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Frictional dissipation at the interface of a two-layer quasi-geostrophic flow

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Summary. — In two-layer ocean circulation models the possible dissipation mechanism arising at the interface between the layers is parameterised in terms of the difference between the horizontal velocities of the flow in each layer. We explain and derive such parameterisation by extending the classical Ekman theory, which originally refers to the surface and to the benthic boundary layers, to the interface of a quasi-geostrophic, two-layered flow.

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1. – Introduction

Two-layer models of circulation of geophysical flows constitute the simplest step in the framework of baroclinic models. For this reason, both two-layered atmospheres and oceans were considered in the early numerical circulation models; according to Salmon [1]: "To a surprising extent, the extra-tropical ocean and atmosphere behave like two-layer fluids". The difference between the current fields in each layer are ascribed to a frictional mechanism, of the same kind of Ekman's boundary layers, but acting in the proximity of the interface between the layers. For instance, in the oceanographic frame, the propagation of the motion with depth has been investigated first by Rhines and Young [2] just by means of a two-layer model equipped with frictional dissipation at the interface. The same problem has been reconsidered, with further details, by Pedlosky in [3]. Usually, this kind of dissipation is parameterised by means of an assumption *ad hoc*, while, in the present investigation, it is derived along the same line adopted to infer the Ekman pumping in the classical theory of Ekman's boundary layers.

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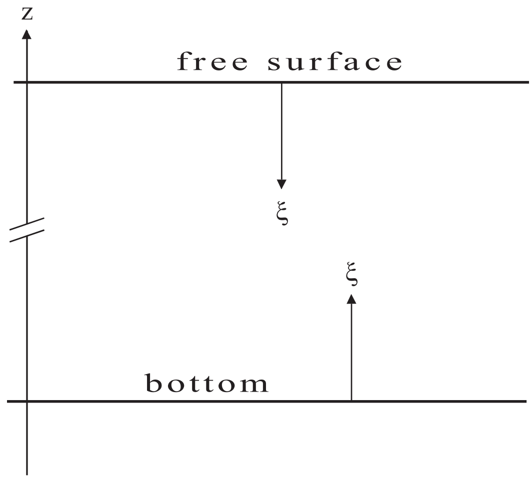

Fig. 1. – Sketch of the positions of the stretched coordinates ξ, relative to the upper and benthic Ekman layers, in the classical model.

2. – Review of the classical Ekman theory

The classical Ekman theory is a basic component, for instance, of the homogeneous model of wind-driven ocean circulation but it suggests also the framework to describe the special dissipative mechanism which typically arises in two-layer circulation models, close to the interface between the layers. For this reason, in the present section the fundamental mathematical content of the theory is outlined, in the context of the quasi-geostrophic dynamics.

In both the upper and benthic Ekman layers the non-dimensional momentum equations are [4]

$$v = v_0(x,y) - \frac{1}{2}\frac{\partial^2 u}{\partial \xi^2}\,, \tag{2.1}$$

$$u = u_0(x,y) + \frac{1}{2}\frac{\partial^2 v}{\partial \xi^2}\,. \tag{2.2}$$

In (2.1) and (2.2), (u_0, v_0) is the geostrophic current of the interior, (u, v) is the current in the considered layer and ξ is the related stretched vertical coordinate pointing, in any case, towards the geostrophic interior (fig. 1).

Equations (2.1) and (2.2) can be decoupled to give

$$\frac{\partial^4 u}{\partial \xi^4} + 4u = 4u_0, \tag{2.3}$$

$$\frac{\partial^4 v}{\partial \xi^4} + 4v = 4v_0. \tag{2.4}$$

The general integrals of (2.3) and (2.4) are

$$u = C_1 \exp[(-1-i)\xi] + C_2 \exp[(-1+i)\xi] + C_3 \exp[(1+i)\xi] + C_4 \exp[(1-i)\xi] + u_0, \tag{2.5}$$

$$v = D_1 \exp[(-1-i)\xi] + D_2 \exp[(-1+i)\xi] + D_3 \exp[(1+i)\xi] + D_4 \exp[(1-i)\xi] + v_0, \tag{2.6}$$

respectively, where $C_i = C_i(x, y)$ and $D_i = D_i(x, y)$. The matching conditions

$$\lim_{\xi\to\infty} u(\xi) = u_0 \quad \text{and} \quad \lim_{\xi\to\infty} v(\xi) = v_0$$

demand $C_3 = C_4 = D_3 = D_4 = 0$, and therefore (2.5) and (2.6) simplify into

$$u = \exp[-\xi][K_1 \sin(\xi) + K_2 \cos(\xi)] + u_0, \tag{2.7}$$

$$v = \exp[-\xi][K_3 \sin(\xi) + K_4 \cos(\xi)] + v_0, \tag{2.8}$$

respectively, where $K_i = K_i(x, y)$. Substitution of (2.7) and (2.8) into (2.1) and (2.2) gives K_3 and K_4 in terms of K_1 and K_2 so one obtains

$$u = \exp[-\xi][K_1 \sin(\xi) + K_2 \cos(\xi)] + u_0, \tag{2.9}$$

$$v = \exp[-\xi][-K_2 \sin(\xi) + K_1 \cos(\xi)] + v_0. \tag{2.10}$$

The coefficients K_1 and K_2 are selected by the remaining boundary condition, as shown below.

Unlike the usual approach, we define the vector function

$$\vec{V} = K_1\hat{\imath} - K_2\hat{\jmath} \tag{2.11}$$

which implies

$$\hat{k} \times \vec{V} = K_2\hat{\imath} + K_1\hat{\jmath}. \tag{2.12}$$

Using (2.11) and (2.12), in vector notation (2.9) and (2.10) are syntetized as

$$\vec{u} = \exp[-\xi] \left[\vec{V} \sin(\xi) + \hat{k} \times \vec{V} \cos(\xi)\right] + \vec{u}_0. \tag{2.13}$$

Although eq. (2.13) is valid for a homogeneous flow, we stress that it will be useful also in dealing with the frictional dissipation at the interface of a two-layer model, as will be shown in the following.

In the framework of the classical Ekman theory, the substitution of

$$\vec{V} = -\frac{\alpha}{2} \left(\hat{k} \times \vec{\tau} + \vec{\tau}\right) \tag{2.14}$$

into (2.13) allows to obtain the solution in the upper layer. The definitions of α and $\vec{\tau}$ are found, for instance, in [4]. Analogously, in the case of the lower layer, the substitution of

$$\vec{V} = \hat{k} \times \vec{u}_0 \tag{2.15}$$

leads to the solution in the benthic layer [4].

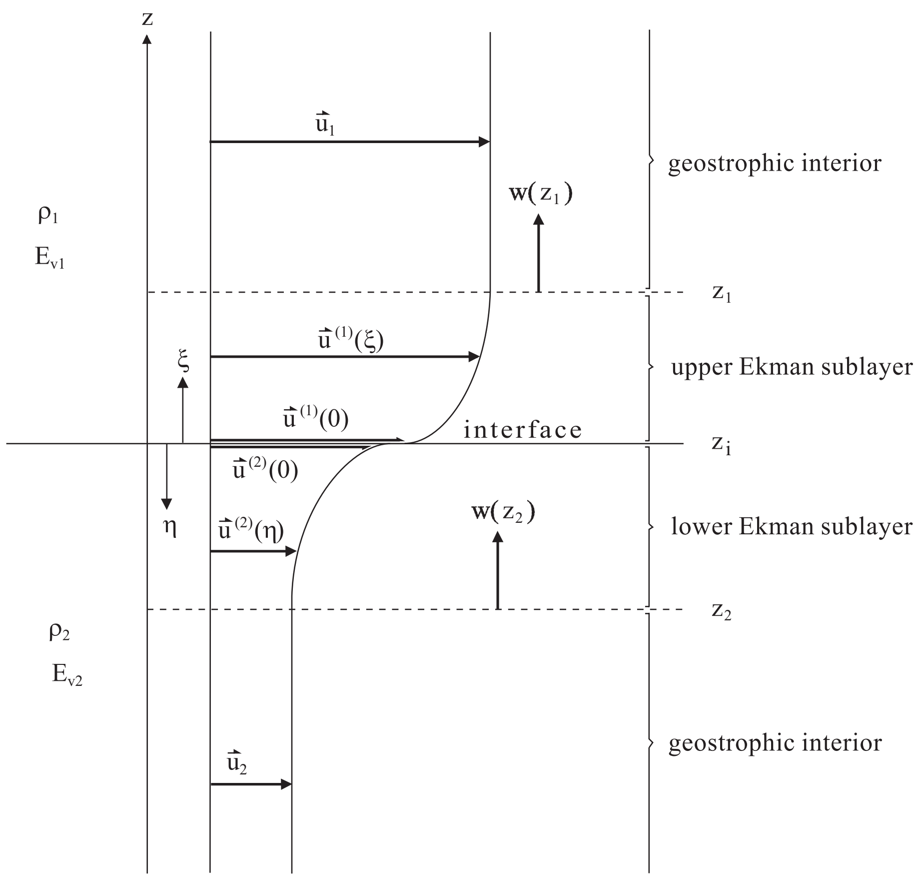

Fig. 2. – Scheme of the geometry, in a vertical plane, of the two-layer model in its nondimensional version. Above $z = z_i$ the fluid density is ρ_1, while below the density is $\rho_2 (> \rho_1)$. Above $z = z_1$ and below $z = z_2$ the currents are geostrophic, while, for $z_2 \le z \le z_1$, the profiles of the depth-dependent horizontal current are given by (4.5) and (4.11). The vertical currents $w(z_1)$ and $w(z_2)$ influence the vorticity of the flow above $z = z_1$ and below $z = z_2$, thus yielding the evolution equations (5.3) and (5.4).

3. – The quasi-geostrophic two-layer model

As mentioned in the Introduction, the simplest model which exhibits baroclinic features of real fluids is the two-layer model. Here, we resort to a very schematic picture of such model (see fig. 2) and retain, into it, only the aspects related to the dynamics of the interface. In what follows, nondimensional quantities are used.

The fluid is included between the sea floor, located in $z = 0$ (flat bottom) and the upper surface $z = 1$ (rigid lid). Unlike the single-layer model, an impermeable and deformable interface, placed at

$$z = z_i(x, y, t), \tag{3.1}$$

separates the lighter water ($\rho = \rho_1$) above z_i from the heavier one ($\rho = \rho_2 > \rho_1$) below

z_i. If the fluid is at rest, $z_1 = \text{const}$, otherwise the deformation of the interface influences the motion of the fluid of each layer and induces a mutual interaction between them. In the latter case, the difference between the geostrophic velocities $\vec{u}_1$ and $\vec{u}_2$ of the layers, far enough from z_i, can be ascribed to a dissipative mechanism, quite similar to that described in sect. **2**. It is assumed to take place above and below the interface, in the proximity of it. While above a certain depth $z_1(> z_i)$ the current $\vec{u}_1$ is geostrophic and, analogously, below another certain depth $z_2(< z_i)$ the current $\vec{u}_2$ is also geostrophic, in both the sublayers $z_i < z \leq z_1$ and $z_2 \leq z < z_i$ the horizontal currents, say $\vec{u}^{(1)}$ and $\vec{u}^{(2)}$ respectively, are not geostrophic because they undergo dissipation. Thus, according to the Ekman theory, $\vec{u}^{(1)}$ and $\vec{u}^{(2)}$ are depth dependent and have a vertical component. In other words, z_1 is the transition depth between the upper geostrophic layer and the underlying sublayer, while z_2 is the transition depth between the lower geostrophic layer and the overhanging sublayer. The vertical velocity, say w, of the fluid in the sublayers propagate up to $z = z_1$ and down to $z = z_2$, thus forcing, through $w_1 = w(z_1)$ and $w_2 = w(z_2)$, the potential vorticity of the geostrophic flows above z_1 and below z_2, respectively. This is, qualitatively, the mechanism by means of which $\vec{u}_1$ and $\vec{u}_2$ are influenced by the friction which develops in the proximity of z_i. Finally, we anticipate that the assumption that friction disappears in the case in which $\vec{u}_1 = \vec{u}_2$ is sufficient to close the model.

To simplify the discussion, both wind forcing and bottom dissipation are disregarded since their presence is not essential for the purposes of this investigation.

4. – The current in the Ekman sublayers

This section is devoted to the determination of the vertical velocities $w_1 = w(z_1)$ and $w_2 = w(z_2)$. We introduce preliminarily the upward stretched coordinate (fig. 2)

$$\xi = \frac{z - z_i}{E_{\mathrm{V1}}^{1/2}} \tag{4.1}$$

and the downward stretched coordinate

$$\eta = \frac{z_i - z}{E_{\mathrm{V2}}^{1/2}}, \tag{4.2}$$

where E_{V1} and E_{V2} are the vertical Ekman numbers of the upper and lower sublayer, respectively.

With reference to the upper one, and in full analogy with (2.13), the current $\vec{u}^{(1)}(\xi)$ in this sublayer is

$$\vec{u}^{(1)}(\xi) = \exp[-\xi]\left[\vec{V}\sin(\xi) + \hat{k}\times\vec{V}\cos(\xi)\right] + \vec{u}_1 \tag{4.3}$$

and, in particular,

$$\vec{u}^{(1)}(0) = \hat{k}\times\vec{V} + \vec{u}_1. \tag{4.4}$$

Equation (4.4) yields $\vec{V} = -\hat{k}\times(\vec{u}^{(1)}(0) - \vec{u}_1)$, whence

$$\vec{u}^{(1)}(\xi) = \exp[-\xi]\left[-\hat{k}\times\left(\vec{u}^{(1)}(0) - \vec{u}_1\right)\sin(\xi) + \left(\vec{u}^{(1)}(0) - \vec{u}_1\right)\cos(\xi)\right] + \vec{u}_1. \tag{4.5}$$

Equation (4.5) implies

$$\vec{\nabla} \cdot \vec{u}^{(1)} = \exp[-\xi] \sin(\xi) \hat{k} \cdot \vec{\nabla} \times \left(\vec{u}^{(1)}(0) - \vec{u}_1 \right), \tag{4.6}$$

where $\vec{\nabla}$ is the horizontal gradient operator. Using the incompressibility equation $\frac{\partial w^{(1)}}{\partial \xi} + E_{V1}^{1/2} \vec{\nabla} \cdot \vec{u}^{(1)} = 0$, written in terms of the variables (x, y, ξ), and (4.6), the vertical velocity in the Ekman sublayer is derived

$$w^{(1)}(\xi) = -E_{V1}^{1/2} \hat{k} \cdot \vec{\nabla} \times \left(\vec{u}^{(1)}(0) - \vec{u}_1 \right) \int_0^{\xi} \exp[-\xi'] \sin(-\xi') \mathrm{d}\xi' + w^{(1)}(\xi = 0). \tag{4.7}$$

In (4.7) we identify $w^{(1)}(\xi = 0)$ with the vertical velocity of the interface, *i.e.*

$$w^{(1)}(\xi = 0) = \frac{\mathrm{D} z_i}{\mathrm{D} t}. \tag{4.8}$$

Finally, the vertical velocity w_1 at the transition depth $z = z_1$ between the geostrophic interior and the underlying sublayer is given by $w(z_1) = \lim_{\xi \to \infty} w^{(1)}(\xi)$, that is to say

$$w(z_1) = -\frac{1}{2} E_{V1}^{1/2} \hat{k} \cdot \vec{\nabla} \times \left(\vec{u}^{(1)}(0) - \vec{u}_1 \right) + \frac{\mathrm{D} z_i}{\mathrm{D} t}. \tag{4.9}$$

Consider then the lower sublayer, where (4.2) holds. Here, again according to (2.13), the current $\vec{u}^{(2)}(\eta)$ is

$$\vec{u}^{(2)}(\eta) = \exp[-\eta] \left[\vec{V} \sin(\eta) + \hat{k} \times \vec{V} \cos(\eta) \right] + \vec{u}_2. \tag{4.10}$$

In this case $\vec{V} = -\hat{k} \times (\vec{u}^{(2)}(0) - \vec{u}_2)$ so (4.10) is equivalent to

$$\vec{u}^{(2)}(\eta) = \exp[-\eta] \left[-\hat{k} \times \left(\vec{u}^{(2)}(0) - \vec{u}_2 \right) \sin(\eta) + \left(\vec{u}^{(2)}(0) - \vec{u}_2 \right) \cos(\eta) \right] + \vec{u}_2. \tag{4.11}$$

From the incompressibility equation $\frac{\partial w^{(2)}}{\partial \eta} - E_{V2}^{1/2} \vec{\nabla} \cdot \vec{u}^{(2)} = 0$ written in terms of the variables (x, y, η) and with $\vec{\nabla} \cdot \vec{u}^{(2)}$ evaluated from (4.11), the vertical velocity in the sublayer is obtained

$$w^{(2)}(\eta) = E_{V2}^{1/2} \hat{k} \cdot \vec{\nabla} \times \left(\vec{u}^{(2)}(0) - \vec{u}_2 \right) \int_0^{\eta} \exp[-\eta'] \sin(-\eta') \mathrm{d}\eta' + \frac{\mathrm{D} z_i}{\mathrm{D} t}. \tag{4.12}$$

Finally, the vertical velocity w_2 at the transition depth $z = z_2$ between the geostrophic interior and the overhanging sublayer is given by $w(z_2) = \lim_{\eta\to\infty} w^{(2)}(\eta)$, that is to say

$$w(z_2) = \frac{1}{2}E_{V2}^{1/2}\hat{k}\cdot\vec{\nabla}\times\left(\vec{u}^{(2)}(0) - \vec{u}_2\right) + \frac{\mathrm{D}z_i}{\mathrm{D}t}\,. \tag{4.13}$$

As anticipated, in the case $\vec{u}_1 = \vec{u}_2$ the above dissipative mechanism is expected to disappear: the simplest way to achieve this is by putting

$$\vec{\nabla}\times\vec{u}^{(1)}(0) = \vec{\nabla}\times\vec{u}_2 \tag{4.14}$$

and

$$\vec{\nabla}\times\vec{u}^{(2)}(0) = \vec{\nabla}\times\vec{u}_1. \tag{4.15}$$

Note that $\vec{u}^{(1)}(0)$ differs from $\vec{u}_2$ in the gradient of a scalar and the same holds true for $\vec{u}^{(2)}(0)$ and $\vec{u}_1$. Under assumptions (4.14) and (4.15), the velocities (4.9) and (4.13) become

$$w(z_1) = \frac{1}{2}E_{V1}^{1/2}\hat{k}\cdot\vec{\nabla}\times\left(\vec{u}_1 - \vec{u}_2\right) + \frac{\mathrm{D}z_i}{\mathrm{D}t} \tag{4.16}$$

and

$$w(z_2) = \frac{1}{2}E_{V2}^{1/2}\hat{k}\cdot\vec{\nabla}\times\left(\vec{u}_1 - \vec{u}_2\right) + \frac{\mathrm{D}z_i}{\mathrm{D}t}\,. \tag{4.17}$$

respectively. In terms of the geostrophic stream functions ψ_1 and ψ_2, such that $\vec{u}_1 = \hat{k}\times\vec{\nabla}\psi_1$ and $\vec{u}_2 = \hat{k}\times\vec{\nabla}\psi_2$, eqs. (4.16) and (4.17) take, respectively, the form

$$w(z_1) = \frac{1}{2}E_{V1}^{1/2}\nabla^2(\psi_1 - \psi_2) + \frac{\mathrm{D}z_i}{\mathrm{D}t} \tag{4.18}$$

and

$$w(z_2) = \frac{1}{2}E_{V2}^{1/2}\nabla^2(\psi_1 - \psi_2) + \frac{\mathrm{D}z_i}{\mathrm{D}t}\,. \tag{4.19}$$

Note that in the absence of frictional dissipation, (4.18) and (4.19) become

$$w(z_1) = w(z_2) = \frac{\mathrm{D}z_i}{\mathrm{D}t}$$

in accordance with the kinematics of an impermeable interface between two fluids. We recall [4] that

$$z_i = \frac{f_0UL}{g'H}(\psi_2 - \psi_1), \tag{4.20}$$

where U, L and H are characteristic of the scale of the model and g' is the reduced gravity. We stress that $\frac{\mathrm{D}z_i}{\mathrm{D}t}$ is a nondimensional quantity, so the explicit form of the nondimensional Lagrangian derivative $\frac{\mathrm{D}}{\mathrm{D}t}$ must be determined. Starting from the dimensional Lagrangian derivative $\frac{\mathrm{D}}{\mathrm{D}t_*}$, we have

$$\frac{\mathrm{D}}{\mathrm{D}t_*} = \frac{\partial}{\partial t_*} + \vec{u}_* \cdot \vec{\nabla}_* = \frac{1}{T}\frac{\partial}{\partial t} + \frac{U}{L}\vec{u}\cdot\vec{\nabla} = \frac{U}{L}\left(\frac{L}{UT}\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla}\right) \tag{4.21}$$

and, in (4.21), we identify the nondimensional Lagrangian derivative with

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{L}{UT}\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla}. \tag{4.22}$$

In the case in which the local time scale $T = (\beta_0 L)^{-1}$, (4.22) becomes

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\beta_0 L^2}{U}\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla}. \tag{4.23}$$

Finally, putting together (4.20) and (4.23), we obtain

$$\frac{\mathrm{D}z_i}{\mathrm{D}t} = \frac{\beta_0 L}{f_0} F\left[\frac{\partial}{\partial t}(\psi_2 - \psi_1) + \left(\frac{\delta_I}{L}\right)^2 \vec{u}\cdot\vec{\nabla}(\psi_2 - \psi_1)\right], \tag{4.24}$$

where the Froude number $F = \frac{f_0^2 L^2}{gH}$ and the inertial boundary layer thickness $\delta_I = (U/\beta_0)^{1/2}$.

5. – The dissipation at the interface in the quasi-geostrophic vorticity dynamics

Consider first the upper geostrophic layer included between the rigid lid, in which, by definition the vertical velocity is zero, and the depth z_1 where (4.18) holds. The dimensional vorticity equation is

$$\frac{\partial \varsigma_1}{\partial t} + \vec{u}_1 \cdot \vec{\nabla}(\varsigma_1 + \beta_0 y) + f_0 w_{1*}/H_1 = 0, \tag{5.1}$$

where ς_1 is the relative vorticity, $\varsigma_1 + f_0 + \beta_0 y$ is the total vorticity, $w_{1*} = UH_1 w_1/L$ is the dimensional vertical velocity at the depth z_1 and H_1 is the thickness of the upper geostrophic layer. The nondimensional version derived from (5.1), written in terms of the geostrophic stream functions ψ_1 and w_1 given by (4.18), has the form

$$\frac{1}{f_0 T}\frac{\partial}{\partial t}\nabla^2\psi_1 + \frac{U}{f_0 L}J(\psi_1, \nabla^2\psi_1) + \frac{\beta_0 L}{f_0}\frac{\partial \psi_1}{\partial x} + \frac{E_{\mathrm{V}1}^{1/2}}{2}\nabla^2(\psi_1 - \psi_2) + \frac{\mathrm{D}z_i}{\mathrm{D}t} = 0. \tag{5.2}$$

Under the assumption of the local time scale $T = (\beta_0 L)^{-1}$ and using (4.24), eq. (5.2) can be written in the final form

$$\frac{\partial}{\partial t}\left(\nabla^2\psi_1 + F_1(\psi_2 - \psi_1)\right) + (\delta_I/L)^2 J(\psi_1, \nabla^2\psi_1 + F_1\psi_2) + \frac{\partial \psi_1}{\partial x} + \frac{\delta_{\mathrm{V}1}}{L}\nabla^2(\psi_1 - \psi_2) = 0, \tag{5.3}$$

where the Froude number $F_1 = \frac{f_0^2 L^2}{g' H_1}$ and $\frac{\delta_{\mathrm{V}1}}{L} = \frac{f_0 E_{\mathrm{V}1}^{1/2}}{2\beta_0 L}$.

In the same way, starting from (4.19) one obtains the vorticity equation of the lower layer that is

$$(5.4)\quad \frac{\partial}{\partial t}\left(\nabla^2\psi_2-F_2(\psi_2-\psi_1)\right)+(\delta_I/L)^2 J(\psi_2,\nabla^2\psi_2+F_2\psi_1)+\frac{\partial\psi_2}{\partial x}+\frac{\delta_{\mathrm{V}2}}{L}\nabla^2(\psi_2-\psi_1)=0,$$

where $F_2=\frac{f_0^2L^2}{g'H_i}$ is the Froude number of the lower layer and $\frac{f_0E_{\mathrm{V}2}^{1/2}}{2\beta_0 L}=\frac{\delta_{\mathrm{V}2}}{L}$.

To summarise, the dissipative mechanism arising at the interface between the layers enters into the quasi-geostrophic vorticity equations (5.3) and (5.4) through the terms $\frac{\delta_{\mathrm{V}i}}{L}\nabla^2(\psi_i-\psi_j)$ with $(i,j)=(1,2)$ for the upper layer and $(i,j)=(2,1)$ for the lower one.

6. – Application to coupled Rossby waves

To show an application of the two-layer model of sect. **5**, here the inertial behaviour of coupled Rossby waves, governed by (5.3) and (5.4), is considered under the following simplifying assumptions:

- The dynamics is linear, that is $(\frac{\delta_I}{L})^2$ is negligibly small.
- The layers have the same thickness, so $F_1=F_2\equiv F$.
- The sublayers have the same thickness so $\frac{\delta_{S1}}{L}=\frac{\delta_{S2}}{L}\equiv r$.

Therefore the vorticity equation for the upper layer, derived from (5.3), is

$$(6.1)\qquad \frac{\partial}{\partial t}\left(\nabla^2\psi_1+F(\psi_2-\psi_1)\right)+\frac{\partial\psi_1}{\partial x}=r\nabla^2(\psi_2-\psi_1)$$

and that for the lower layer, derived from (5.4), is

$$(6.2)\qquad \frac{\partial}{\partial t}\left(\nabla^2\psi_2-F(\psi_2-\psi_1)\right)+\frac{\partial\psi_2}{\partial x}=-r\nabla^2(\psi_2-\psi_1).$$

Because of linearity, trial wavelike solutions of the kind

$$(6.3)\qquad \Phi_1=A_1\exp[i(kx+ny-\sigma t)]$$

and

$$(6.4)\qquad \Phi_2=A_2\exp[i(kx+ny-\sigma t)]$$

are considered. Then

$$(6.5)\qquad \psi_1=\mathrm{Re}\{\Phi_1\}$$

and

$$(6.6)\qquad \psi_2=\mathrm{Re}\{\Phi_2\}$$

satisfy the same equations (6.1) and (6.2) and constitute the physical stream functions of the model. In (6.3) and (6.4) the amplitudes A_1 and A_2 are constant while the frequency σ is determined by substituting (6.3) and (6.4) in (6.1) and (6.2), respectively. The result is given by solving the system of algebraic equations

$$\begin{cases} \left(i\sigma(k^2+n^2)+i\sigma F+ik-r(k^2+n^2)\right)A_1+\left(-i\sigma F+r(k^2+n^2)\right)A_2=0, \\ \left(-i\sigma F+r(k^2+n^2)\right)A_1+\left(i\sigma(k^2+n^2)+i\sigma F+ik-r(k^2+n^2)\right)A_2=0. \end{cases} \tag{6.7}$$

Nontrivial solutions of (6.7) demand

$$\det\begin{pmatrix} i\sigma(k^2+n^2)+i\sigma F+ik-r(k^2+n^2) & -i\sigma F+r(k^2+n^2) \\ -i\sigma F+r(k^2+n^2) & i\sigma(k^2+n^2)+i\sigma F+ik-r(k^2+n^2) \end{pmatrix}=0,$$

whence σ is singled out from

$$i\sigma(k^2+n^2)+i\sigma F+ik-r(k^2+n^2)=-i\sigma F+r(k^2+n^2) \tag{6.8}$$

or from

$$i\sigma(k^2+n^2)+i\sigma F+ik-r(k^2+n^2)=i\sigma F-r(k^2+n^2). \tag{6.9}$$

Hence, the admissible dispersion relations are found to be

$$\sigma=-\frac{k}{k^2+n^2+2F}-2i\frac{r(k^2+n^2)}{k^2+n^2+2F} \tag{6.10}$$

and

$$\sigma=-\frac{k}{k^2+n^2}\,. \tag{6.11}$$

The real part of σ shows that Rossby waves are concerned. Dispersion relation (6.10) implies

$$A_1+A_2=0, \tag{6.12}$$

while (6.11) implies

$$A_1-A_2=0. \tag{6.13}$$

Therefore, in case (6.10), the stream functions (6.5) and (6.6) are given by

$$\psi_1=A_1\exp\left[-2\frac{r(k^2+n^2)}{k^2+n^2+2F}t\right]\cos\left(kx+ny-\frac{k}{k^2+n^2+2F}t\right) \tag{6.14}$$

and

$$\psi_2=-\psi_1, \tag{6.15}$$

respectively. The time decay of the waves in both the layers is provided by the exponential factor appearing in (6.14). Note that the decay rate depends on the wave numbers and thus on the wavelength, as discussed in [4] for the single-layer flow.

On the other hand, in case (6.11) stream functions (6.5) and (6.6) coincide and are given by

$$\psi_1 = \psi_2 = A_1 \cos\left(kx + ny - \frac{k}{k^2 + n^2} t\right). \tag{6.16}$$

Hence, $\nabla^2(\psi_2 - \psi_1)$ is trivially zero, so the system does not decay in time.

7. – Conclusions

The main result of this paper consists in the *derivation* of eqs. (4.18) and (4.19), which are mostly *postulated*, along the same line adopted to infer the Ekman pumping in the classical theory of Ekman's boundary layers. However, unlike the latter case, a definite boundary condition for the flow close to the interface does not seem to be deducible, so an assumption *ad hoc*, that is (4.14) and (4.15), has been introduced to achieve (4.18), (4.19). Finally, an application to quasi-geostrophic coupled Rossby waves has explained the role of this kind of dissipation in the framework of a very simple baroclinic model.

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