IL NUOVO CIMENTO Vol. 31 C, N. 3 Maggio-Giugno 2008
DOI 10.1393/ncc/i2008-10315-4

On the benthic Ekman layer

F. Crisciani([1]), G. Badin([2]) and R. Mosetti([3])

([1]) *CNR, Istituto di Scienze Marine - Trieste, Italy*
([2]) *Department of Earth and Ocean Sciences, Liverpool University - Liverpool, UK*
([3]) *OGS, Istituto Nazionale di Oceanografia e di Geofisica Sperimentale - Trieste, Italy*

(ricevuto il 6 Novembre 2008; revisionato il 12 Gennaio 2009; approvato il 27 Febbraio 2009; pubblicato online il 16 Marzo 2009)

Summary. — A review of the standard model of the benthic Ekman layer is presented and reformulated in terms of relative vorticity in place of horizontal current. In this context, the possibility to use mixed boundary conditions to model this layer is explored. The related model solutions can be cast into two main groups: the first is a generalization of the classical Ekman result, while the second one has some unexpected features which are problematic with respect to the Ekman pumping process. An investigation on the finite amplitude stability via the Lyapunov method shows that solutions belonging to the first group are stable while, in the second group, solutions are unstable. This fact poses a physical constraint to the set of the admissible boundary conditions. Finally, a connection between mass transport and boundary condition at the sea floor is numerically investigated.

PACS `92.10.Dh` – Deep ocean processes.
PACS `47.20.Ft` – Instability of shear flows (*e.g.*, Kelvin-Helmholtz).

1. – Introduction

Bottom friction is the mechanism by means of which Ekman pumping is produced in the proximity of the matching depth of the benthic Ekman layer with the fluid interior situated above it. Ekman pumping is carried out by a suitable vertical velocity which yields a vorticity sink; it plays a basic role for instance to dissipate the vorticity put into the water body by the wind in order that a steady circulation regime can be reached [1].

The flow field of the turbulent benthic layer is, traditionally, the solution of a well-established model [1]; however the possibility to generalise such model has been recently pointed out by Vallis [2] who noted that, in a turbulent flow, the no-slip condition at the bottom may be inappropriate, thus suggesting an alternative, more general boundary condition. The note of Vallis has inspired to the authors an investigation on the same problem, which is the subject of this note. In particular, the stability analysis of the above cited basic state as a function *only* of the boundary condition at the sea floor is carried out in a dynamical regime in which the basic state is stable if the classical no-slip boundary condition is imposed.

The paper is organized as follows. Section **2** is a review of the standard benthic Ekman layer. In sect. **3** a reformulation of the same model in the relative vorticity-vertical velocity fields is presented but mixed boundary conditions, as suggested originally by Vallis, are considered. In particular, the unexpected possibility that the vertical velocity changes its sign is observed. To explain this phenomenon, in sect. **4** the stability for finite amplitude perturbation in Lyapunov sense of the flow field is explored and it is shown that, in the case in which vertical velocity changes its sign, the flow turns out to be unstable. In sect. **5** a connection between the boundary conditions and the mass-transport is discussed based on the results of a homogeneous wind-driven quasigeostrophic model with bottom friction.

2. – The standard model of the benthic Ekman layer

In this section the classical model of the benthic Ekman layer is summarised in its nondimensional form.

In general, the depth-dependent flow (u, v, w) of the benthic layer is the solution of the following governing equations [1]:

$$u = u_0 + \frac{1}{2}\frac{\partial^2 v}{\partial \xi^2}\,, \tag{2.1}$$

$$v = v_0 - \frac{1}{2}\frac{\partial^2 u}{\partial \xi^2}\,, \tag{2.2}$$

$$\frac{\partial w}{\partial \xi} = -E_V^{1/2}\vec{\nabla}\cdot\vec{u}, \tag{2.3}$$

where $\vec{u}_0 = (u_0, v_0)$ is the geostrophic current of the interior above the benthic layer, $\xi = E_V^{-1/2} z$ is the stretched vertical coordinate $(0 \le \xi < \infty)$, positive upwards, $z = 0$ is the vertical coordinate of the bottom which is assumed to be flat, the constant E_V is the vertical Ekman number and $\vec{\nabla}$ is the *horizontal* gradient operator. The no mass-flux boundary condition at the bottom yields

$$w = 0 \quad \text{in } \xi = 0, \tag{2.4}$$

while the no-slip condition means

$$\vec{u}_H = \vec{0} \quad \text{in } \xi = 0. \tag{2.5}$$

Finally, the matching of $\vec{u}_H$ with $\vec{u}_0$ for large ξ is expressed by the asymptotic conditions

$$\lim_{\xi\to\infty} u(x, y, \xi) = u_0(x, y), \tag{2.6}$$

$$\lim_{\xi\to\infty} v(x, y, \xi) = v_0(x, y). \tag{2.7}$$

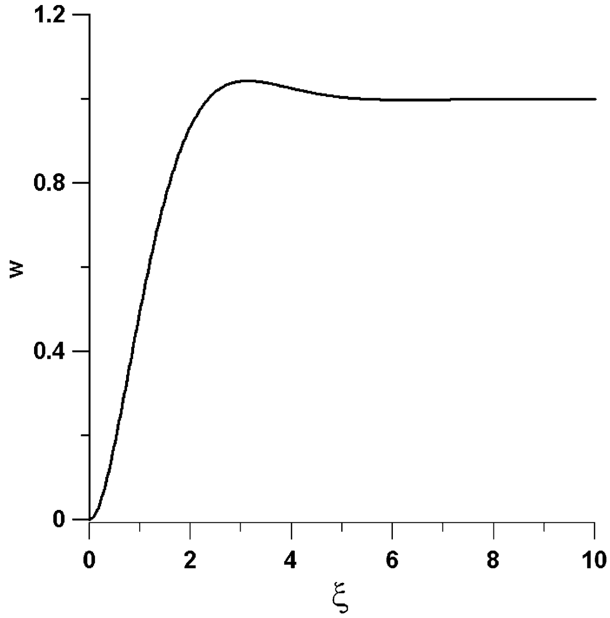

Fig. 1. – Plot of profile of the vertical velocity with $p = 0$.

The solution of problem (2.1)-(2.7) is given by [1]

$$\vec{u}_H = \vec{u}_0 + \exp[-\xi]\left\{\hat{k} \times \vec{u}_0 \sin(\xi) - \vec{u}_0 \cos(\xi)\right\}, \tag{2.8}$$

$$w = \frac{E_V^{1/2}}{2}\varsigma_0 \left\{1 - \exp[-\xi][\cos(\xi) + \sin(\xi)]\right\}. \tag{2.9}$$

A noticeable feature of (2.9), depicted in fig. 1 that coincides with fig. 4.5.1 of [1], is that, since $1 - \exp[-\xi][\cos(\xi) + \sin(\xi)] \geq 0$ $\forall \xi \geq 0$, w and ς_0 are positively correlated throughout the whole benthic layer. In particular, the well-known relationship which expresses bottom friction in terms of the vorticity ς_0 of the geostrophic layer

$$\lim_{\xi \to \infty} w = \frac{E_V^{1/2}}{2}\varsigma_0 \tag{2.10}$$

is recovered.

3. – The $\varsigma - w$ model of the benthic Ekman layer

Equations (2.1) and (2.2) are preliminarily written in vector form as

$$\vec{u}_H = \vec{u}_0 - \frac{1}{2}\frac{\partial^2}{\partial \xi^2}\hat{k} \times \vec{u}_H. \tag{3.1}$$

Then, take first the horizontal divergence of (3.1), recalling that $\vec{\nabla} \cdot \vec{u}_0 = 0$ and the identity $\vec{\nabla} \cdot (\hat{k} \times \vec{u}) = -\hat{k} \cdot \vec{\nabla} \times \vec{u}$; thus, as $\vec{\nabla} \cdot \vec{u}_H = \vec{\nabla} \cdot \vec{u}$ and $\hat{k} \cdot \vec{\nabla} \times \vec{u}_H = \hat{k} \cdot \vec{\nabla} \times \vec{u}$, one finds

$$\vec{\nabla} \cdot \vec{u} = \frac{1}{2} \frac{\partial^2}{\partial \xi^2} \hat{k} \cdot \vec{\nabla} \times \vec{u}. \tag{3.2}$$

By using (2.3) and the shorthand notation

$$\varsigma \equiv \hat{k} \cdot \vec{\nabla} \times \vec{u} \tag{3.3}$$

for relative vorticity, eq. (3.2) becomes

$$-E_V^{-1/2} \frac{\partial w}{\partial \xi} = \frac{1}{2} \frac{\partial^2 \varsigma}{\partial \xi^2}. \tag{3.4}$$

Second, evaluate the vertical component of the curl of (3.1), setting, in analogy with (3.3), $\varsigma_0 \equiv \hat{k} \cdot \vec{\nabla} \times \vec{u}_0$ and recalling the identity $\hat{k} \cdot \vec{\nabla} \times (\hat{k} \times \vec{u}) = \vec{\nabla} \cdot \vec{u}$ which is valid because $\vec{\nabla}$ is the *horizontal* gradient operator. The resulting equation is

$$\varsigma = \varsigma_0 - \frac{1}{2} \frac{\partial^2}{\partial \xi^2} \vec{\nabla} \cdot \vec{u},$$

that is, according to (2.3),

$$\varsigma = \varsigma_0 + \frac{E_V^{-1/2}}{2} \frac{\partial^3 w}{\partial \xi^3}. \tag{3.5}$$

Equations (3.4) and (3.5) allow to establish the model of the benthic Ekman layer in terms of the relative vorticity ς and the vertical velocity w of the flow. The general integrals of the coupled equations (3.4) and (3.5) are

$$\varsigma = \varsigma_0 + \exp[-\xi][A \cos(\xi) + B \sin(\xi)] \tag{3.6}$$

and

$$w = \frac{E_V^{1/2}}{2} \left\{ B - A + \exp[-\xi][(A - B) \cos(\xi) + (A + B) \sin(\xi)] \right\}. \tag{3.7}$$

In (3.6) and (3.7), $A = A(x, y)$ and $B = B(x, y)$ are real functions to be singled out by means of the boundary conditions which are discussed below.

Mathematically, the second-order ordinary differential equations (2.1), (2.2) can be equipped, in place of (2.5), with homogeneous and linear boundary conditions of the kind

$$q \vec{u}_H + p \frac{\partial \vec{u}_H}{\partial \xi} = 0 \quad \text{in } \xi = 0, \tag{3.8}$$

where q and p are constants. Obviously, the choice $p = 0$, $q \neq 0$ is equivalent to (2.5) so it determines again solution (2.8), (2.9). On the contrary, the choice $q = 0$, $p \neq 0$ gives the solution

$$\vec{u}_H = \vec{u}_0 \tag{3.9}$$

throughout the benthic layer (the check is trivial). Thus, in this case, the boundary layer correction to (u_0, v_0) is not requested and the Ekman pumping cannot be put into action, so lateral diffusion of relative vorticity must be introduced into the dynamics of the fluid interior as an alternative mechanism of vorticity dissipation. Hence, only the class of boundary conditions

$$\vec{u}_H + p\frac{\partial \vec{u}_H}{\partial \xi} = 0 \quad \text{in } \xi = 0 \tag{3.10}$$

bears actually physical interest in the present context. From (3.10) the related boundary conditions in terms of ς and w can be derived as follows.

Application of the operator "$\hat{k} \cdot \vec{\nabla}\times$" to (3.10) gives, using position (3.3),

$$\varsigma + p\frac{\partial \varsigma}{\partial \xi} = 0 \quad \text{in } \xi = 0. \tag{3.11}$$

On the other hand, application of the operator "$\vec{\nabla}\cdot$" to the same equation yields, using also (2.3),

$$\frac{\partial w}{\partial \xi} + p\frac{\partial^2 w}{\partial \xi^2} = 0 \quad \text{in } \xi = 0. \tag{3.12}$$

Substitution of (3.11) and (3.12) into (3.6) and (3.7) allows to determine A and B, that is

$$A = \varsigma_0 \frac{p-1}{2p^2 - 2p + 1}\,, \qquad B = -\varsigma_0 \frac{p}{2p^2 - 2p + 1}\,. \tag{3.13}$$

Finally, with the aid of (3.13), the vorticity field is determined

$$\varsigma = \varsigma_0 \left\{ 1 + \frac{\exp[-\xi]}{2p^2 - 2p + 1} [(p-1)\cos(\xi) - p\sin(\xi)] \right\}. \tag{3.14}$$

In the special case $p = 0$, (3.14) coincides with the vorticity which can be inferred from (2.8).

In the same way, from (3.7) and (3.13), the field of vertical velocity turns out to be

$$w = \frac{E_V^{1/2}}{2} \frac{\varsigma_0}{2p^2 - 2p + 1} \left\{ (1-2p)[1 - \exp[-\xi]\cos(\xi)] - \exp[-\xi]\sin(\xi) \right\}. \tag{3.15}$$

In the case $p = 0$, velocity (3.15) coincides with (2.9).

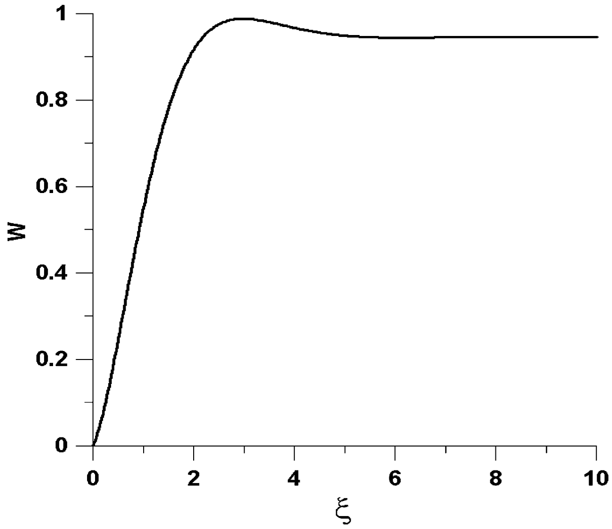

Fig. 2. – Plot of profile of the vertical velocity with $p = -0.2$.

Does the positive correlation between w and ς_0, pointed out at the end of sect. **2**, hold also for $p \neq 0$?

To answer, the possibilities $p \leq 0$ and $p > 0$ must be investigated separately and the further restriction $p < 1/2$ is assumed. This, since in the case $p \geq 1/2$ w and ς_0 are negatively correlated, as (3.15) shows, thus giving an unphysical result.

Take first $p \leq 0$. In the special case $p = -0.2$, the vertical velocity derived from (3.15) is reported in fig. 2. Consider the factor

$$(1-2p)[1-\exp[-\xi]\cos(\xi)] - \exp[-\xi]\sin(\xi) \tag{3.16}$$

appearing in (3.15). As $1-2p > 1$ and $1-\exp[-\xi]\cos(\xi) \geq 0$,

$$\begin{aligned}(1-2p)[1-\exp[-\xi]\cos(\xi)] - \exp[-\xi]\sin(\xi) \geq \\ 1-\exp[-\xi]\cos(\xi) - \exp[-\xi]\sin(\xi) \geq 0 \quad \forall\xi \geq 0.\end{aligned} \tag{3.17}$$

Therefore w and ς_0 are positively correlated $\forall\xi \geq 0$. Note that, for p running from $-\infty$ to 0, w increases monotonically from 0 to $\frac{E_V^{1/2}}{2}\varsigma_0$.

Second, take $0 < p < 1/2$ and consider again (3.16). For $0 < \xi \ll 1$, by using the truncated Taylor expansions $\exp[-\xi]\cos(\xi) \approx 1-\xi$ and $\exp[-\xi]\sin(\xi) \approx \xi - \xi^2$, one can resort to the approximation

$$(1-2p)[1-\exp[-\xi]\cos(\xi)] - \exp[-\xi]\sin(\xi) \approx \xi^2 - 2p\xi$$

to conclude

$$(1-2p)[1-\exp[-\xi]\cos(\xi)] - \exp[-\xi]\sin(\xi) < 0 \quad \text{for } 0 < \xi < 2p. \tag{3.18}$$

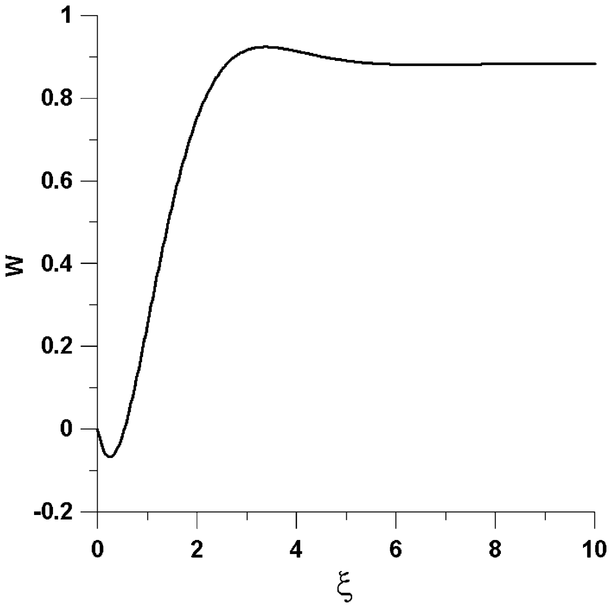

Fig. 3. – Plot of profile of the vertical velocity with $p = 0.2$.

On the other hand,

$$\lim_{\xi\to\infty}\{(1-2p)[1-\exp[-\xi]\cos(\xi)]-\exp[-\xi]\sin(\xi)\}=1-2p>0. \tag{3.19}$$

Unlike (3.17), relationships (3.18) and (3.19) imply that $w(\xi)$ is negative for $0 < \xi < 2p$ and positive for ξ large enough: for $p = 0.2$ Figure 3 shows the profile of the vertical velocity with depth. Thus, for $0 < p < 1/2$, the positive correlation between w and ς_0 fails in a sublayer close to the bottom while it holds at least for ξ large enough. This fact demands a further discussion which is developed in next sections.

4. – Stability analysis

4˙1. *Asymptotic stability of the benthic layer for* $p \le 0$. – We stress that eqs. (2.1) and (2.2) constitute the $O(1)$ terms of the momentum equations

$$\varepsilon_T\frac{\partial u}{\partial t}+\varepsilon(\vec{u}\cdot\vec{\nabla})u-v+\delta w=-\frac{\partial p}{\partial x}+\frac{E_H}{2}\nabla_H^2 u+\frac{1}{2}\frac{\partial^2 u}{\partial\xi^2}\,, \tag{4.1}$$

$$\varepsilon_T\frac{\partial v}{\partial t}+\varepsilon(\vec{u}\cdot\vec{\nabla})v+u=-\frac{\partial p}{\partial y}+\frac{E_H}{2}\nabla_H^2 v+\frac{1}{2}\frac{\partial^2 v}{\partial\xi^2}\,, \tag{4.2}$$

where the perturbation pressure p is that geostrophic, so $\frac{\partial p}{\partial x} = v_0$ and $\frac{\partial p}{\partial y} = -u_0$. One can conceive a configuration in which the "basic" flow (u, v, w) is steady and a time-dependent disturbance, say $\vec{\tilde{u}} \equiv (\tilde{u}, \tilde{v}, \tilde{w})$, is superposed to it. The time scale of the disturbance is assumed to be "short" enough to make $\varepsilon_T = O(1)$, that is $T = O(f_0^{-1}) = O(10^4\,\mathrm{s})$, while $\varepsilon < O(1)$. In this dynamic regimen, setting for simplicity $\varepsilon_T = 1$, the horizontal momentum equations for the perturbed flow $(u+\tilde{u}, v+\tilde{v}, w+\tilde{w})$ derived from (4.1) and (4.2) are

$$\frac{\partial}{\partial t}(u+\tilde{u}) - v - \tilde{v} = -v_0 + \frac{1}{2}\frac{\partial^2}{\partial \xi^2}(u+\tilde{u}), \tag{4.3}$$

$$\frac{\partial}{\partial t}(v+\tilde{v}) + u + \tilde{u} = u_0 + \frac{1}{2}\frac{\partial^2}{\partial \xi^2}(v+\tilde{v}), \tag{4.4}$$

and the incompressibility equation is

$$\frac{\partial}{\partial \xi}(w+\tilde{w}) = -E_V^{1/2}\vec{\nabla}\cdot\left(\vec{u}+\vec{\tilde{u}}\right). \tag{4.5}$$

From (2.1)-(2.3) and (4.3)-(4.5) the equations for the disturbance $(\tilde{u}, \tilde{v}, \tilde{w})$ are easily derived:

$$\frac{\partial \tilde{u}}{\partial t} - \tilde{v} = \frac{1}{2}\frac{\partial^2 \tilde{u}}{\partial \xi^2}, \tag{4.6}$$

$$\frac{\partial \tilde{v}}{\partial t} + \tilde{u} = \frac{1}{2}\frac{\partial^2 \tilde{v}}{\partial \xi^2}, \tag{4.7}$$

$$\frac{\partial \tilde{w}}{\partial \xi} = -E_V^{1/2}\vec{\nabla}\cdot\vec{\tilde{u}}. \tag{4.8}$$

Equation (4.6) and (4.7) are synthesized in vector form as

$$\frac{\partial}{\partial t}\hat{k}\times\vec{\tilde{u}} - \vec{\tilde{u}} = \frac{1}{2}\frac{\partial^2}{\partial \xi^2}\hat{k}\times\vec{\tilde{u}}. \tag{4.9}$$

Thus, setting $\tilde{\varsigma} = \hat{k}\cdot\vec{\nabla}\times\vec{\tilde{u}}$, the governing equations of the disturbance are (4.8) and (4.9). Then, application of the operator "$\vec{\nabla}\cdot$" to (4.9) yields

$$\frac{\partial \tilde{\varsigma}}{\partial t} + \vec{\nabla}\cdot\vec{\tilde{u}} = \frac{1}{2}\frac{\partial^2 \tilde{\varsigma}}{\partial \xi^2}, \tag{4.10}$$

whence, by using (4.8) into (4.10), we obtain

$$E_V^{1/2}\frac{\partial \tilde{\varsigma}}{\partial t} - \frac{\partial \tilde{w}}{\partial \xi} = \frac{E_V^{1/2}}{2}\frac{\partial^2 \tilde{\varsigma}}{\partial \xi^2}. \tag{4.11}$$

Moreover, application of the operator "$\hat{k}\cdot\vec{\nabla}\times$" to (4.9) gives

$$\frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{\tilde{u}} - \tilde{\varsigma} = \frac{1}{2}\frac{\partial^2}{\partial \xi^2}\vec{\nabla}\cdot\vec{\tilde{u}} \tag{4.12}$$

and, using (4.8) into (4.12), the equation

$$E_V^{-1/2}\frac{\partial}{\partial t}\frac{\partial \tilde{w}}{\partial \xi} + \tilde{\varsigma} = \frac{E_V^{-1/2}}{2}\frac{\partial^2}{\partial \xi^2}\frac{\partial \tilde{w}}{\partial \xi} \tag{4.13}$$

follows. Equations (4.11) and (4.13) are fit for establishing the time growth rate of a suitable norm of any disturbance, say $\|\vec{\tilde{u}}\|$, in order to infer the stability of the basic flow (u, v, w) relatively to the considered norm. To do this, suitable information about the behaviour of $\tilde{\varsigma}$ and $\tilde{w}$ in $\xi = 0$ and for $\xi \to \infty$ is in order.

Linearity of conditions (3.11) and (3.12) implies

$$\tilde{\varsigma} + p\frac{\partial \tilde{\varsigma}}{\partial \xi} = 0 \quad \text{in } \xi = 0, \tag{4.14}$$

$$\frac{\partial \tilde{w}}{\partial \xi} + p\frac{\partial^2 \tilde{w}}{\partial \xi^2} = 0 \quad \text{in } \xi = 0. \tag{4.15}$$

Equation (3.14) implies $\lim_{\xi\to\infty} \varsigma = \varsigma_0 \Rightarrow \lim_{\xi\to\infty}(\varsigma + \tilde{\varsigma}) = \varsigma_0$ whence

$$\lim_{\xi\to\infty} \tilde{\varsigma} = 0 \tag{4.16}$$

and, analogously, (3.15) implies

$$\lim_{\xi\to\infty} \frac{\partial \tilde{w}}{\partial \xi} = 0. \tag{4.17}$$

After these preliminaries, eq. (4.11) is multiplied by $\tilde{\varsigma}$, (4.13) by $\frac{\partial \tilde{w}}{\partial \xi}$ and the results are added together to give, after little algebra,

$$\frac{\partial}{\partial t}\left[E_V^{1/2}\tilde{\varsigma}^2 + E_V^{-1/2}\left(\frac{\partial \tilde{w}}{\partial \xi}\right)^2\right] = E_V^{1/2}\frac{\partial}{\partial \xi}\left(\tilde{\varsigma}\frac{\partial \tilde{\varsigma}}{\partial \xi}\right) + E_V^{-1/2}\frac{\partial}{\partial \xi}\left(\frac{\partial \tilde{w}}{\partial \xi}\frac{\partial^2 \tilde{w}}{\partial \xi^2}\right) - \left[E_V^{1/2}\left(\frac{\partial \tilde{\varsigma}}{\partial \xi}\right)^2 + E_V^{-1/2}\left(\frac{\partial^2 \tilde{w}}{\partial \xi^2}\right)^2\right]. \tag{4.18}$$

It is quite obvious to define the norm

$$\left\|\vec{\tilde{u}}\right\| \equiv \left[\int_\Omega \left(E_V^{1/2}\tilde{\varsigma}^2 + E_V^{-1/2}\left(\frac{\partial \tilde{w}}{\partial \xi}\right)^2\right) \mathrm{d}\Omega'\right]^{1/2}, \tag{4.19}$$

where $\Omega = D \times [0 \le \xi < \infty[$ is the fluid domain, D being the projection of Ω on the beta-plane, to which the no mass-flux condition across its boundary ∂D applies. Space integration of (4.18) on Ω yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\vec{\tilde{u}}\right\|^2 = \int_D \left\{E_V^{1/2}\left[\tilde{\varsigma}\frac{\partial \tilde{\varsigma}}{\partial \xi}\right]_0^\infty + E_V^{-1/2}\left[\frac{\partial \tilde{w}}{\partial \xi}\frac{\partial^2 \tilde{w}}{\partial \xi^2}\right]_0^\infty\right\} \mathrm{d}x\mathrm{d}y - \int_\Omega \left\{E_V^{1/2}\left(\frac{\partial \tilde{\varsigma}}{\partial \xi}\right)^2 + E_V^{-1/2}\left(\frac{\partial^2 \tilde{w}}{\partial \xi^2}\right)^2\right\} \mathrm{d}\Omega', \tag{4.20}$$

where (4.14)-(4.17) apply to evaluate $[\tilde{\varsigma}\frac{\partial\tilde{\varsigma}}{\partial\xi}]_0^\infty$ and $[\frac{\partial\tilde{w}}{\partial\xi}\frac{\partial^2\tilde{w}}{\partial\xi^2}]_0^\infty$. On the whole

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\vec{\tilde{u}}\right\|^2 = p\int_D \left\{E_V^{1/2}\left(\left[\frac{\partial\tilde{\varsigma}}{\partial\xi}\right]_0\right)^2 + E_V^{-1/2}\left(\left[\frac{\partial^2\tilde{w}}{\partial\xi^2}\right]_0\right)^2\right\}\mathrm{d}x\mathrm{d}y - \int_\Omega \left\{E_V^{1/2}\left(\frac{\partial\tilde{\varsigma}}{\partial\xi}\right)^2 + E_V^{-1/2}\left(\frac{\partial^2\tilde{w}}{\partial\xi^2}\right)^2\right\}\mathrm{d}\Omega' \tag{4.21}$$

and, from (4.21), one concludes

$$p \le 0 \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\left\|\vec{\tilde{u}}\right\| < 0. \tag{4.22}$$

Therefore, if the parameter p of (3.10) is *not positive*, then the current field in the benthic layer is stable in the norm (4.19).

Under hypothesis $p \le 0$ a stronger result can be proved, that is the asymptotic stability of the basic state. To achieve this, consider the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\vec{\tilde{u}}\right\|^2 \le -\int_\Omega \left\{E_V^{1/2}\left(\frac{\partial\tilde{\varsigma}}{\partial\xi}\right)^2 + E_V^{-1/2}\left(\frac{\partial^2\tilde{w}}{\partial\xi^2}\right)^2\right\}\mathrm{d}\Omega' \tag{4.23}$$

that immediately follows from (4.21) and, in particular, the integral

$$\int_0^\infty \left[E_V^{1/2}\left(\frac{\partial\tilde{\varsigma}}{\partial\xi}\right)^2 + E_V^{-1/2}\left(\frac{\partial^2\tilde{w}}{\partial\xi^2}\right)^2\right]\mathrm{d}\xi \tag{4.24}$$

which is a step of the evaluation of the 3D integral appearing on the r.h.s. of (4.23). We first prove that (4.24) is bounded from below by a quantity proportional to $\int_0^\infty[E_V^{1/2}\tilde{\varsigma}^2 + E_V^{-1/2}(\frac{\partial\tilde{w}}{\partial\xi})^2]\mathrm{d}\xi$. To achieve this, we stress that, strictly speaking, the relation $\xi = E_V^{-1/2}z$ between the vertical coordinates ξ and z implies $E_V^{-1/2}\int_0^1 \mathrm{d}z = \int_0^{E_V^{-1/2}}\mathrm{d}\xi$ while, due to the smallness of E_V, the boundary layer approximation allows us to write $\int_0^\infty \mathrm{d}\xi$ in place of $\int_0^{E_V^{-1/2}}\mathrm{d}\xi$. At the same time, again due to the smallness of E_V, relations (4.16) and (4.17) mean

$$\tilde{\varsigma}\left(x, y, E_V^{-1/2}, t\right) = 0, \tag{4.25}$$

$$\frac{\partial}{\partial\xi}\tilde{w}\left(x, y, E_V^{-1/2}, t\right) = 0, \tag{4.26}$$

respectively. Hence the inequalities [3]

$$\int_0^{E_V^{-1/2}} \tilde{\varsigma}^2\mathrm{d}\xi \le C_1\int_0^{E_V^{-1/2}}\left(\frac{\partial\tilde{\varsigma}}{\partial\xi}\right)^2\mathrm{d}\xi, \tag{4.27}$$

$$\int_0^{E_V^{-1/2}} \left(\frac{\partial\tilde{w}}{\partial\xi}\right)^2\mathrm{d}\xi \le C_2\int_0^{E_V^{-1/2}}\left(\frac{\partial^2\tilde{\varsigma}}{\partial\xi^2}\right)^2\mathrm{d}\xi \tag{4.28}$$

hold true for suitable positive values of the constants C_1 and C_2. Within the boundary layer approximation, based on (4.27) and (4.28), the inequality

$$\int_0^\infty \left[E_V^{1/2}\tilde{\varsigma}^2 + E_V^{-1/2}\left(\frac{\partial\tilde{w}}{\partial\xi}\right)^2\right]\mathrm{d}\xi \leq C\int_0^\infty \left[E_V^{1/2}\left(\frac{\partial\tilde{\varsigma}}{\partial\xi}\right)^2 + E_V^{-1/2}\left(\frac{\partial^2\tilde{w}}{\partial\xi^2}\right)^2\right]\mathrm{d}\xi \tag{4.29}$$

can be established, where $C = \max(C_1, C_2)$. In turn, (4.29) implies

$$-\int_\Omega \left[E_V^{1/2}\left(\frac{\partial\tilde{\varsigma}}{\partial\xi}\right)^2 + E_V^{-1/2}\left(\frac{\partial^2\tilde{w}}{\partial\xi^2}\right)^2\right]\mathrm{d}\Omega' \leq -C^{-1}\int_\Omega \left[E_V^{1/2}\tilde{\varsigma}^2 + E_V^{-1/2}\left(\frac{\partial\tilde{w}}{\partial\xi}\right)^2\right]\mathrm{d}\Omega',$$

that is to say

$$-\int_\Omega \left[E_V^{1/2}\left(\frac{\partial\tilde{\varsigma}}{\partial\xi}\right)^2 + E_V^{-1/2}\left(\frac{\partial^2\tilde{w}}{\partial\xi^2}\right)^2\right]\mathrm{d}\Omega' \leq -C^{-1}\left\|\vec{\tilde{u}}\right\|^2. \tag{4.30}$$

Finally, substitution of (4.30) into (4.23) gives the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\vec{\tilde{u}}\right\|^2 \leq -C^{-1}\left\|\vec{\tilde{u}}\right\|^2 \tag{4.31}$$

whose time integration on $[0, t]$ yields

$$\left\|\vec{\tilde{u}}(t)\right\| \leq \left\|\vec{\tilde{u}}(0)\right\|\exp[-t/2C]. \tag{4.32}$$

Finally, (4.32) implies

$$\lim_{t\to\infty}\left\|\vec{\tilde{u}}(t)\right\| = 0, \tag{4.33}$$

so, from (4.22) and (4.33), the *asymptotic* stability of the basic state is proved.

4·2. *Instability of the benthic layer for* $p > 0$. – The instability analysis relies on the hypothesis

$$0 < p(< 1/2) \tag{4.34}$$

and is based on the determination of a special class of perturbations whose norm (4.19) diverges in time. To this purpose, reconsider (4.11) and (4.13) and look for fields of the kind

$$\tilde{\varsigma} = a(x, y, t)\exp[-\xi/p], \tag{4.35}$$

$$\frac{\partial\tilde{w}}{\partial\xi} = b(x, y, t)\exp[-\xi/p], \tag{4.36}$$

so (4.35) satisfies both (4.14) and (4.16) while (4.36) satisfies both (4.15) and (4.17), whatever $a(x,y,t)$ and $b(x,y,t)$ may be. Then, substitution of (4.35) and (4.36) into (4.11) and (4.13) yields the coupled equations

$$E_V^{1/2}\frac{\partial a}{\partial t} - b = \frac{E_V^{1/2}}{2p^2}a, \tag{4.37}$$

$$E_V^{-1/2}\frac{\partial b}{\partial t} + a = \frac{E_V^{-1/2}}{2p^2}b. \tag{4.38}$$

Multiplication of (4.37) by a, (4.38) by b and the subsequent addition gives

$$\frac{\partial}{\partial t}\left(E_V^{1/2}a^2 + E_V^{-1/2}b^2\right) = \frac{1}{p^2}\left(E_V^{1/2}a^2 + E_V^{-1/2}b^2\right). \tag{4.39}$$

Integration of (4.39) on Ω with $a^2 = \tilde{\varsigma}^2 \exp[2\xi/p]$, $b^2 = (\frac{\partial \tilde{w}}{\partial \xi})^2 \exp[2\xi/p]$ yields, in terms of the norm (4.19),

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\vec{\tilde{u}}\right\|^2 = \frac{1}{p^2}\left\|\vec{\tilde{u}}\right\|^2, \tag{4.40}$$

and time integration of (4.40) on $[0,t]$ yields

$$\left\|\vec{\tilde{u}}(t)\right\| = \left\|\vec{\tilde{u}}(0)\right\| \exp\left[\frac{t}{2p^2}\right]. \tag{4.41}$$

Hence

$$\sup_{t>0}\left\|\vec{\tilde{u}}(t)\right\| = \infty \tag{4.42}$$

and the instability is thus proved.

Note that the integration of (4.36) with respect to ξ, under the no mass-flux condition across the bottom, gives

$$\tilde{w} = pb(x,y,t)\{1 - \exp[-\xi/p]\}. \tag{4.43}$$

Equation (4.43) shows that, *in the case of instability*, the disturbance crosses the whole benthic layer and penetrates into the geostrophic interior where $\tilde{w} \approx pb(x,y,t)$.

Equations (4.37) and (4.38) can give further details on the structure of the disturbance leading to instability. In fact, these equations can be decoupled and the related general integrals are

$$a(x,y,t) = \exp\left[\frac{t}{2p^2}\right]\{a_1\exp[it] + a_2\exp[-it]\}, \tag{4.44}$$

$$b(x,y,t) = iE_V^{1/2}\exp\left[\frac{t}{2p^2}\right]\{a_1\exp[it] - a_2\exp[-it]\}, \tag{4.45}$$

where $a_1 = a_1(x, y)$ and $a_2 = a_2(x, y)$ are, for the time being, arbitrary complex functions of their arguments. However, as the physical fields are real, these functions are linked by the relation

$$a_1 = a_2^*, \tag{4.46}$$

so one obtains

$$a(x, y, t) = 2 \exp\left[\frac{t}{2p^2}\right] \mathrm{Re}\{a_1 \exp[it]\}, \tag{4.47}$$

$$b(x, y, t) = -2E_V^{1/2} \exp\left[\frac{t}{2p^2}\right] \mathrm{Im}\{a_1 \exp[it]\}, \tag{4.48}$$

respectively. From (4.35) and (4.47) the vorticity of the disturbance takes the form

$$\tilde{\varsigma} = 2 \exp\left[\frac{t}{2p^2} - \frac{\xi}{p}\right] \mathrm{Re}\{a_1 \exp[it]\}, \tag{4.49}$$

while, from (4.36) and (4.48)

$$\frac{\partial \tilde{w}}{\partial \xi} = -2E_V^{1/2} \exp\left[\frac{t}{2p^2} - \frac{\xi}{p}\right] \mathrm{Im}\{a_1 \exp[it]\} \tag{4.50}$$

follows. The latter equation implies

$$\tilde{w} = \int_0^\xi \frac{\partial \tilde{w}}{\partial \xi'} \mathrm{d}\xi' = 2pE_V^{1/2} \left\{\exp\left[\frac{t}{2p^2} - \frac{\xi}{p}\right] - \exp\left[\frac{t}{2p^2}\right]\right\} \mathrm{Im}\{a_1 \exp[it]\}. \tag{4.51}$$

We see that all the fields (4.49), (4.50) and (4.51) propagate vertically upward with speed $c = E_V^{1/2}/2p$. In fact the exponential $\exp[\frac{t}{2p^2} - \frac{\xi}{p}]$ appearing in these equations can be written as $\exp[-\frac{E_V^{-1/2}}{p}(z - \frac{E_V^{1/2}}{2p}t)]$ and this quantity represents the non-dispersive propagation of the profile $\exp[-\frac{E_V^{-1/2}}{p}z]$ towards decreasing depths with speed $c = E_V^{1/2}/2p$.

Finally, we stress that the actual form of $a_1 = a_1(x, y)$ is determined by the initial conditions of $\tilde{\varsigma}$ and $\tilde{w}$. In fact, (4.49) implies

$$\mathrm{Re}\{a_1\} = \exp[\xi/p]\tilde{\varsigma}(t = 0)/2, \tag{4.52}$$

while (4.51) yields

$$\mathrm{Im}\{a_1\} = \{\exp[-\xi/p] - 1\}^{-1} \frac{\tilde{w}(t = 0)}{2pE_V^{1/2}}. \tag{4.53}$$

Comments on stability of the Ekman layers

Like the most of fluids in boundary layer systems, also the Ekman layer models undergo mathematical and laboratory [4] investigations about their dynamic stability, mainly related to the parameterisation of turbulence in terms of Reynolds number (Re). Early papers [5-7] on this subject show the existence of a critical Reynolds number (Re_C)

below which the system turns out to be stable while it exhibits typical unstable modes for $\mathrm{Re} > \mathrm{Re}_C$. In the framework of theoretical investigations, the current of the Ekman layer representing the basic state is taken only depth dependent, without the Ekman pumping which, in the oceanographic context, is responsible of the coupling with the geostrophic interior. Moreover, the analysis performed by the classical Orr-Sommerfeld equation fits only for infinitesimal disturbances. The authors did not find any paper concerning with finite amplitude perturbations. In the present investigation the basic current of the benthic layer is taken into account with its dependence of the horizontal coordinates and under the assumption of finite amplitude disturbances, in the framework of Lyapunov method. We stress that the results of Dudis and Davis [7] show that stability is realized for Reynolds' numbers Re below a certain critical threshold $\mathrm{Re}_C \approx 18.3$. The length and time scales considered by these authors are (following the original notation) $L = (\nu/\Omega)^{1/2}$ and $T = L^2/\nu$ while the velocity U of the basic state is a constant, say V_0. Hence, Reynolds' number is

$$\mathrm{Re} = V_0 L/\nu. \tag{4.54}$$

On the other hand, by using $T = L^2/\nu$ into (4.54) one obtains $\mathrm{Re} = UT/L$ that is $\mathrm{Re} = \frac{U}{\Omega L}\Omega T$. As the advective Rossby number $\varepsilon = \frac{U}{\Omega L}$ and the temporal Rossby number $\varepsilon_T = \frac{1}{\Omega T}$, one concludes

$$\mathrm{Re} = \varepsilon/\varepsilon_T. \tag{4.55}$$

In our analysis $\varepsilon \ll \varepsilon_T$ is considered, so, because of (4.55), condition $\mathrm{Re} < \mathrm{Re}_C$ found by Dudis and Davis is certainly satisfied. We conclude that our results concerning the stability of the Ekman layer do not contradict the authors above.

5. – Connection between the bottom boundary condition of the benthic layer and the mass-transport is a quasi-geostrophic, bottom-dissipated, wind-driven ocean

The standard nondimensional model of a steady, single layer, quasigeostrophic, bottom-dissipated, wind-driven ocean is governed by the well-known equation [8]

$$\left(\frac{\delta_I}{L}\right)^2 J\left(\psi, \nabla^2\psi\right) + \frac{\partial \psi}{\partial x} = \hat{k}\cdot\vec{\nabla}\times\vec{\tau} - \frac{\delta_S}{L}\nabla^2\psi. \tag{5.1}$$

The link between (5.1) and the benthic layer lies in the term $\frac{\delta_S}{L}\nabla^2\psi$ which represents the vertical velocity w_1 of the Ekman pumping at the base of the geostrophic interior. In fact, at the basin scale the non-dimensional vertical velocity $w \approx \frac{\beta_0 L}{f_0} w_1$ whence, recalling also (2.10), one obtains

$$\lim_{\xi\to\infty} w_1 = \frac{f_0}{\beta_0 L}\lim_{\xi\to\infty} w = \frac{f_0}{\beta_0 L}\frac{E_V^{1/2}}{2}\varsigma_0 \equiv \frac{\delta_S}{L}\nabla^2\psi. \tag{5.2}$$

In the case of (3.15),

$$\lim_{\xi\to\infty} w = \frac{E_V^{1/2}}{2}\frac{1-2p}{2p^2-2p+1}\varsigma_0, \tag{5.3}$$

and (5.2), with (5.3) in place of $\frac{E_V^{1/2}}{2}\varsigma_0$, gives

$$\lim_{\xi\to\infty} w_1 = \frac{f_0}{\beta_0 L}\lim_{\xi\to\infty} w = \frac{f_0}{\beta_0 L}\frac{E_V^{1/2}}{2}\frac{1-2p}{2p^2-2p+1}\varsigma_0 \equiv \frac{\delta_S}{L}\frac{1-2p}{2p^2-2p+1}\nabla^2\psi. \tag{5.4}$$

Thus, because of (5.4), eq. (5.1) becomes

$$\left(\frac{\delta_I}{L}\right)^2 J(\psi,\nabla^2\psi) + \frac{\partial\psi}{\partial x} = \hat{k}\cdot\vec{\nabla}\times\vec{\tau} - \frac{\delta_S}{L}\frac{1-2p}{2p^2-2p+1}\nabla^2\psi, \tag{5.5}$$

where, based on the results of sect. **4**, only the range $p \le 0$ is considered. To be concrete, we solve numerically (5.5) in the fluid domain

$$D = [0 \le x \le 1] \times [0 \le y \le 1] \tag{5.6}$$

with the forcing

$$\hat{k}\cdot\vec{\nabla}\times\vec{\tau} = -\sin(\pi y) \tag{5.7}$$

which is representative of an idealised subtropical gyre.

In (5.5), the Jacobian is evaluated using the Arakawa scheme [9], which conserves kinetic energy and enstrophy, while the relative vorticity is inverted using the method of successive overrelaxations [10].

Once δ_S/L and δ_I/L are suitably fixed, solutions of (5.5) with (5.7) are found for different values of $p \le 0$. We denote with

$$\psi = \psi(x, y; \delta_S/L, \delta_I/L, p) \tag{5.8}$$

these solutions.

Integration of (5.5) on a domain included into (5.6) and encircled by a closed streamline C yields the dissipation integral

$$\oint_C \vec{\tau}\cdot\hat{t}\,\mathrm{d}s = \frac{\delta_S}{L}\frac{1-2p}{2p^2-2p+1}\oint_C \vec{u}_0\cdot\hat{t}\,\mathrm{d}s \tag{5.9}$$

where the geostrophic current $\vec{u}_0 = \hat{k}\times\vec{\nabla}\psi$ while $\mathrm{d}s$ is the differential arclength along C.

Thus, as $\oint_C \vec{\tau}\cdot\hat{t}\,\mathrm{d}s = O(1)$ in accordance with (5.7), the smallness of the damping factor $\frac{1-2p}{2p^2-2p+1}$ on the r.h.s. of (5.9) accentuates that of the whole product $\frac{\delta_S}{L}\frac{1-2p}{2p^2-2p+1}$ (with respect to the case $p = 0$) so the latter must be compensated by a *further reinforcement* of the intensification of $\vec{u}_0$ along a portion of the streamline C (which is known to be located in the north-western side of every oceanic basin). All this can be quantified by evaluating the transport

$$M(\delta_S/L, \delta_I/L, p) = \max_{(x,y)\in D}\psi(x, y; \delta_S/L, \delta_I/L, p), \tag{5.10}$$

where δ_S/L and δ_I/L are fixed while p varies into a suitable interval.

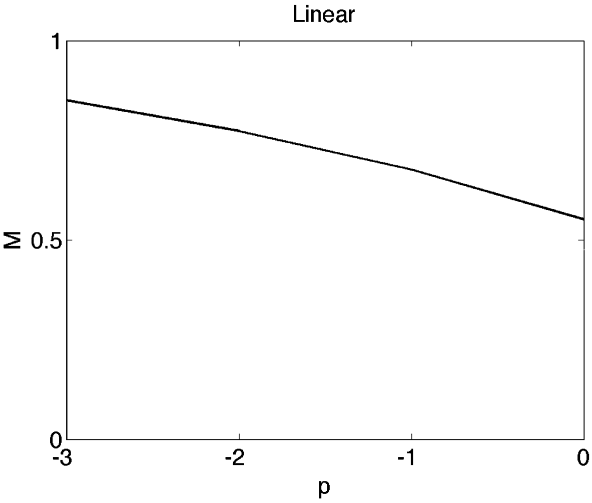

Fig. 4. – Plot of relation (5.11) in the range $-3 \le p \le 0$.

Linear case. The choice $\delta_S/L = 7 \times 10^{-2}$, $(\delta_I/L)^2 = 0$ is considered. Different values of p lead to the relation

$$p \to M(7 \times 10^{-2}, 0, p) \tag{5.11}$$

which is illustrated in fig. 4. In this case, $-3 \le p \le 0$ while lower values of this parameter produce non-physical circulation patterns. The monotonic decreasing behaviour of (5.11) is evident although not very pronounced.

Nonlinear case. Under the assumption $\delta_S = \delta_I$, the previous value $\delta_S/L = 7 \times 10^{-2}$

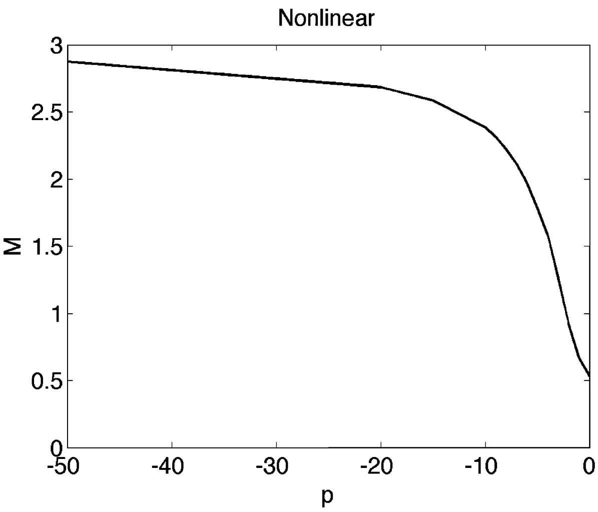

Fig. 5. – Plot of relation (5.12) in the range $-50 \le p \le 0$.

is retained, whence the estimate $(\delta_I/L)^2 = 5 \times 10^{-3}$ follows. The monotonic decreasing behaviour of

$$p \to M(7 \times 10^{-2}, 7 \times 10^{-2}, p) \tag{5.12}$$

is again evident (fig. 5), but with a different qualitative behaviour with respect to the linear case. In fact, it can be seen that for $-10 \le p \le 0$, M decreases rapidly as p increases, while, for $p < -10$, M decreases more slowly as p increases. In the latter circumstance, the dissipation term is far smaller than the nonlinear term and the dynamics of the geostrophic interior is reminiscent of that of Niiler's model [11, 12].

REFERENCES

[1] Pedlosky J., *Geophysical Fluid Dynamics* (Springer-Verlag, New York) 1987.
[2] Vallis G. K., *Atmospheric and Oceanic Fluid Dynamics* (Cambridge University Press, Cambridge) 2006.
[3] Cavallini F. and Crisciani F., *J. Inequal. Appl.*, **5** (2000) 343.
[4] Arons A. B., Ingersoll A. P. and Green T., *Tellus*, **XIII** (1961) 31.
[5] Stern M. E., *Tellus*, **XII** (1960) 399.
[6] Barcilon V., *Tellus*, **XVII** (1965) 53.
[7] Dudis J. J. and Davis S. H., *J. Fluid Mech.*, **47** (1971) 405.
[8] Pedlosky J., *Ocean Circulation Theory* (Springer-Verlag, New York) 1996.
[9] Arakawa A., *J. Comput. Phys.*, **1** (1966) 119.
[10] Flannery B. P., Teukolsky S. A. and Vetterling W. T., *Numerical Recipes in FORTRAN 77: The Art of Scientific Computing*, 2nd edition (Cambridge University Press, Cambridge) 1992.
[11] Crisciani F., *J. Phys. Oceanogr.*, **28** (1998) 218.
[12] Crisciani F. and Ozgokmen T. M., *Dyn. Atmos. Oceans*, **33** (2001) 135.