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Simple way of introducing the optical theorem for non-spherical particles

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Summary. — In case of non-spherical particles, by considering the elements of the amplitude matrix **J**, the form of the optical theorem for the cross-section is obtained in a general way which is faster than how is shown in usual textbooks. It takes into account polarization of incident radiation. Polarized radiation scattered in the forward direction by a thin layer of the medium is added to the incident plane wave. Interference leads to the optical theorem. Dichroism can be avoided by considering two modes of polarization, which can be defined from the elements of the **J** matrix. In this paper: upper-case bold symbols denote vectors or matrices; lower-case bold symbols denote unit vectors.

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1. – Cross-sections for non-spherical particles

The presence of non-spherical particles has been evidenced in natural media, such as atmosphere (ice particles in the high clouds) [1] and biological tissues, *e.g.*, dentine [2] or cornea fibrils [3].

K. Shifrin carried out and headed long and deep studies on the optics of marine water, taking into account non-spherical shapes of suspended particles [4].

Propagation of an electromagnetic plane wave in a medium with suspension of non-spherical particles has particular aspects. They can be dealt with by using a formalism based on considering the Mueller (phase) matrix **M**, as is shown in ref. [1], Chapt. 1. The 16 elements K_{mn} of the **M** matrix are obtained in [5, Sect. 1.VI] from the 4 element f_{mn}

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of the amplitude matrix **J**, which connects the incident wave field $\mathbf{E}^{\rm i}$ to the scattered wave field $\mathbf{E}^{\rm s}$.

The total optical cross-section Ce for a particle is shown in [5, Sect. 1.VII], as a combination of the elements of the Stokes vector of the incident field $\mathbf{E}^{\rm i}$ with the first row elements of **M** taken in the forward direction.

This short note aims at showing an alternative form of Ce, which is simpler and connects directly the two components of $\mathbf{E}^{\rm i}$ with the four elements of the **J** matrix. This particular form of the optical theorem takes into account polarization for the case of non-spherical particles.

Starting with the case of a medium containing one type of particles, the relationship between the incident (i) and scattered (s) fields (transverse components in the far field) via the amplitude scattering matrix (Jones matrix) can be written as

$$\mathbf{E}^{\rm s} = \mathbf{J}\mathbf{E}^{\rm i}$$

with

$$\mathbf{J} = \frac{\exp[ikr]}{r}\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}, \tag{1}$$

where in general the elements f_{mn} depend on the directions of arrival and scattering.

Given the incident field: $\mathbf{E}^{\rm i} = E_0\mathbf{u}\exp[ikz]$, with the unit vector $\mathbf{u} = a_x\mathbf{x} + a_y\mathbf{y}$ defining polarization, we show that the optical theorem for the particle extinction cross-section Ce can be written as

$$Ce = (4\pi/k)\,\mathrm{Im}(f_{11}|a_x|^2 + f_{12}a_xa_y + f_{21}a_xa_y^* + f_{22}|a_y|^2). \tag{2}$$

This is the explicit form of the relationship given by textbooks such as in [5], where the relative phase of incident and scattered field is not always evidenced.

In eq. (2) a_x has been taken as reference for phase, and the matrix elements f_{mn} are taken in the forward direction. For non-spherical particles Ce depends on the direction of incidence. The aim of this paper is to give a proof of eq. (2) which is faster than that usually given in textbooks.

For a medium with a suspension of identical particles with equal orientation the linear extinction coefficient of the medium is given as

$$\sigma_{\rm e} = NCe, \tag{3}$$

with N the number of particles per unit volume.

Equations (2) and (3) can be obtained as follows. Let us define as x, y $(\mathbf{x}, \mathbf{y})$ axes of the reference system those along two directions for which the 2×2 scattering matrix J has the elements f_{11}, f_{12}, f_{21}, f_{22}, which are functions of the direction **s** with respect to the incident wave direction **z**. Let the incident wave field $\mathbf{E}_{\rm i}$ (unitary amplitude) have the complex components a_x, a_y along the x- and y-axes, respectively.

$(|a_x|^2 + |a_y|^2 = 1$. Thus polarization is defined.) Consider an elementary layer (width $\mathrm{d}z$) where the scatterers are contained, and the scattered field on an (x, y)-plane at a distance D (plane D) from the layer. In a generic point on this plane the scattered field has also a component parallel to the z-axis. Apart from this component, which is here neglected, the component $\mathrm{d}\mathbf{E}'^{\rm s}$ in the D plane of the scattered field, due to a volume element $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ of the layer, is related to the components of the incident field by the linear relationship

$$\text{(4)} \qquad \mathrm{d}\mathbf{E}'^{\mathrm{s}} = \exp\left[ik(\rho^2+D^2)^{1/2}\right](\rho^2+D^2)^{1/2}((a_x f'_{11}+a_y f'_{12})\mathbf{x} \\ +(a_x f'_{21}+a_y f_{22}{}')\mathbf{y})N\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$$

$\rho=(x^2+y^2)^{1/2}$, N the number of particles per unit volume. The parameters f'_{mn} in eq. (4) are linear coefficients, depending on positions x, y and D. They connect the field components relative to the same x, y axes both for the incident and scattered fields. In the forward direction they coincide with the f_{mn}.

$\mathrm{d}\mathbf{E}'^{\mathrm{s}}$ has to be added to the incident wave.

If the layer is extended laterally (actually to infinite for the next considerations), one can apply the principle of stationary phase to obtain by integration the field scattered by the whole layer:

$$\text{(5)} \qquad \mathrm{d}\mathbf{E}'^{\mathrm{s}} = (2i\pi/k)\exp[ikD]((a_x f_{11}+a_y f_{12})\mathbf{x}+(a_x f_{21}+a_y f_{22})\mathbf{y})N\mathrm{d}z,$$

where the f_{mn} parameters are the elements of the scattering matrix $\mathbf{J}$ taken in the forward direction. The total field is thus obtained as

$$\mathbf{E}=\mathbf{E}^{\mathrm{i}}+\mathrm{d}\mathbf{E}'^{\mathrm{s}}\,.$$

The applied principle takes into account the value of the integrand at the point where the phase is stationary, that is at $\rho=0$. Due to the applied principle $\mathrm{d}\mathbf{E}^{\mathrm{s}}$ is parallel to $\mathbf{E}^{\mathrm{i}}$.

Since the added $\mathrm{d}\mathbf{E}^{\mathrm{s}}$ is uniform over the plane at D, it can be considered as the field of a plane wave travelling in the z direction and added to the undisturbed wave.

As for power W per unit area at the distance D, taking into account the differential, one has (apart from a constant factor $1/(2Z)$, with Z medium specific impedance)

$$\text{(6)} \qquad W=|\mathbf{E}^{\mathrm{i}}+\mathrm{d}\mathbf{E}^{\mathrm{s}}|^2=|\mathbf{E}^{\mathrm{i}}|^2+2\,\mathrm{Re}(\mathbf{E}^{\mathrm{i}}\cdot\mathrm{d}\mathbf{E}^{\mathrm{s}*}).$$

By considering eq. (5) one has the increment per unit width of the medium ($\mathrm{d}z=1$)

$$\text{(7)} \qquad \mathrm{d}W=(4\pi/k)N\,\mathrm{Im}(f^*_{11}|a_x|^2+f^*_{12}a_x a^*_y+f^*_{21}a^*_x a_y+f^*_{22}|a_y|^2)= \\ =-(4\pi/k)N\,\mathrm{Im}(f_{11}|a_x|^2+f_{12}a^*_x a_y+f_{21}a_x a^*_y+f_{22}|a_y|^2)$$

(with f_{mn} in the forward direction).

If the considered incident power is unitary, one obtains the linear extinction coefficient. From eq. (7) one can verify that the extinction cross-section for the particles in the medium is in agreement with eqs. (2) and (3).

2. – Polarization of the propagating field

Dichroism is a well-known effect which consists of polarization change for light beams propagating in a medium with non-spherical shapes, if particles are not oriented at random [5, Sect. 1.VIII]. However it is always possible to define two polarization "modes", for which polarization is maintained.

In [5, 6] one finds that the effect can be related to the properties of the Mueller matrix (4×4 matrix) pertaining to the medium, and it is only absent in the case of two particular polarization states, which we here name as polarization MODES (eigenvectors in [6]).

These polarization states are obtained in [6, Sect. 3.8] by considering the 4×4 phase matrix, and imposing that the four elements of the Stokes vector of the propagating beam are attenuated in the same proportion. This note shows that they are obtained by considering the properties of the 2×2 Jones matrix (**J** matrix) of the scatterers.

One can impose that the polarization of the scattered field in the forward direction is equal to that of the incident field. That is the ratio of complex amplitude components is the same as that of the incident field:

$$(a_x f_{11} + a_y f_{12})/(a_x f_{21} + a_y f_{22}) = a_x/a_y\,.$$

We thus obtain a second-order equation for the ratio a_x/a_y, and the $\mathbf{E}_x$ and $\mathbf{E}_y$ component of the incident field. By simple algebraic passages one obtains the two modes, which, apart from constant factors to be introduced for normalization, are

$$\mathbf{E}_1 = \mathbf{x} - 2\mathbf{y} f_{21}/(f_{22} - f_{11} + R), \tag{8a}$$

$$\mathbf{E}_2 = \mathbf{y} + 2\mathbf{x} f_{12}/(f_{22} - f_{11} + R), \tag{8b}$$

with $R = ((f_{22} - f_{11})^2 + 4 f_{12} f_{21})^{1/2}$, and the elements f_{mn} of the amplitude matrix are taken in the forward direction.

One can see that eqs. (8a) and (8b) for the polarization modes (directly obtained from the **J** matrix elements) correspond to those of [6, sect. 3.8].

3. – Polydispersion of particles

Since the scattered field components are taken in the strictly forward direction, the expressions for the modes and the linear extinction coefficient of the medium are obtained from those for the monodispersions, with the following substitutions (i: index for the kind of particle):

$$g_{11},\ g_{12},\ g_{21},\ g_{22} \text{ in the place of } f_{11},\ f_{12},\ f_{21},\ f_{22} \tag{9}$$

with:: $g_{mn} = \sum_i N_i f_{mni}$ and N_i the number of particles of kind i per unit volume.

The extinction coefficient becomes

$$\sigma_{\mathrm{e}} = \sum_i N_i Ce_i \tag{10}$$

(indexes i for different particles and different orientation). The polarization modes are obtained from eqs. (8a) and (8b) by substituting the quantities g_{mn} in place of f_{mn}. One can see that eqs. (8a) and (8b) for the polarization modes (directly obtained from the **J** matrix elements) correspond to those of [6, sect. 3.8] where they were obtained by considering the extinction matrix for Stokes vectors.

For eq. (9) it has been taken into account that in the forward direction the relative phases contributions to scattered field from the particles are only due to the f_{mn} matrix elements, and not to differences in paths.

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