able to capture the basic effects of the stratosphere-troposphere interactions on circulation dynamics. It should be interesting to investigate whether this behaviour is also supported by re-analysed data.

## Appendix

The dynamical system to be solved consists of eqs. (8), (9), (13) and (14) (see sect. **2**). Note that the thermodynamic equation for the eddy component at the two boundaries can be written in a general form as

$$\text{(A.1)}\quad \partial_{tz}\varphi^{(1)} + \big(U^{(1)} - \partial_y \Phi^{(1)}\big)\partial_{xz}\varphi^{(1)} - \big(\Lambda - \partial_{yz}\Phi^{(1)}\big)\partial_x\varphi^{(1)} + \\ + \delta_{\mathrm{E}}\nabla_H^2\varphi^{(1)} = 0, \qquad \text{at } z = 0,$$

$$\text{(A.2)}\quad \partial_{tz}\varphi^{(1)} + \big(U^{(1)} - \partial_y \Phi^{(1)}\big)\partial_{xz}\varphi^{(1)} - \big(\Lambda - \partial_{yz}\Phi^{(1)}\big)\partial_x\varphi^{(1)} = \\ = \gamma\big[\partial_{tz}\varphi^{(2)} + \big(U^{(2)} - \partial_y \Phi^{(2)}\big)\partial_{xz}\varphi^{(2)} - \big(a_0\Lambda - \partial_{yz}\Phi^{(2)}\big)\partial_x\varphi^{(2)}\big], \quad \text{at } z = H_{\mathrm{T}},$$

where the quantities between the round parentheses are the zonal wind and the zonal wind vertical shear plus their respective corrections.

The system of eqs. (8), (9), (13), (14) can be written in a matrix form as

$$\text{(A.3)}\qquad C(\partial_t x) = F(x),$$

where $x$ is a complex vector with components: $x(1) = A_{0,lb}$, $x(2) = B_{0,lb}$, $x(3) = A_{k,l}$, $x(4) = B_{k,l}$. $C$ is a real matrix $(4 \times 4)$ and $F$ is a complex vector.

The matrix $C$ has all the elements equal to zero except

$$\text{(A.4)}\quad \begin{aligned} C(1,1) &= \alpha^{(1)}{}_{0,lb},\\ C(2,1) &= \alpha^{(1)}{}_{0,lb}\cosh\big(\alpha^{(1)}{}_{0,lb}H_{\mathrm{T}}\big) + \gamma\alpha^{(2)}{}_{0,lb}\sinh\big(\alpha^{(1)}{}_{0,lb}H_{\mathrm{T}}\big),\\ C(2,2) &= \alpha^{(1)}{}_{0,lb}\sinh\big(\alpha^{(1)}{}_{0,lb}H_{\mathrm{T}}\big) + \gamma\alpha^{(2)}{}_{0,lb}\cosh\big(\alpha^{(1)}{}_{0,lb}H_{\mathrm{T}}\big),\\ C(3,3) &= \alpha^{(1)}{}_{k,l},\\ C(4,3) &= \alpha^{(1)}{}_{k,l}\cosh\big(\alpha^{(1)}{}_{k,l}H\big) + \gamma\alpha^{(2)}{}_{k,l}\sinh\big(\alpha^{(1)}{}_{k,l}H_{\mathrm{T}}\big),\\ C(4,4) &= \alpha^{(1)}{}_{k,l}\sinh\big(\alpha^{(1)}{}_{k,l}H_{\mathrm{T}}\big) + \gamma\alpha^{(2)}{}_{k,l}\cosh\big(\alpha^{(1)}_{k,l}H_{\mathrm{T}}\big), \end{aligned}$$

while the components of $F$ are

$$
\begin{aligned}
F(1) &= \delta_E lb^2 x(2) - 2iklI_1\alpha^{(1)}{}_{k,l}[x(4)x^*(3) - x^*(4)x(3)] - \alpha^{(1)}{}_{0,lb}/\tau_0 x(1),\\
F(2) &= -2iklI_1\alpha^{(1)}{}_{k,l}[x(4)x^*(3) - x^*(4)x(3)] - \big[x(1)\cosh\big(\alpha^{(1)}{}_{0,lb}H_T\big)+\\
&\quad + x(2)\sinh\big(\alpha^{(1)}{}_{0,lb}H_T\big)\big]\alpha^{(1)}{}_{0,lb}/\tau_H - \gamma\big[x(1)\sinh\big(\alpha^{(1)}{}_{0,lb}H_T\big)+\\
&\quad + x(2)\cosh\big(\alpha^{(1)}{}_{0,lb}H_T\big)\big]\alpha^{(2)}{}_{0,lb}/\tau_H,\\
F(3) &= -ik\big\{\alpha^{(1)}{}_{k,l}[U_0 + lbI_2 x(2)]x(3) - \big[\Lambda + lbI_2\alpha^{(1)}{}_{0,lb}x(1)\big]x(4)-\\
&\quad - \delta_{\mathrm{E}}(k^2+l^2)/(ik)x(4)\big\},\\
F(4) &= -ik\big\{\alpha^{(1)}{}_{k,l}[U_0 + \Lambda H_{\mathrm{T}}+\\
&\quad + lbI_2\big(x(1)\sinh\big(\alpha^{(1)}{}_{0,lb}H_{\mathrm{T}}\big) + x(2)\cosh\big(\alpha^{(1)}{}_{0,lb}H_{\mathrm{T}}\big)\big)\big]\\
&\quad \big[x(3)\cosh\big(\alpha^{(1)}{}_{k,l}H_{\mathrm{T}}\big) + x(4)\sinh\big(\alpha^{(1)}{}_{k,l}H_{\mathrm{T}}\big)\big]-\\
&\quad - \big[\Lambda + lbI_2\alpha^{(1)}{}_{0,lb}\big(x(1)\cosh\big(\alpha^{(1)}{}_{0,lb}H_{\mathrm{T}}\big)+\\
&\quad + x(2)\sinh\big(\alpha^{(1)}{}_{0,lb}H_{\mathrm{T}}\big)\big)\big][x(3)\sinh(\alpha^{(1)}{}_{k,l}H_{\mathrm{T}})+\\
&\quad + x(4)\cosh\big(\alpha^{(1)}{}_{k,l}H_{\mathrm{T}}\big)\big] + \gamma\big[\alpha^{(2)}{}_{k,l}(U_0 + \Lambda H_{\mathrm{T}}\big)+\\
&\quad + a_0\Lambda + lbI_2\big(\big[\alpha^{(2)}{}_{k,l} - \alpha^{(2)}{}_{0,lb}\big)\big(x(1)\sinh\big(\alpha^{(1)}{}_{0,lb}H_{\mathrm{T}}\big)+\\
&\quad + x(2)\cosh\big(\alpha^{(1)}{}_{0,lb}H_{\mathrm{T}}\big)\big)\big]\big[x(3)\sinh\big(\alpha^{(1)}{}_{k,l}H_{\mathrm{T}}\big)+\\
&\quad + x(4)\cosh(\alpha^{(1)}{}_{k,l}H_{\mathrm{T}})]\}.
\end{aligned}
\tag{A.5}
$$

Here the asterisk denotes the complex conjugate. Finally, $I_1$ and $I_2$ are the following integrals:

$$
I_1 = 1/L_x(2/L_y)^{3/2}\int_0^{Ly}\sin(ly)\cos(ly)\cos(lby)\mathrm{d}y,
$$

$$
I_2 = (2/L_y)^{3/2}\int_0^{Ly}\sin^2(ly)\sin(lby)\mathrm{d}y.
$$

If $l = lb$, $I_1 = 1/L_x(2/L_y)^{3/2}2/(3l)$ and $I_2 = 2L_xI_1$.

* * *

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