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A note on the full non-linear stability of inviscid, planar flows with constant relative vorticity

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Summary. — The non-linear stability of inviscid, planar flows with constant relative vorticity is proved in the context of the quasi-geostrophic shallow-water theory, for simply connected fluid domains of arbitrary shape. First, the result is obtained relative to the enstrophy and kinetic energy norms and, then, it is extended to a "generalised energy" norm which is expressed through the former.

PACS 47.20.Ky – Nonlinearity, bifurcation, and symmetry breaking.
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1. – Introduction

In this paper we study the full non-linear stability of a special class of two-dimensional, uniformly rotating flows of inviscid, incompressible fluids, governed by the shallow-water equations in the quasi-geostrophic approximation. The main feature of the considered basic states is their constant relative vorticity, a problem formally raised by [1], in a simply connected domain of arbitrary shape. The considered basic flows turn out to be nonlinearly stable with respect to both the enstrophy and energy norms and hence to a generalized norm expressed as a function of the former. The stability of constant vorticity flows has been already considered in the context of Eulerian flows [2]. However, the results reported here represent a generalization in a context of geophysical relevance.

2. – Model equations

The governing equations of a uniformly rotating, single-layer incompressible and inviscid fluid on the f-plane are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g\frac{\partial \eta}{\partial x}\,, \tag{2.1}$$

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$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -g\frac{\partial \eta}{\partial y}\,, \tag{2.2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{2.3}$$

In (2.1) and (2.2), $\eta = \eta(x, y, t)$ is the free surface elevation while the Coriolis parameter f is constant. Under the hypothesis

$$-H \le z \le \eta, \tag{2.4}$$

where $z = -H$ represents the flat bottom, and assuming that the horizontal velocity is depth independent, *i.e.*

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0, \tag{2.5}$$

eq. (2.3) can be integrated vertically to yield

$$\frac{\partial \eta}{\partial t} + \vec{\nabla}\cdot\left[\vec{u}(\eta + H)\right] = 0, \tag{2.6}$$

where $\vec{\nabla}$ is the horizontal gradient operator. The fluid domain D on the f-plane is closed and simply connected; hence the condition that there is no mass-flux across the rigid boundary ∂D holds, so

$$\vec{u}\cdot\hat{n} = 0 \quad \text{along} \quad \partial D, \tag{2.7}$$

where $\hat{n}$ is the unit vector locally normal to ∂D. Then, integration of (2.6) on D with the aid of the divergence theorem together with (2.7) expresses the conservation of the total mass of the fluid in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_D \eta\,\mathrm{d}x\mathrm{d}y = 0. \tag{2.8}$$

Among all the motions governed by (2.1), (2.2) and (2.6), only a special subset of those which tend to follow the geostrophic balance are investigated in the present context. To achieve this, the so-called quasi-geostrophic scaling is applied to (2.1), (2.2) and (2.6) and the following non-dimensional (primed) variables are introduced:

$$(x, y) = L(x', y'), \quad t = \frac{L}{U}t', \quad (u, v) = U(u', v'), \quad \eta = \frac{fUL}{g}\eta'. \tag{2.9}$$

Then, in terms of the Rossby number $Ro = U/fL$ and under the further assumption that the Froude number $F = f^2L^2/(gH) = O(Ro)$, after dropping the primes, the non-dimensional governing equations are

$$Ro\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)-v=-\frac{\partial \eta}{\partial x}\,, \tag{2.10}$$

$$Ro\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)+u=-\frac{\partial \eta}{\partial y}\,, \tag{2.11}$$

$$Ro^2\frac{\partial \eta}{\partial t}+\vec{\nabla}\cdot\left[\vec{u}\left(1+Ro^2\eta\right)\right]=0. \tag{2.12}$$

For instance, in the marine framework, one can take $U=O(10^{-1}\,\mathrm{ms}^{-1})$, $L=O(10^5\,\mathrm{m})$ and $H=O(10^3\,\mathrm{m})$, whence $Ro=F=10^{-2}$.

Setting

$$u=-\frac{\partial \psi}{\partial y}+O(Ro),\quad v=\frac{\partial \psi}{\partial x}+O(Ro)\quad \text{and}\quad \eta=\psi+O(Ro), \tag{2.13}$$

the governing equation for the stream function ψ turns out to be (a detailed procedure is expounded, for instance, in [3])

$$\frac{\partial}{\partial t}\nabla^2\psi+J\left(\psi,\nabla^2\psi\right)=0, \tag{2.14}$$

where, in Cartesian coordinates, the Jacobian determinant $J(a,b)\equiv\frac{\partial a}{\partial x}\frac{\partial b}{\partial y}-\frac{\partial a}{\partial y}\frac{\partial b}{\partial x}$.

Condition (2.7) is equivalent to

$$\psi=0\quad\forall(x,y)\in\partial D,\ \forall t, \tag{2.15}$$

where the non-dimensional fluid domain D is hereafter understood. Finally, the non-dimensional version of (2.8) at the geostrophic level of approximation is

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_D\psi\,\mathrm{d}x\mathrm{d}y=0. \tag{2.16}$$

3. – A class of time-independent solutions

A particular class of steady solutions of problem (2.14), (2.15) is given by flows with constant relative vorticity, *i.e.* such that

$$\nabla^2\psi_0=Q_0, \tag{3.1}$$

and

$$\psi_0=0\quad\forall(x,y)\in\partial D, \tag{3.2}$$

where Q_0 is any constant. Obviously, in this case (2.16) is trivially satisfied. The physical meaning of the constant Q_0 can be easily derived from (3.1). Multiplication of (3.1) by ψ_0 and the integration on D, together with the use of (3.2), gives

$$\int_D\vec{\nabla}\psi_0\cdot\vec{\nabla}\psi_0\mathrm{d}x\mathrm{d}y=-Q_0\int_D\psi_0\mathrm{d}x\mathrm{d}y. \tag{3.3}$$

The l.h.s. of (3.3) is twice the total kinetic energy, per unit of mass, of the flow, say

$$\int_D \vec{\nabla}\psi_0 \cdot \vec{\nabla}\psi_0 \mathrm{d}x\mathrm{d}y = 2K_0. \tag{3.4}$$

On the other hand, the vertical component of the integrated angular momentum per unit of mass of the flow is

$$M_{0_z} = \hat{k} \cdot \int_D \vec{r} \times \vec{u}_0 \mathrm{d}x\mathrm{d}y, \tag{3.5}$$

where $\vec{r} = x\hat{\imath} + y\hat{\jmath}$, with $(\hat{\imath}, \hat{\jmath}, \hat{k})$ right handed, and $\vec{u}_0 = \hat{k} \times \vec{\nabla}\psi_0$. Integration of (3.5) with the aid of (3.2) yields

$$M_{0_z} = -2 \int_D \psi_0 \mathrm{d}x\mathrm{d}y. \tag{3.6}$$

Finally, using (3.4) and (3.6) in (3.3), gives

$$K_0 = \frac{Q_0 M_{0_z}}{4}. \tag{3.7}$$

Equation (3.7) expresses the constant Q_0 as a function of the ratio between the kinetic energy and the angular momentum of the flow and implies that M_{0_z} has the same sign as Q_0.

Problem (3.1), (3.2) can be solved analytically if the shape of D is simple enough, for instance by considering

$$D = [0 \le x \le 1] \times [0 \le y \le 1]. \tag{3.8}$$

In case (3.8), the solution of (3.1), (3.2) is given by [4]

$$\psi_0(x, y) = -Q_0 \int_0^1 \int_0^1 G(x, y, \xi, \lambda) \mathrm{d}\xi \mathrm{d}\lambda, \tag{3.9a}$$

where

$$G(x, y, \xi, \lambda) = 4 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\sin(\pi n x) \sin(\pi m y) \sin(\pi n \xi) \sin(\pi m \lambda)}{\pi^2 (n^2 + m^2)}. \tag{3.9b}$$

Solution (3.9a), (3.9b) looks like a vortex locked onto the largest possible scale consistent with the dynamics and the boundary conditions (see, for example, [5]). Since for a two-dimensional flow to be dynamically unstable, energy must pass to larger as well as to smaller scales by the disturbance, thus ψ_0 is expected to be stable to other eigenmode perturbations that have smaller scales. This is the main reason why the stability, in the sense of Lyapunov, of the "basic state" defined implicitly as the solution of problem (3.1), (3.2) is worthy of investigation. Note also that Arnold's theorem [6] does not hold in case (3.1).

4. – The stability problem

The classical approach of the stability theory [7] starts from the substitution of the time-dependent perturbed state

$$\psi(x, y, t) = \psi_0(x, y) + \phi(x, y, t) \tag{4.1}$$

into (2.14) and (2.15), thus yielding the following problem for the generic perturbation ϕ:

$$\frac{\partial}{\partial t}\nabla^2\phi + J(\psi_0 + \phi, \nabla^2\phi) = 0 \quad \forall(x, y) \in D,\ \forall t, \tag{4.2}$$

$$\phi = 0, \quad \forall(x, y) \in \partial D,\ \forall t. \tag{4.3}$$

According to Lyapunov, the basic flow ψ_0 is stable relatively to a certain norm, say $\|\cdot\|$, if the norm of the perturbation $\|\phi\|$ is arbitrarily small for all the subsequent times after that initial, say in $t = 0$, provided that $\|\phi\|$ be small enough initially. In particular, and this is the case of our interest, stability holds if $\|\phi\|$ is conserved in time whatever ϕ may be, that is if

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\phi\| = 0 \quad \forall\phi \in S, \tag{4.4}$$

where S is a suitable functional space to which perturbations belong. In fact (4.4) implies

$$\|\phi(t)\| = \|\phi(0)\| \quad \forall t > 0,$$

so that $\|\phi(0)\| < \varepsilon \Rightarrow \|\phi(t)\| < \varepsilon\ \forall t > 0$, no matter how small ε is, and thus the criterion of Lyapunov is immediately satisfied.

To derive from (4.2) a conservation principle for a suitable norm of the generic disturbance ϕ, the integrals involving the non-linear part of these equations, *i.e.* the Jacobian $J(\phi, \nabla^2\phi)$, must be zero. We anticipate here that this result can be achieved at least in two ways. The first one makes use of the time-dependent functional

$$\|\phi\|_E \equiv \left(\int_D \left(\nabla^2\phi\right)^2 \mathrm{d}x\mathrm{d}y\right)^{1/2}, \tag{4.5}$$

which is also named the enstrophy norm, and consists in multiplying both the terms of (4.2) by $\nabla^2\phi$ and then integrating each product on D with the aid of the no mass-flux boundary condition. The second one makes use of the so-called energy norm,

$$\|\phi\|_K \equiv \left(\int_D \vec{\nabla}\phi \cdot \vec{\nabla}\phi\, \mathrm{d}x\mathrm{d}y\right)^{1/2}, \tag{4.6}$$

and consists in multiplying both the terms of (4.2) by $\psi_0+\phi$, and integrating each product on D, again with the aid of no mass-flux boundary condition. Once that two separate conservation principles are obtained, it is possible to introduce a class of equivalent norms as functions of both $\|\phi\|_E$ and $\|\phi\|_K$, relatively to which the basic state is stable.

4˙1. *Stability in the enstrophy norm.* – It is useful to anticipate the following identity:

$$J(a,b) = \vec{\nabla} \cdot \left(a\vec{\nabla} b \times \hat{k} \right), \tag{4.7}$$

where $\hat{k}$ is the unit vector normal to the f-plane. Now, multiplication of (4.2) by $\nabla^2\phi$ yields

$$\frac{\partial}{\partial t} \left(\nabla^2\phi\right)^2 + J\left(\psi_0 + \phi, \left(\nabla^2\phi\right)^2\right) = 0 \tag{4.8}$$

and the integration of (4.8) on D gives

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_D \left(\nabla^2\phi\right)^2 \mathrm{d}x\mathrm{d}y + \int_D J\left(\psi_0 + \phi, \left(\nabla^2\phi\right)^2\right) \mathrm{d}x\mathrm{d}y = 0. \tag{4.9}$$

Because of (4.7), the divergence theorem and (2.15)

$$\int_D J\left(\psi_0 + \phi, \left(\nabla^2\phi\right)^2\right) \mathrm{d}x\mathrm{d}y = \oint_{\partial D} \left(\psi_0 + \phi\right) \vec{\nabla} \left(\nabla^2\phi\right)^2 \times \hat{k} \cdot \hat{n} \mathrm{d}s = \oint_{\partial D} \psi \vec{\nabla} \left(\nabla^2\phi\right)^2 \cdot \hat{t} \mathrm{d}s = 0,$$

where $\hat{t}$ is the unit vector locally tangent to ∂D. Hence (4.9) simplifies into the conservation statement

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_D \left(\nabla^2\phi\right)^2 \mathrm{d}x\mathrm{d}y = 0 \quad \forall\phi,$$

which implies, according to (4.5),

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\phi\|_E = 0. \tag{4.10}$$

Hence, the unconditional non-linear stability of ψ_0 in the enstrophy norm is established by (4.10).

4˙2. *Stability in the energy norm.* – Multiplication of (4.2) by $\psi = \psi_0 + \phi$ gives

$$\left(\psi_0 + \phi\right) \nabla^2 \frac{\partial\phi}{\partial t} + \frac{1}{2} J\left(\psi^2, \nabla^2\phi\right) = 0. \tag{4.11}$$

By using the identity

$$a\nabla^2 b = b\nabla^2 a + \vec{\nabla} \cdot \left(a\vec{\nabla} b - b\vec{\nabla} a \right)$$

and (4.7) in the form

$$J(a^2, b) = \vec{\nabla} \cdot \left(a^2\vec{\nabla} b \times \hat{k} \right),$$

eq. (4.11) can be written as

$$\nabla^2 \left(\psi_0 + \phi\right) \frac{\partial\phi}{\partial t} + \vec{\nabla} \cdot \left(\psi\vec{\nabla}\frac{\partial\phi}{\partial t} - \frac{\partial\phi}{\partial t}\vec{\nabla}\psi \right) + \frac{1}{2}\vec{\nabla} \cdot \left(\psi^2\vec{\nabla}\left(\nabla^2\phi\right) \times \hat{k} \right) = 0 \tag{4.12}$$

and integration of (4.12) on D with the aid of the divergence theorem together with (2.15) and (4.3) yields

$$\int_D \nabla^2 (\psi_0 + \phi) \frac{\partial \phi}{\partial t} \mathrm{d}x\mathrm{d}y = 0. \tag{4.13}$$

Because of (3.1), eq. (4.13) is equivalent to

$$Q_0 \int_D \frac{\partial \phi}{\partial t} \mathrm{d}x\mathrm{d}y + \int_D \nabla^2 \phi \frac{\partial \phi}{\partial t} \mathrm{d}x\mathrm{d}y = 0,$$

that is to say

$$Q_0 \frac{\mathrm{d}}{\mathrm{d}t} \int_D \phi \, \mathrm{d}x\mathrm{d}y + \int_D \left[\vec{\nabla} \cdot \left(\frac{\partial \phi}{\partial t} \vec{\nabla} \phi \right) - \frac{1}{2} \frac{\partial}{\partial t} \left| \vec{\nabla} \phi \right|^2 \right] \mathrm{d}x\mathrm{d}y = 0. \tag{4.14}$$

Now, because of the steadiness of the basic state, eq. (2.16) implies

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_D \phi \, \mathrm{d}x\mathrm{d}y = 0 \tag{4.15}$$

so, using (4.15) in (4.14) and recalling also (4.3), eq. (4.14) takes the form of the conservation statement

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_D \left| \vec{\nabla} \phi \right|^2 \mathrm{d}x\mathrm{d}y = 0. \tag{4.16}$$

In terms of (4.6), eq. (4.16) yields

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\phi\|_K = 0. \tag{4.17}$$

We stress that enstrophy conservation is obtained also in [2], eq. (11), as a preliminary result to achieve the stability in the energy norm. However, unlike (4.4), in [2] an inequality of the kind

$$\|\phi\|_K \leq \text{constant}$$

is invoked to prove stability, but this procedure may be problematic since it is not clear how δ controls ε as the classical Lyapunov relation

$$\|\phi(0)\| < \delta(\varepsilon) \Rightarrow \|\phi(t)\| < \varepsilon \quad \forall t > 0$$

demands.

5. – Stability in a generalized energy norm

Statements (4.10) and (4.17) imply that the solution of problem (3.1), (3.2) is stable both in the norm (4.5) and in the norm (4.6). These results hold in the full non-linear context and are independent of Q_0 and the shape of the fluid domain D.

As in fluid dynamics no unique norm exists to which the stability/instability of a given basic flow can be referred, the use of so-called generalized norms is suggested [7,8]. In the problem here considered, (4.10) and (4.17) can be unified by resorting to a generalized norm $\|\cdot\|$ of the kind

$$\|\phi\| = \left(C_1\|\phi\|_E^2 + C_2\|\phi\|_K^2\right)^{1/2}, \tag{5.1}$$

where C_1 and C_2 are non-negative constants and $C_1C_2 > 0$. Conservation of $\|\phi\|$, and hence the stability of ψ_0 in norm (5.1), trivially comes from (4.10) and (4.17) whatever C_1 and C_2 may be. Inversely, the norms singled out from (5.1) for different values of C_1 and C_2 are equivalent, so the stability with respect to one of them implies the stability with respect to all of them.

To prove the equivalence, two of them are considered, say $\|\phi\|(C_1', C_2')$ and $\|\phi\|(C_1'', C_2'')$. By means of the positions

$$m' = \min\{C_1', C_2'\}, \; M' = \max\{C_1', C_2'\}, \; m'' = \min\{C_1'', C_2''\}, \; M'' = \max\{C_1'', C_2''\}$$

and

$$\|\phi\|(1,1) = \left(\|\phi\|_E^2 + \|\phi\|_K^2\right)^{1/2}, \tag{5.2}$$

one observes that

$$m'\|\phi\|^2(1,1) \le \|\phi\|^2\,(C_1', C_2') \le M'\|\phi\|^2(1,1) \tag{5.3}$$

and

$$m''\|\phi\|^2(1,1) \le \|\phi\|^2\,(C_1'', C_2'') \le M''\|\phi\|^2(1,1). \tag{5.4}$$

Inequalities (5.3) and (5.4) imply

$$\|\phi\|^2\,(C_1', C_2') \le M'\|\phi\|^2(1,1) = \frac{M'}{m''}m''\|\phi\|^2(1,1) \le \frac{M'}{m''}\|\phi\|^2\,(C_1'', C_2'') \tag{5.5}$$

and

$$\|\phi\|^2\,(C_1'', C_2'') \le M''\|\phi\|^2(1,1) = \frac{M''}{m'}m'\|\phi\|^2(1,1) \le \frac{M''}{m'}\|\phi\|^2\,(C_1', C_2'). \tag{5.6}$$

Finally, inequalities (5.5) and (5.6) prove that each couple of norms of the above-defined class is constituted by equivalent norms.

REFERENCES

[1] Blumen W., *J. Atmos. Sci.*, **25** (1968) 929.
[2] Petroni R. and Vulpiani A., *Nuovo Cimento B*, **78** (1983) 1.
[3] Pedlosky J., *Geophysical Fluid Dynamics* (Springer Verlag) 1987.
[4] Polyanin A. D., *Handbook of Linear Partial Differential Equations for Engineers and Scientists* (Chapman & Hall/CRC) 2002.
[5] Pedlosky J., *Ocean Circulation Theory* (Springer Verlag) 1996.
[6] Arnold V. I., *Sov. Math. Dokl.*, **6** (1965) 773; *Am. Math. Soc. Trans.*, **79** (1969) 267.
[7] Drazin P. G. and Reid W. R., *Hydrodynamic Stability* (Cambridge University Press) 1981.
[8] Straughan B., *The Energy Method, Stability, and Nonlinear Convection* (Springer Verlag) 1992.