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An inner additional condition for solving Munk-like ocean circulation models

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Summary. — The problem of determining the additional boundary conditions, in the framework of the wind-driven single-gyre ocean circulation with lateral diffusion of relative vorticity, is explored with reference to the western boundary and in the context of the classical linear Munk model. As, of necessity, each model solution satisfies a certain inner additional condition, the latter is used to select special kinds of partial-slip boundary conditions, one of which assures the dynamic stability of the linear solution. Moreover, as the boundary conditions are left unaffected by nonlinearity, the same condition can be applied also to the nonlinear model. In particular, an open question about the flow energetics, reported in the literature, is solved by using the present results.

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1. – Introduction

Lateral diffusion of relative vorticity was first introduced by W. Munk [1] in his archetypal model of steady wind-driven ocean circulation to parameterise vorticity dissipation. Although nowadays this model is considered nothing but a classical result in the framework of the so-called homogeneous models of the wind-driven ocean circulation, both Munk's model and those which followed from it exhibit a still unresolved indeterminateness. In fact, the term related to lateral diffusion of relative vorticity raises the order of the governing differential equation thus demanding additional conditions, besides those of no mass-flux, to single out a unique model solution. In the standard approach, *all* the additional conditions have the form of *boundary* conditions, such as no-slip (null tangential velocity) or free-slip (null relative vorticity) or super-slip (null flux of relative vorticity) or hyper-slip (null flux of total vorticity) or, finally, partial-slip (linear and homogeneous superposition of the first three conditions) to be applied along the border

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of the fluid domain. Each additional boundary condition has its own kinematical meaning and influence on the model output, but no criterion is known to select, according to a shared physical viewpoint, the "true" or the "best" additional condition (if any) or, at least, to restrict their set. Here we focus our attention to the western additional condition and put forward an alternative *inner* one. Within the β-plane approximation, we start from the fact that, for every gyre and for every latitude y included in the fluid domain, a turning longitude ξ necessarily exists at which the meridional current of the gyre vanishes. In general, ξ is determined by the full model solution. Unlike this, in the present alternative ξ is fixed *a priori* and the meridional velocity is consistently forced to vanish at ξ, thus releasing any additional condition at the western coastline. The assumption of a traditional sinusoidal wind-stress curl, spanning the whole latitudinal extension of the gyre, considerably simplifies the mathematics and allows us to transform the vorticity equation into an ordinary differential equation whose integral depends parametrically on ξ. In turn, to obtain physically meaningful solutions, ξ must be restricted to a suitable longitudinal interval and, for each ξ of this interval, the model solution is in one-to-one relation with ξ itself. Then, given any model solution parameterised by ξ, the additional condition at the western boundary can be *inferred* from it. In particular, special partial-slip conditions are able to determine every ξ-solution, that is to say, every physically meaningful solution of the linear model. Moreover, additional conditions of this kind assure the dynamic nonlinear stability in the energy norm of every solution of the linear model. It is also possible to find the value of ξ that minimizes the difference between observed and computed (in the linear context) values of basic quantities in the western boundary layer; so, from this viewpoint, the "best" additional condition can be actually derived.

As boundary conditions are left unchanged even if the advection of relative vorticity is added in the vorticity equation, the same additional condition previously inferred is still valid also in the nonlinear context. This fact plays a noticeable role to carry out the energetics of the gyre, which, in general, is sensitive to the kind of additional condition. Unlike other additional conditions, which are problematic in the computation of the gyre energy in a closed fluid domain, the special partial-slip conditions that are considered here successfully lead lateral friction to drain energy, as it is physically expected.

2. – The Munk-like models and the inner condition

We start from the linear, non-dimensional vorticity equation governing the steady wind-driven circulation (notation as in Pedlosky [2] is understood):

$$\frac{\partial \psi}{\partial x} = w_E(x, y) + \varepsilon \nabla^4 \psi \tag{2.1}$$

in the square fluid domain

$$D = [0 \le x \le 1] \times [0 \le y \le 1] \tag{2.2}$$

of the β-plane. The shape of (2.2) means that D is included between two circles of latitude (*i.e.* at $y = 0$ and $y = 1$) along which the wind-stress curl is zero (see the following equation (2.4)) and two coastlines in the meridional direction; this geometry considerably simplifies the subsequent model solution.

The no mass-flux condition across the boundary ∂D is expressed, in terms of the stream function ψ, by

$$\psi(x,y) = 0 \quad \forall (x,y) \in \partial D. \tag{2.3}$$

To describe a rudimentary subtropical single gyre included into (2.2), the simple traditional assumption

$$w_E(x,y) = -\sin(\pi y) \tag{2.4}$$

is made about the vertical component $w_E = \hat{k} \cdot \vec{\nabla} \times \vec{\tau}$ of the wind-stress curl.

Equation (2.1) is the linear approximation of the fully nonlinear vorticity equation, which takes into account the inertial behaviour of the flow; therefore, the eddy viscosity coefficient ε in (2.1) is constrained by the estimate

$$\varepsilon = O(10^{-3}). \tag{2.5}$$

A classic result of geophysical fluid dynamics states that, far enough from the western boundary and in the linear or weakly nonlinear regime, the transport is adequately represented by the so-called Sverdrup flow. In the present context, because of (2.4), the related stream function ψ_{Sv} is

$$\psi_{\mathrm{Sv}}(x,y) = (1-x)\sin(\pi y). \tag{2.6}$$

Since (2.6) implies

$$\nabla^2 \psi_{\mathrm{Sv}} = -\pi^2 \psi_{\mathrm{Sv}}, \tag{2.7}$$

the identity

$$\nabla^2 \psi_{\mathrm{Sv}} = 0 \quad \forall (x,y) \in C \tag{2.8}$$

holds true along the zonal and eastern parts of ∂D, say C. An equation of the same form as (2.8) is assumed for the overall model solution ψ, setting

$$\nabla^2 \psi = 0 \quad \forall (x,y) \in C. \tag{2.9}$$

In the latter case, (2.9) constitutes the *additional* boundary condition (so-called free-slip) that will be imposed along C for solving the model governed by (2.1) [3].

Finally, unlike all the Munk-like models considered in the literature, we renounce every *a priori* additional condition at the western boundary in favour of the *inner additional condition*

$$\left(\frac{\partial \psi}{\partial x}\right)_{x=\xi} = 0, \quad \forall y \in [0,1]. \tag{2.10}$$

In (2.10), $\xi \in \,]0,1[$ is, for a given single-gyre flow, the unique longitude at which the meridional current $\frac{\partial\psi}{\partial x}$ necessarily reverses its sign, *i.e.*

$$\frac{\partial\psi}{\partial x} > 0 \text{ (northward flow)} \quad \forall x < \xi \quad \text{and} \quad \frac{\partial\psi}{\partial x} < 0 \text{ (southward flow)} \quad \forall x < \xi. \tag{2.11}$$

We stress that (2.10) is nothing but a consequence of Rolle's theorem applied to ψ in the interval $0 < x < 1$, while the uniqueness of ξ, according to (2.11), is demanded by the phenomenology of the actual flow patterns.

To summarise, the present model is composed of eq. (2.1), the no mass-flux boundary condition (2.3) along ∂D, the additional boundary condition (2.9) along C and the inner additional condition (2.10) for a certain longitude ξ. Note that, because of (2.3), (2.9) is equivalent to

$$\left(\frac{\partial^2\psi}{\partial y^2}\right)_{y=0} = \left(\frac{\partial^2\psi}{\partial y^2}\right)_{y=1} = 0 \tag{2.12}$$

and

$$\left(\frac{\partial^2\psi}{\partial x^2}\right)_{x=1} = 0. \tag{2.13}$$

The trial stream function

$$\psi(x,y) = \Phi(x)\sin(\pi y) \tag{2.14}$$

identically satisfies condition (2.12); moreover its substitution into (2.1) leads to the following ordinary differential equation for the sole Φ (primes and Roman superscripts denote x-derivatives):

$$\Phi' = -1 + \varepsilon\left(\Phi^{\mathrm{IV}} - 2\pi^2\Phi'' + \pi^4\Phi\right). \tag{2.15}$$

Equation (2.15) is supplemented by the no mass-flux boundary condition at the meridional boundaries coming from (2.3), *i.e.*

$$\Phi(0) = 0 \quad \text{and} \quad \Phi(1) = 0, \tag{2.16}$$

by the inner additional condition (2.10), *i.e.*

$$\Phi'(\xi) = 0 \tag{2.17}$$

and by the free-slip boundary condition (2.13), *i.e.*

$$\Phi''(1) = 0. \tag{2.18}$$

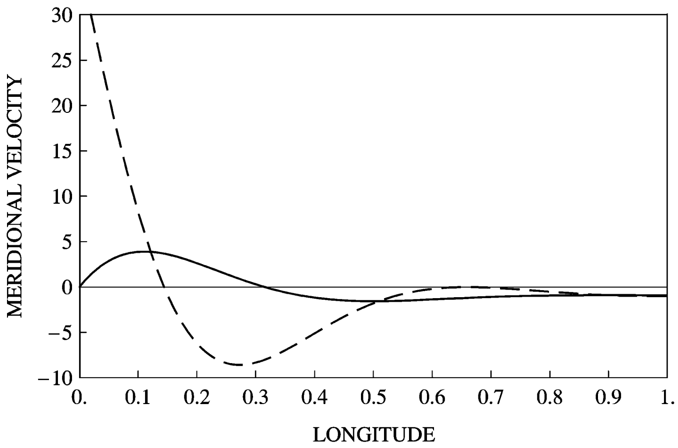

Fig. 1. – Mid-basin ($y = 1/2$) meridional velocities $\Phi'(x;\xi)$ related to the extreme values of ξ (hatched line is for $\xi = 0.144$, continuous line is for $\xi = 0.315$). The marked difference between these profiles is ascribed only to the different values of the turning longitude in solving problem (2.15)-(2.18).

3. – The range of validity of the inner condition

To stress that the solution of problem (2.15)-(2.18) depends parametrically on the turning longitude ξ, we denote its solution with $\Phi(x;\xi)$ rather than $\Phi(x)$. This problem (for instance with $\varepsilon = 10^{-3}$ in accordance with (2.5)) can be solved in principle for every $\xi \in]0,1[$; but only the solutions satisfying the double constraint (2.11) are qualitatively consistent with the phenomenology of the subtropical single gyres. In other words, inequalities (2.11) constitute a test of physical validity for $\Phi(x;\xi)$, and a numerical analysis of all the mathematically admissible solutions shows that the test is passed if and only if

$$0.144 \leq \xi \leq 0.315. \tag{3.1}$$

Interval (3.1) is closer to the western boundary than to the eastern one, in accordance with the westward intensification of the meridional flow, which is implicit in the dynamics governed by (2.1). We expect that, for $\varepsilon \neq 10^{-3}$ and/or $w_E(x,y) \neq -\sin(\pi y)$, the range (3.1) slightly changes; but we disregard different choices of ε and w_E. We shall return to this point at the end of sect. **5**. In any case, by varying ξ into (3.1), all the solutions of the above posed model are obtained. In particular, a noticeable feature of the special solution $\psi = \Phi(x;0.315)\sin(\pi y)$ is

$$\Phi'(0;0.315) = 0, \tag{3.2}$$

that is to say $\psi = \Phi(x;0.315)\sin(\pi y)$ satisfies the classical no-slip boundary condition at the western boundary. The plots of $\Phi'(x;0.315)$ and $\Phi'(x;0.144)$ represent the mid-latitude meridional velocities in the extreme cases of (3.1) and are depicted in fig. 1.

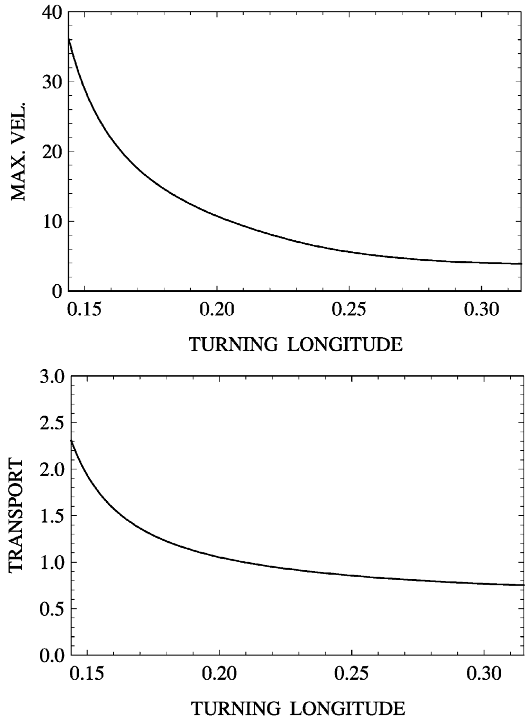

Fig. 2. – Upper panel: maximum mid-basin ($y = 1/2$) northward velocity (4.1) as a function of ξ. Lower panel: mid-basin ($y = 1/2$) northward transport (4.2) as a function of ξ.

4. – On the "best" inner condition

The main feature of the simple circulation model governed by (2.1) lies in its ability to explain the westward intensification of the wind-driven current. However, the computation of basic dynamic variables in the western boundary layer, such as the northward transport M or the northward velocity v as well as the width δ of the layer itself, depends markedly on the additional condition imposed at the western boundary. Therefore, not every additional condition equally contributes to westward intensification, and we expect that the same fact happens also if, alternatively, the inner additional condition (2.10) is used with different values of ξ. Hence, we can single out the turning longitude ξ of (3.1) that minimizes the difference between the observed and theoretical values taken by M, v and δ. The crudeness of the linearity assumption prevents the achievement of a minimization which is quantitatively satisfactory; but, in spite of this, a univocal selection of the best (in the above said sense) turning longitude can be attained.

The maximum northward current referred to the mid-basin latitude (*i.e.* for $y = 1/2$)

$$v(\xi) = \max_{0 \le x \le \xi} \Phi'(x; \xi), \tag{4.1}$$

and the northward transport at the same latitude

$$M(\xi) = \int_0^\xi \Phi'(x;\xi)\mathrm{d}x = \Phi(\xi;\xi), \tag{4.2}$$

are evaluated by means of the solution of problem (2.15)-(2.18), while the width of the western boundary layer is simply

$$\delta(\xi) = \xi, \tag{4.3}$$

where the turning longitude ξ satisfies (3.1). The plots of (4.1) and (4.2) are reported in fig. 2. In particular, because of the monotonic decreasing behaviour of v and M, we have

$$v_{\max} \equiv \max_\xi v(\xi) = v(0.144) \approx 36, \tag{4.4}$$

$$M_{\max} \equiv \max_\xi M(\xi) = M(0.144) \approx 2.3, \tag{4.5}$$

while, because of (3.1),

$$\min_\xi \delta(\xi) = 0.144. \tag{4.6}$$

To evaluate the dimensional values of (4.4)-(4.6), we introduce the typical interior velocity U and the interior horizontal length scale L, where

$$U = O(10^{-2}\,\mathrm{m/s}), \tag{4.7}$$

$$L = O(10^6\,\mathrm{m}). \tag{4.8}$$

Moreover, the dimensional Sverdrup balance yields the estimate

$$ULH = O\left(\frac{\tau}{\rho\beta}\right) = O\left(10^7\,\mathrm{m^3/s}\right), \tag{4.9}$$

where we consider the depth of the motion $H = O(10^3\,\mathrm{m})$, the wind-stress $\tau = O(10^{-1}\,\mathrm{Pa})$, the sea water density $\rho = O(10^3\,\mathrm{kg/m^3})$ and the planetary vorticity gradient $\beta = O(10^{-11}\,\mathrm{m^{-1}s^{-1}})$.

On the whole, eqs. (4.4)-(4.9) show that, for $\xi = 0.144$, the model yields the maximum dimensional northward velocity, *i.e.* $Uv_{\max} \approx 0.36\,\mathrm{m/s}$, the maximum dimensional northward transport, *i.e.* $ULHM_{\max} \approx 23\,\mathrm{Sv}$ and the minimum dimensional boundary layer width, *i.e.* $L\delta_{\min} \approx 140\,\mathrm{km}$. On the other hand, the orders of magnitude of the observed values are

$$v_* = O(1\,\mathrm{m/s}) > Uv_{\max},\ M_* = O(3\,\mathrm{Sv}) > ULHM_{\max},\ \delta_* = O(50\,\mathrm{km}) < L\delta_{\min}, \tag{4.10}$$

so the conclusion is that, even if every solution of the linear model with $0.144 \le \xi \le 0.315$ underestimates both the northward velocity and the northward transport and overestimates the western boundary layer width, the inner additional condition that minimizes these differences is for

$$\xi = 0.144. \tag{4.11}$$

5. – Connection between the inner and the western boundary conditions

In the range (3.1), the meridional velocity at the western boundary is strictly positive, namely,

$$\Phi'(0;\xi) > 0. \tag{5.1}$$

Then, in such range, setting

$$Q_{21}(\xi) = \Phi''(0;\xi)/\Phi'(0;\xi) \tag{5.2}$$

and

$$Q_{31}(\xi) = \Phi'''(0;\xi)/\Phi'(0;\xi), \tag{5.3}$$

the identities

$$Q_{21}(\xi)\Phi'(0;\xi) - \Phi''(0;\xi) = 0 \tag{5.4}$$

and

$$Q_{31}(\xi)\Phi'(0;\xi) - \Phi'''(0;\xi) = 0 \tag{5.5}$$

follow, respectively. Therefore, if the previous problem (2.15)-(2.18) is re-stated by substituting in place of (2.17) one of the following partial-slip conditions (the notation $\Phi(x)$ is here restored because ξ disappears from the model if (2.17) is absent)

$$\lambda\Phi'(0) - \Phi''(0) = 0, \tag{5.6}$$

or

$$\lambda\Phi'(0) - \Phi'''(0) = 0, \tag{5.7}$$

then the latter problem admits a unique solution provided that (5.6) can be written in the form (5.4) or, alternatively, (5.7) can be written in the form (5.5). In case (5.6), comparison with (5.4) shows that this happens if the equation

$$Q_{21}(\xi) = \lambda \tag{5.8}$$

in the unknown ξ has a solution in the interval (3.1). In case (5.7), comparison with (5.5) shows that the same condition involves the equation

$$Q_{31}(\xi) = \lambda. \tag{5.9}$$

From fig. 3 we ascertain that $Q_{21}(\xi)$ is a monotonically increasing function of its argument, such that $Q_{21}(0.144) = -8.57$, while $Q_{31}(\xi)$ is a monotonically decreasing function and $Q_{31}(0.144) = -4.71$, so eqs. (5.8) and (5.9) have, at most, one solution. We see that the partial-slip condition (5.6) is equivalent to (2.17) provided that $\lambda \geq -8.57$, while the partial-slip condition (5.7) is equivalent to (2.17) provided that $\lambda \leq -4.71$.

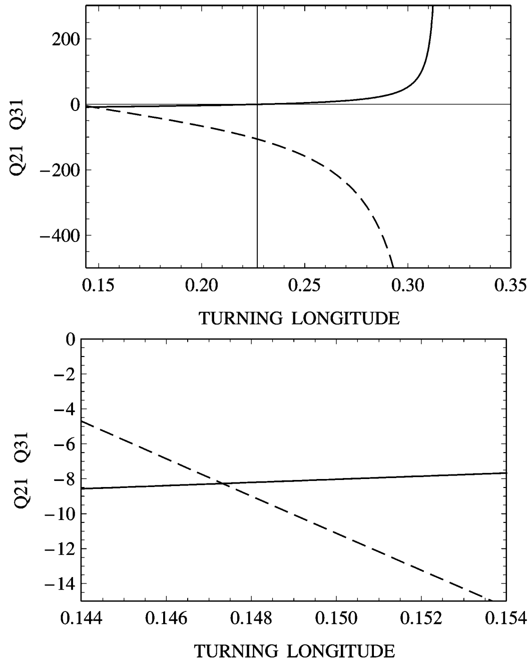

Fig. 3. – Behaviour of the functions $Q_{21}(\xi)$ (continuous line) and $Q_{31}(\xi)$ (hatched line), defined by (5.3) and (5.4), respectively. The upper panel refers to the full domain (3.1) while the lower panel shows the details in the proximity of $\xi = 0.144$. These functions are monotonic and diverge in the limit for $\xi \to 0.315$. The longitude $\bar{\xi} = 0.227$, marked in the upper panel, is that related to the free-slip condition, as shown in sect. **5**.

Figure 3 also shows that (5.6) and (5.7) are reciprocally equivalent, in the sense that if a certain solution is obtained by means of the boundary condition

$$\lambda_a \Phi'(0) - \Phi''(0) = 0 \tag{5.10}$$

for a suitable λ_a, then the same solution follows by substituting (5.10) with the condition

$$\lambda_b \Phi'(0) - \Phi'''(0) = 0, \tag{5.11}$$

where λ_b is such that

$$Q_{21}^{-1}(\lambda_a) = Q_{31}^{-1}(\lambda_b), \tag{5.12}$$

and *vice versa*. In particular, because of the relations $Q_{21}^{-1}(-8.57) = Q_{31}^{-1}(-4.71) = 0.144$, both the boundary conditions

$$8.57\Phi'(0) + \Phi''(0) = 0 \tag{5.13}$$

and

$$4.71\Phi'(0) + \Phi'''(0) = 0 \tag{5.14}$$

single out the same solution $\Phi(x; 0.144)$, whose derivative is shown in fig. 1.

The free-slip condition

$$\nabla^2\psi = 0 \quad \text{at } x = 0, \tag{5.15}$$

with ψ given by (2.14), takes at the western boundary the form

$$\Phi''(0) = 0. \tag{5.16}$$

Equation (5.16) is met by (5.6) with $\lambda = 0$ and the related turning longitude, say $\bar{\xi}$, is given by

$$\bar{\xi} = Q_{21}^{-1}(0) = 0.227. \tag{5.17}$$

The super-slip boundary condition

$$\hat{n} \cdot \vec{\nabla}(\nabla^2\psi) = 0, \quad \text{at } x = 0 \tag{5.18}$$

with ψ given by (2.14) and $\hat{n} = -\hat{\imath}$, takes the form

$$\pi^2\Phi'(0) - \Phi'''(0) = 0. \tag{5.19}$$

Comparison of (5.19) with (5.7) leads to the identification

$$\lambda = \pi^2 \tag{5.20}$$

but we know that no solution exists if, in (5.7), we set $\lambda > -4.71$. The conclusion is that (5.18) cannot be applied to the present model. However, a previous investigation [4] suggests that a suitable increase of the eddy viscosity coefficient ε modifies the shape of $Q_{31}(\xi)$ in such a way that the equation

$$Q_{31}(\xi) = \pi^2 \tag{5.21}$$

has a solution. For instance, if $\varepsilon = 10^{-2}$ is taken in place of (2.5), then the range (3.1) changes into

$$0.248 < \xi < 0.519 \tag{5.22}$$

and one obtains

$$Q_{31}(0.248) = 24.3(> \pi^2). \tag{5.23}$$

This example shows that condition (5.18) can be applied only in the presence of relatively high turbulent viscosity, that is, it demands a relatively large boundary layer width, which makes the related linear model further on far from reality.

6. – Additional conditions and relative vorticity

We reconsider the additional condition (5.6) under the hypothesis $\Phi'(0) > 0$, so only the no-slip condition is excluded in the present discussion. Thus (5.6) implies

$$\frac{1}{\lambda}(\nabla^2\psi)_{x=0} > 0 \quad (\lambda \geq -8.57), \tag{6.1}$$

whence the alternative

$$\lambda > 0 \quad \text{and} \quad (\nabla^2\psi)_{x=0} > 0, \tag{6.2}$$

or

$$\lambda < 0 \quad \text{and} \quad (\nabla^2\psi)_{x=0} < 0 \tag{6.3}$$

follows.

The phenomenon described by (6.2) and (6.3) is due to a bowl of negative relative vorticity which is in any case present in the western area of the fluid domain and whose precise position depends on ξ and induces a consequent change of the sign in the parameter λ appearing in the additional boundary condition (5.6). For negative λ, the bowl

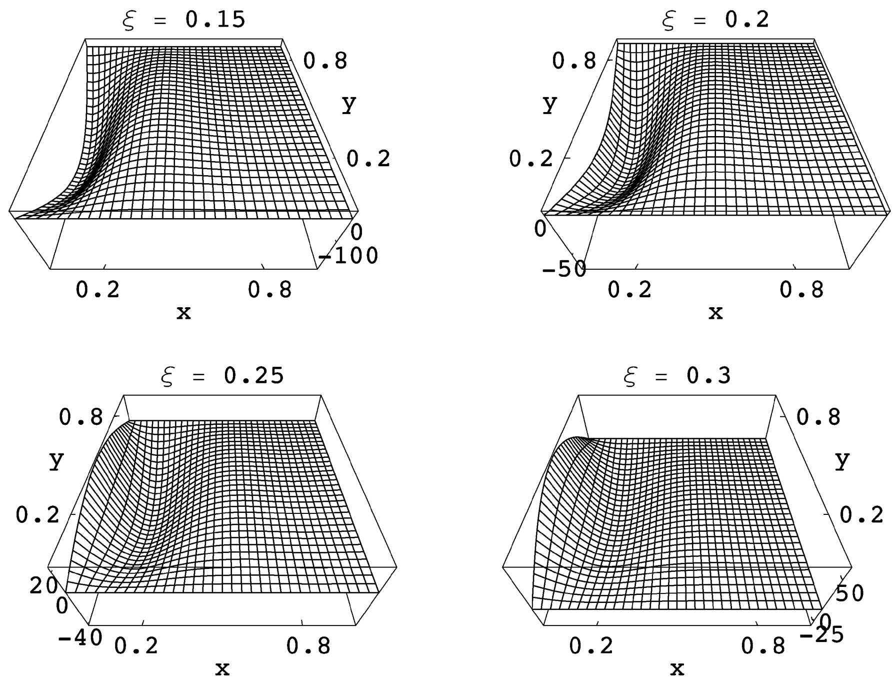

Fig. 4. – 3D plot representation of the relative vorticity $\nabla^2\psi$ of the model solution for increasing values of ξ. Note the different positions of the bowl with negative relative vorticity: for instance, for $\xi = 0.15$ and $\xi = 0.20$ (upper panels), the bowl crosses the western boundary so $\nabla^2\psi < 0$ in $x = 0$, while, for $\xi = 0.25$ and $\xi = 0.30$ (lower panels), the bowl is located on the r.h.s. of the western boundary, so $\nabla^2\psi > 0$ in $x = 0$.

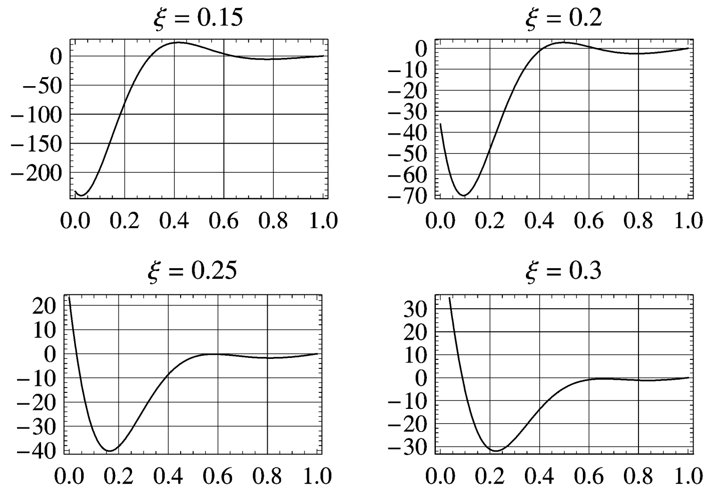

Fig. 5. – Mid-basin ($y = 1/2$) profiles of the surfaces depicted in fig. 4. For increasing values of ξ the minimum of (negative) relative vorticity moves eastward and, at the same time, the relative vorticity at the western boundary increases from a markedly negative value (top left) to a positive one (bottom right).

crosses the western boundary so, in particular, $\nabla^2\psi < 0$ at the western boundary. On the other hand, for positive λ, the bowl of negative vorticity is wholly located in the basin interior; so, necessarily, $\nabla^2\psi > 0$ at the western boundary. Various positions of the bowl are reported in figs. 4 and 5.

7. – Dynamic stability of the model solutions

The time-dependent version of (2.1) is [2]

$$\frac{\partial}{\partial t}\nabla^2\Psi + \frac{\partial\Psi}{\partial x} = w_E(x,y) + \varepsilon\nabla^4\Psi. \tag{7.1}$$

To analyse the stability in the energy norm of the solution (2.14) of problem (2.15), (2.16), (2.18) and (5.6), the perturbed solution

$$\Psi = \psi(x,y) + \varphi(x,y,t) \tag{7.2}$$

is substituted into (7.1) thus yielding the following evolution equation for the disturbance φ:

$$\frac{\partial}{\partial t}\nabla^2\varphi + \frac{\partial\varphi}{\partial x} = \varepsilon\nabla^4\varphi. \tag{7.3}$$

Linearity of (7.1) allows the decoupling of the basic state from the disturbance, as eq. (7.3) shows.

The boundary conditions of φ are the no mass-flux across ∂D, *i.e.*

$$\varphi(x,y,t) = 0 \quad \forall (x,y) \in \partial D, \ \forall t, \tag{7.4}$$

the free-slip along C, *i.e.*

$$\nabla^2\varphi = 0 \quad \forall (x,y) \in C,\ \forall t \tag{7.5}$$

and (5.6), which here becomes

$$\lambda \frac{\partial \varphi}{\partial x} - \frac{\partial^2 \varphi}{\partial x^2} = 0 \quad \text{at } x = 0,\ \forall y \in [0,1],\ \forall t. \tag{7.6}$$

To obtain the time growth rate of the energy norm of the perturbation, defined by

$$E(t) \equiv \left(\int_D \vec{\nabla}\varphi \cdot \vec{\nabla}\varphi \mathrm{d}x\,\mathrm{d}y \right)^{1/2}, \tag{7.7}$$

eq. (7.3) is multiplied by φ, each product is integrated on the fluid domain D with the aid of (7.4) and the repeated use of the divergence theorem. The resulting equation is

$$E\frac{\mathrm{d}E}{\mathrm{d}t} = \varepsilon \oint_{\partial D} \nabla^2\varphi \vec{\nabla}\varphi \cdot \hat{n}\,\mathrm{d}s - \varepsilon \int_D (\nabla^2\varphi)^2 \mathrm{d}x\,\mathrm{d}y. \tag{7.8}$$

If the free-slip or no-slip conditions, *i.e.* $\nabla^2\psi = 0$ or $\vec{\nabla}\varphi \cdot \hat{n} = 0$ respectively, are applied *throughout* the boundary ∂D, then the circuit integral

$$I = \oint_{\partial D} \nabla^2\varphi \vec{\nabla}\varphi \cdot \hat{n}\,\mathrm{d}s \tag{7.9}$$

is zero and the stability in the energy norm follows from the inequality

$$\frac{\mathrm{d}E}{\mathrm{d}t} < 0 \tag{7.10}$$

which results from (7.8). Indeed, because of Poincaré-Wirtinger's inequality (*e.g.*, [5])

$$-\int_D (\nabla^2\varphi)^2 \mathrm{d}x\,\mathrm{d}y < -\alpha_D E^2, \tag{7.11}$$

where $\alpha_D = 2\pi^2$ if D is given by (2.2), the r.h.s. of (7.8) can be bounded so that

$$\frac{\mathrm{d}E}{\mathrm{d}t} < -\varepsilon\alpha_D E \quad (\alpha_D > 0). \tag{7.12}$$

Time integration of (7.12) yields

$$E(t) < E(0)\exp[-\varepsilon\alpha_D t] \tag{7.13}$$

and hence the *asymptotic* stability in the energy norm is derived.

In the present model, because of conditions (7.5) and (7.6), integral (7.9) is not zero but, rather, it takes the form

$$I = -\int_0^1 \left(\frac{\partial^2\varphi}{\partial x^2}\frac{\partial\varphi}{\partial x}\right)_{x=0} \mathrm{d}y. \tag{7.14}$$

Now, (7.6) implies

$$\lambda\left(\frac{\partial\varphi}{\partial x}\right)^2_{x=0} = \left(\frac{\partial\varphi}{\partial x}\frac{\partial^2\varphi}{\partial x^2}\right)_{x=0} \tag{7.15}$$

and the substitution of (7.15) into (7.14) gives

$$I = -\lambda\int_0^1 \left(\frac{\partial\varphi}{\partial x}\right)^2_{x=0} \mathrm{d}y. \tag{7.16}$$

Because of (7.16), eq. (7.8) becomes

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\vec{\nabla}\varphi\right\|^2 = -\varepsilon\left\{\lambda\int_0^1\left(\frac{\partial\varphi}{\partial x}\right)^2_{x=0}\mathrm{d}y + \int_D(\nabla^2\varphi)^2\mathrm{d}x\,\mathrm{d}y\right\} \tag{7.17}$$

and we conclude from (7.17) that, if $\lambda \geq 0$, then (7.10) holds true and the basic state is stable. As (7.17) together with (7.11) imply

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\vec{\nabla}\varphi\right\|^2 < -\varepsilon\int_D(\nabla^2\varphi)^2\mathrm{d}x\,\mathrm{d}y < -\varepsilon\alpha_D\left\|\vec{\nabla}\varphi\right\|^2,$$

also the asymptotic stability immediately follows for non-negative λ.

Recalling moreover that (5.6) presupposes $\lambda \geq -8.57$, we consider separately the range

$$-8.57 \leq \lambda < 0 \tag{7.18}$$

whose analysis is less immediate. By means of the Fourier expansion (recall (7.4))

$$\varphi = \sum_{m,n\geq 1} a_{mn}(t)\sin(m\pi x)\sin(n\pi y) \tag{7.19}$$

we evaluate

$$\int_0^1\left(\frac{\partial\varphi}{\partial x}\right)^2_{x=0}\mathrm{d}y = \frac{\pi^2}{2}\sum_{m,n}a^2_{mn}(t)m^2, \tag{7.20}$$

$$\int_D\left(\nabla^2\varphi\right)^2\mathrm{d}x\,\mathrm{d}y = \frac{\pi^4}{4}\sum_{m,n}a^2_{mn}(t)\left(m^2+n^2\right)^2. \tag{7.21}$$

With the aid of (7.20) and (7.21), eq. (7.17) becomes

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\vec{\nabla}\varphi\right\|^2 = -\varepsilon\frac{\pi^2}{2}\sum_{m,n} a_{mn}^2(t)\left\{\lambda m^2 + \frac{\pi^2}{2}\left(m^2+n^2\right)^2\right\} \tag{7.22}$$

and the sign of the l.h.s. of (7.22) can be investigated in terms of the quantities

$$C_{mn} \equiv \lambda m^2 + \frac{\pi^2}{2}\left(m^2+n^2\right)^2 \tag{7.23}$$

appearing on the r.h.s. of the same equation. Consider

$$C_{11} = \lambda + 2\pi^2. \tag{7.24}$$

First, trivially, $C_{11} > 0$ if and only if

$$\lambda > -2\pi^2. \tag{7.25}$$

Second, elimination of λ between (7.23) and (7.24) yields

$$C_{mn} = \left(C_{11} - 2\pi^2\right)m^2 + \frac{\pi^2}{2}\left(m^2+n^2\right)^2 \geq C_{11} + \frac{\pi^2}{2}\left(m^2-1\right)^2 \geq C_{11},$$

that is

$$C_{mn} \geq C_{11} \quad \forall m \geq 1,\ n \geq 1. \tag{7.26}$$

Thus, if (7.25) is true, (7.26) follows and (7.22) implies (7.10). On the whole, *a fortiori*, stability holds in the full interval

$$-8.57 < \lambda < +\infty. \tag{7.27}$$

We are not able to prove the asymptotic stability of the solution of the linear model in the interval $-8.57 < \lambda < 0$ because of the impossibility to prove that the term

$$\varepsilon|\lambda|\int_0^1\left(\frac{\partial\varphi}{\partial x}\right)^2_{x=0}\mathrm{d}y - \varepsilon\int_D\left(\nabla^2\varphi\right)^2\mathrm{d}x\,\mathrm{d}y,$$

appearing on the r.h.s. of (7.17), is less than $-\varepsilon\int_D(\nabla^2\varphi)^2\mathrm{d}x\,\mathrm{d}y$.

In conclusion, every model solution satisfying (2.11) is dynamically stable in the energy norm (7.7) whatever the turning longitude ξ may be.

We stress that this conclusion is based on (5.6) rather than (5.7), so the former should be preferred to the latter in spite of their equivalence, in the linear context, as shown in sect. **5**.

8. – Application to nonlinear Munk models

The method of the inner additional condition relies on the factorisation (2.14) of the stream function. This possibility is due to the linearity of the governing vorticity equation (2.1) and it is precluded if the nonlinear version

$$RJ\left(\psi,\nabla^2\psi\right)+\frac{\partial\psi}{\partial x}=w_E(x,y)+\varepsilon\nabla^4\psi\quad(R>0)\tag{8.1}$$

of (2.1) is taken into account. However, as boundary conditions (no mass-flux and additional) are left unchanged even if the advection of relative vorticity is added in the vorticity equation, then (5.6), written in terms of ψ rather than Φ, is still valid when (8.1) is considered in place of (2.1). In other words, in the framework of (8.1), the additional boundary condition at the western boundary that coincides with (5.6) in the linear counterpart is

$$\left(\lambda\frac{\partial\psi}{\partial x}-\frac{\partial^2\psi}{\partial x^2}\right)_{x=0}=0.\tag{8.2}$$

Further investigations on the solution of model (2.3), (2.9), (8.1) and (8.2) would demand a numerical approach that goes beyond the scope of this note. In any case, additional condition (8.2) plays an important role in the time-dependent energetics of Munk's models, irrespectively of the linear or nonlinear context. In fact, starting from the time-dependent version of (8.1), that is

$$\frac{\partial}{\partial t}\nabla^2\psi+RJ\left(\psi,\nabla^2\psi\right)+\frac{\partial\psi}{\partial x}=w_E(x,y)+\varepsilon\nabla^4\psi\quad(R\geq 0),\tag{8.3}$$

multiplication of (8.3) by ψ, and the subsequent integration on D with the use of (2.3) and the divergence theorem, gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_D\vec{\nabla}\psi\cdot\vec{\nabla}\psi\,\mathrm{d}x\,\mathrm{d}y=\int_D\vec{u}\cdot\vec{\tau}\,\mathrm{d}x\,\mathrm{d}y-\varepsilon\int_D\left(\nabla^2\psi\right)^2\mathrm{d}x\,\mathrm{d}y+\varepsilon\oint_{\partial D}\nabla^2\psi\vec{\nabla}\psi\cdot\hat{n}\,\mathrm{d}s,\tag{8.4}$$

where $\vec{u}=\hat{k}\times\vec{\nabla}\psi$ is the fluid velocity.

Here is an authoritative comment on the last term of (8.4).

In general, the boundary integral of (8.4) *is not sign determinant. If the flow satisfies either the no-slip condition, for which* $\vec{\nabla}\psi\cdot\hat{n}$ *vanishes, or the slip condition, for which* $\nabla^2\psi$ *vanishes, the boundary term in the energy equation would also vanish, and the effect of the horizontal friction would always be a drain of energy. It is certainly a desirable feature of the dynamics for the lateral mixing definitely to lead to a decrease in energy. However, if* other conditions are used in (8.4), *the boundary term does not vanish, and the effect of lateral friction could actually add energy to the circulation* [2]. In spite of this eventuality, conditions (8.2) and (2.9) lead to the correct decrease in energy, as one can check by resorting to a computation quite similar to that of sect. **7**. In fact, because of (2.9), we have

$$\oint_{\partial D}\nabla^2\psi\,\vec{\nabla}\psi\cdot\hat{n}\,\mathrm{d}s=-\int_0^1\left(\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial x^2}\right)_{x=0}\mathrm{d}y\tag{8.5}$$

and using (8.2) in (8.5) we obtain

$$-\int_0^1 \left(\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial x^2}\right)_{x=0} \mathrm{d}y = -\lambda \int_0^1 \left(\frac{\partial\psi}{\partial x}\right)^2_{x=0} \mathrm{d}y. \tag{8.6}$$

Equation (8.6) means that the energy sink of (8.4) can be written as

$$\begin{aligned} -\varepsilon \int_D \left(\nabla^2\psi\right)^2 \mathrm{d}x\,\mathrm{d}y + \varepsilon \oint_{\partial D} \nabla^2\psi\, \vec{\nabla}\psi \cdot \hat{n}\,\mathrm{d}s &= \\ = \quad -\varepsilon \left(\lambda \int_0^1 \left(\frac{\partial\psi}{\partial x}\right)^2_{x=0} \mathrm{d}y + \int_D \left(\nabla^2\psi\right)^2 \mathrm{d}x\,\mathrm{d}y\right) \end{aligned} \tag{8.7}$$

and we know, from sect. **7**, that, if

$$\lambda > -2\pi^2, \tag{7.25}$$

then

$$\lambda \int_0^1 \left(\frac{\partial\psi}{\partial x}\right)^2_{x=0} \mathrm{d}y + \int_D \left(\nabla^2\psi\right)^2 \mathrm{d}x\,\mathrm{d}y \geq 0. \tag{8.8}$$

Thus, we conclude that the enstrophy integral $\int_D(\nabla^2\psi)^2\mathrm{d}x\,\mathrm{d}y$ of (8.4) compensates, under hypothesis (7.25), the contribution of the circuit integral (8.5) so that, on the whole, lateral friction does actually drain energy.

9. – A remark on the forcing

The inner additional condition (2.10) can be applied not only with forcing (2.4), but also in the more general case in which

$$w_E = \sum_n w_n \sin(n\pi y), \quad n \in N^+ \tag{9.1}$$

whatever the Fourier coefficients w_n may be. In fact, in the linear context, the model solution of the flow forced by (9.1) takes the form

$$\psi = \sum_n \Phi_n(x) \sin(n\pi y) \tag{9.2}$$

and condition (2.10) becomes

$$\sum_n \Phi_n'(\xi) \sin(n\pi y) = 0. \tag{9.3}$$

The orthogonality relation $\int_0^1 \sin(n\pi y)\sin(m\pi y)\mathrm{d}y = 0$ for $n \neq m$, applied to (9.3), yields

$$\Phi_n'(\xi) = 0 \quad \forall n \in N^+ \tag{9.4}$$

and (9.4) means that each Fourier coefficient $\Phi_n(x)$ of the full solution (9.2) is demanded to satisfy *the same inner condition* (2.17).

10. – Conclusion

The results of the present investigation, concerning with Munk-like wind-driven ocean circulation models, are summarised in the following list:

1) First of all, we stress the resort to the concept of inner condition in solving problem (2.1), (2.3), (2.10), (2.12), (2.13): this concept is rather unusual in the theory of partial differential equations, but it is somehow similar to the idea underlying the Nicoletti problem of ordinary differential equations (*e.g.*, [6]).

2) In the framework of problem (2.15)-(2.18), all the physically admissible model solutions are obtained by varying the turning longitude ξ, appearing in the inner additional condition (2.17), in the range (3.1). This means that, whatever the additional condition at the western boundary may be applied in place of (2.17), the related model solution necessarily belongs to the above set of solutions.

3) Whatever the choice of ξ in (3.1) may be, the same model solution can be obtained by substituting to (2.17) a suitable one-parameter partial-slip additional condition of the kind (5.6) or (5.7).

4) Every additional condition of the kind (5.6) assures the dynamic stability, in the energy norm, of the related model solution.

5) The same class of partial-slip conditions (5.6) leads to a self-consistent time-dependent energy equation, which is valid also in the nonlinear context, thus positively answering an until now open question in the framework of the homogeneous models of the wind-driven ocean circulation.

* * *

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REFERENCES

[1] Munk W. H., *J. Meteorol.*, **7** (1950) 79.
[2] Pedlosky J., *Ocean Circulation Theory* (Springer) 1996.
[3] Crisciani F., Cavallini F. and Mosetti R., *Nuovo Cimento C*, **30** (2007) 1-10.
[4] Carnevale G. F., Cavallini F. and Crisciani F., *J. Phys. Oceanogr.*, **31** (2001) 2489.
[5] Straughan B., *The Energy Method, Stability and Nonlinear Convection* (Springer-Verlag) 1992.
[6] Piccinini L. C., Stampacchia G. and Vidossich G., *Ordinary Differential Equations in* $\Re^n$ (Springer Verlag, New-York) 1984.