IL NUOVO CIMENTO Vol. 31 C, N. 5-6 Settembre-Dicembre 2008
DOI 10.1393/ncc/i2008-10325-2

Impact of height-dependent drainage forcing on the stable atmospheric boundary layer over a uniform slope

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(ricevuto il 5 Ottobre 2008; approvato il 20 Gennaio 2009; pubblicato online il 27 Marzo 2009)

Summary. — This paper presents a theoretical study of the stably stratified atmospheric boundary layer (SBL) overlying a uniform shallow slope with a gradient of the order of 1:1000. By relaxing the assumption made in a previous study that the slope-induced drainage force is constant across the boundary layer, analysis has been performed that demonstrates that a realistic functional form for the drainage forcing is a term proportional to $(1 - \frac{z}{h})^{1/2}$, where z is the height above the ground and h is the depth of the boundary layer. Modified expressions for the maximum sustainable surface buoyancy flux and Zilitinkevich's ratio are derived.

PACS 92.60.-e – Properties and dynamics of the atmosphere; meteorology.
PACS 92.60.Fm – Boundary layer structure and processes.
PACS 92.60.Gn – Winds and their effects.

1. – Introduction

The motivation for this study arises because stable boundary layers (SBL) are still poorly resolved in large-scale numerical models as their dynamics are complicated [1]. In numerical simulations turbulent mixing at strong stability is artificially maintained by enhanced mixing schemes, yet although the use of such models improves the energy budget, the predicted SBLs are too deep, and the stratification is weakened. Due to the difficulties of simultaneously resolving both small and large scale flow features, large-eddy simulation (LES) are often used for SBL studies [2-4]. Under stable conditions the dispersion of pollutants is suppressed and fog and frost formation may occur. In contrast, the structure of the convective boundary layer is relatively unaffected by moderately uneven terrain [5].

The SBL is particularly sensitive to topography. Brown and Wood [6] used numerical simulations to investigate the impact of moderate scale topography on the SBL. They found that the presence of hills with heights comparable to the undisturbed boundary layer depth leads to enhanced turbulence drag and deepening of the boundary layer.

The literature on the development of the SBL has followed two main strands. One branch comprises theories that are based on universal scalings (such as those based on the concept of a critical Richardson number or on similarity theory), and the other covers non-universal, site specific effects including topography or the presence of gravity waves. Derbyshire and Wood [7] discussed whether these two competing approaches could be reconciled. They demonstrated the need for parameterisations for the effect of unresolved hills on the SBL as even relatively shallow slopes can significantly influence the structure of the SBL. The static stability of the SBL causes the vertical length scales to be more limited than in the case of the convective boundary layer. The consequence of this is that turbulent quantities are influenced by their immediate surroundings. This can be used to obtain similarity equations for turbulence in a stably stratified flow. Local similarity theory predicts that locally normalised quantities are simply functions of a local stability parameter. For large values of the local stability parameter the normalised quantities approach a constant value. Nieuwstadt [8] derived equations for a quasi-steady model of the SBL based on observations at Cabauw, Netherlands and a z-less scaling theory. In this simplified model the only closure assumption was constancy of the flux Richardson number. Derbyshire [9] later showed that Nieuwstadt's original model was consistent only with a given value of the surface heat flux, H. Derbyshire amended Nieuwstadt's theory making it consistent with standard surface layer assumptions. This led to the prediction of the existence of a maximum sustainable surface buoyancy flux.

Several intensive experimental campaigns designed to unravel the secrets of the SBL have taken place in recent years. The nocturnal SBL may either be relatively short-lived, as in mid-latitudes, or may persist for up to several months as in Antarctica. Many studies have focussed on investigating the applicability of the concept of z-less scaling for stable conditions. This involves examining how moments of turbulent fluxes vary with the dimensionless parameter $\zeta = z/L$, where z is the height above the ground and L is the Obukhov length scale. Yagüe *et al.* [10] used data from the mid-latitude SABLES98 experimental campaign which took place over a plateau north of Valladolid in central Spain at the Research Centre for the Lower Atmosphere (CIBA). The objectives of this study were to examine the influence of stability on the flux-profile relationships for momentum and heat. They found that flux profiles increased with increasing stability parameter ζ up to a value of $\zeta = 1$ to 2, above which the values remained constant. Other recent studies using data from the CIBA include Vindel *et al.* [11] who used data from a sonic anemometer to examine the relationship between intermittency and turbulence in the SBL, and Viana *et al.* [12] who used data from an instrumented 100 m mast in conjunction with an array of microbarometers and a tethered balloon in order to analyse the different regimes of atmospheric turbulence that occurred during a single night. Dias *et al.* [13] made measurements of turbulent fluctuations at a height of 2.5 m in stable conditions over grass in order to examine the variability of second- and third-order moments involving vertical wind velocity, humidity and temperature. In general, they observed very little variation of the normalised moments with changes in the dimensionless stability parameter ζ, the exception being the normalised second-order moment of temperature. They thought this variation may be attributed to nonstationary conditions. The atmospheric boundary layer exhibits a range of turbulent and wavelike features (*e.g.*, gravity waves [14], solitary waves [15,16]). Of particular interest is the goal of understanding the intermittent nature of turbulence in the SBL. Tarquis *et al.* [17] used multifractal analysis to examine the structure of wind speed time series from a climatic station deployed in the experimental fields of the Agricultural School of Madrid, Spain. They found that an intermittency parameter could be successfully determined using detrended fluctuation analysis. Tijera *et al.* [18] used calculations of the fractal dimension of wind velocity components in order to develop an effective filter for extracting wave components from the power spectrum. They found that the fractal dimension increased with both length scale and the dissipation rate of kinetic energy.

Simple models of the nocturnal boundary layer are generally formulated in terms of eddy diffusivities which are functions of boundary layer depth and height above ground. Analysis of data from the Cooperative Atmospheric-Surface Exchange Study 1999 prompted a modified approach involving the inclusion of a z-less turbulence formulated in terms of a stability-dependent mixing length and a Prandtl number [19].

As the dispersion of contaminants is suppressed in the SBL, knowledge of turbulent mixing is particularly sought. In the SBL turbulent kinetic energy is generated by the velocity shear and is expended by viscous dissipation and by work against buoyancy forces. The relative strengths of the competing stabilising influence of stratification and the destabilising effects of the velocity shear are characterised by the gradient Richardson number (Ri). Traditional theories based on hydrodynamic instability theory have postulated the existence of a critical Ri, Ri_c, whose values typically range from 0.25 to 1, above which the local shear cannot maintain turbulence. However, recent studies [20-22] have revised this concept by considering a total turbulent energy closure model. The total turbulent energy is a conservative parameter that is maintained by shear in any stratification and subsequently there does not exist an Ri_c for turbulent energetics. This new theory explains the occurrence of turbulence in the deep ocean and free atmosphere where $Ri \gg 1$.

Several recent studies have addressed the effects of shear on the structure of turbulence in the SBL. Hanazaki and Hunt [23] examined the statistics of unsteady turbulence with uniform stratification over a wide range of Ri using rapid distortion theory. Smedman *et al.* [24] analysed data from marine field experiments in the Baltic Sea to examine the effects of shear sheltering in the SBL.

The aim of this paper is to develop a theoretical model of the SBL over a sloping surface in order to further our understanding of the dynamics of the SBL. Maguire and co-workers [25] proposed a theoretical model for the SBL overlying a uniform shallow slope where the drainage force was assumed to be constant across the depth of the boundary layer. In this study, a more realistic scenario is modelled via the inclusion of height-dependent drainage-forcing.

The results of previous studies of the SBL over flat, homogenous terrain [8, 9] and over a shallow uniform slope where the drainage force is assumed to be constant with height [25] are outlined in sect. **2**. In sect. **3** the theory of [25] is extended by incorporating a height-dependent model of drainage-forcing. The conclusions are summarised in sect. **4**.

2. – Background theory

In this paper we will adopt the conventional notation that (u, v, w) are the wind velocity components in the (x, y, z) directions, where gravity, g, acts in the negative z-direction and θ is the potential temperature. An overbar denotes an ensemble average and a primed quantity denotes the deviation from the background state. Previous studies of the SBL have expressed turbulence variables scaled in terms of surface-layer fluxes as a function of z/h, where h is the depth of the boundary layer [26, 27]. The use of such a scaling is only justified if h is a representative length scale of turbulence. Nieuwstadt [8] argued that since vertical motions are restricted in the SBL, a turbulent eddy cannot extend across the whole depth of the SBL and thus the use of h as a characteristic scale is not necessarily valid. Nieuwstadt [8] developed a time-dependent model of the SBL over flat, homogeneous terrain inferred from both theoretical arguments and observational data. The model incorporated a closure hypothesis based on local scaling to obtain vertical profiles of turbulent fluxes. Although this theory implies that vertical profiles of turbulence depend generally on the time history of the SBL, it was possible to derive expressions for the Reynolds stress, τ, and the heat flux, $-\overline{w'\theta'}$, only in the limit of stationary conditions.

Assuming stationary conditions and that for closure, the gradient Richardson number, Ri, and the flux Richardson number R_f are constants (with $Ri = R_f$), Nieuwstadt [8] showed that vertical buoyancy flux, B, decreases linearly with height:

$$B = B_0 \left(1 - \frac{z}{h}\right), \tag{1}$$

where B_0 is the surface buoyancy flux. Derbyshire [9] discovered that Nieuwstadt's model predicts the existence of a maximum sustainable downward surface buoyancy flux, $B_{\max}$, that turbulence can support. He found that

$$B_{\max} \simeq \frac{R_{f_c}}{\sqrt{3}} G^2 |f|, \tag{2}$$

where R_{f_c} is the critical flux Richardson number such that turbulence is suppressed for $R_f > R_{f_c}$, G is the geostrophic wind and f is the Coriolis parameter. Maguire and co-workers [25] extended Derbyshire's theory to consider the effects of a shallow uniform slope (with gradient of the order of 1:1000) on the turbulent structure of the SBL. In this study we follow [8] by assuming that the flux Richardson number, R_f, is constant throughout the boundary layer and that inertial equilibrium exists, *i.e.* steady state is assumed.

Horizontal pressure gradients are induced hydrostatically when buoyancy contours run parallel to a large scale sloping surface. This can be represented via a rotated gravity formalism in which the "horizontal gravity" term represents pressure gradients that are hydrostatically induced [28]. A convenient way to include this slope-induced horizontal pressure gradient is through the geostrophic wind:

$$W_g^{\text{slope}} = W_g + e^{i\phi} \frac{\mathcal{F}\gamma}{if}, \tag{3}$$

where W_g^{slope} is proportional to the full pressure gradient (which incorporates the baroclinic effect of cold air on the slope) and W_g is proportional to the synoptic pressure gradient. ϕ is the angle between the x-axis and the slope-induced force $\mathcal{F}$—the form of $\mathcal{F}$ will be discussed later. γ is the gradient of the sloping surface. The geometry of the slope is shown in fig. 1. The x-axis points in the direction of the surface wind. The geostrophic wind (magnitude G) acts parallel to the (x, y)-plane, the direction shown is arbitrary.

By assuming that the drainage force induced by the slope is constant across the boundary layer [25] predicted the existence of a maximum sustainable buoyancy flux, $B_{\max}^{\text{slope}}$, the magnitude of which is dependent upon the gradient of the slope, γ:

$$B_{\max}^{\text{slope}} = \frac{R_f}{\sqrt{3}} \left|W_g^{\text{slope}}\right|^2 |f|. \tag{4}$$

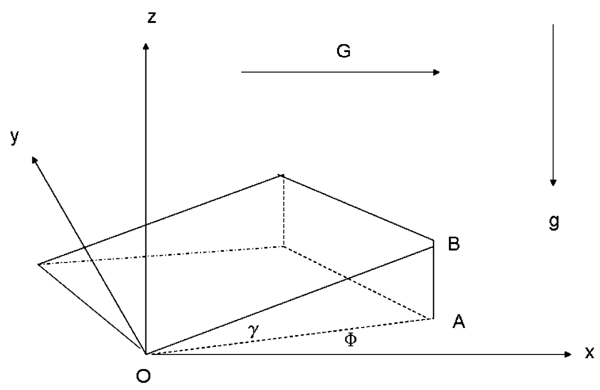

Fig. 1. – Sketch of the coordinate system relative to the sloping surface. The inclination of the slope is the angle subtended by BOA, *i.e.* γ. The slope is aligned at an angle ϕ with respect to the x-axis.

Zilitinkevich [29] showed that the height of the SBL could be expressed entirely in surface layer quantities, *i.e.*

$$h^2 = \sqrt{3} R_f k u_* \frac{L}{f}\,, \tag{5}$$

where L is the Monin-Obukhov length: $L = \frac{u_*^3}{B_0 k}$, k is von Kármán's constant and u_* is the friction velocity.

Equation (5) leads to the conclusion that the Zilitinkevich ratio, Z_i, is constant, where $Z_i = \sqrt{\frac{h^2|f|}{L u_*}}$. [25] showed that this result is not compromised under the assumption of a constant drainage force. Furthermore, they showed that in the limit $\gamma \to 0$, Derbyshire's results for the SBL over a horizontal surface are recovered.

The aim of this paper is to further investigate the influence of a shallow, uniformly sloping surface on the properties of the SBL by relaxing the assumption that the drainage force, $\mathcal{F}$, is constant across the depth of the boundary layer.

3. – Impact of height-dependent drainage forcing

In the work of [25] outlined in sect. **2**, the drainage forcing induced by the underlying sloping surface was assumed to be constant. This assumption will now be relaxed with the aim of understanding the effects of height distribution of the drainage force on bulk properties.

The quasi steady-state momentum equations can be written as

$$0 = f(V - V_g) - \frac{\mathrm{d}\overline{u'w'}}{\mathrm{d}z} + \mathcal{F}_u, \tag{6}$$

$$0 = -f(U - U_g) - \frac{\mathrm{d}\overline{v'w'}}{\mathrm{d}z} + \mathcal{F}_v, \tag{7}$$

where $\mathcal{F}_u$ and $\mathcal{F}_v$ are the components of the force $\mathcal{F}$ representing the drainage forcing induced by the uniform slope parallel to the x- and y-axes, respectively. Following

Derbyshire [9] we introduce a complex number notation for the mean wind velocity:

$$W \equiv U + iV, \tag{8}$$

where U and V are the wind components in the x- and y- horizontal axes, respectively. This allows us to express the relationship between the Reynolds stress, τ, the surface heat flux and the flux Richardson number as

$$\tau^* \frac{\mathrm{d}W}{\mathrm{d}z} = \left(\frac{B_0}{R_f}\right)\left(1 - \frac{z}{h}\right), \tag{9}$$

where $*$ indicates the complex conjugate. Multiplying eq. (7) by i, adding the result to eq. (6) and expressing $\tau = -\overline{u'w'} - i\overline{v'w'}$ gives

$$\frac{\mathrm{d}\tau}{\mathrm{d}z} = if(W - W_g) - (\mathcal{F}_u + i\mathcal{F}_v), \tag{10}$$

where $W_g = (U_g + iV_g)$ represents the synoptic pressure gradient and is written in the form of a geostrophic wind. Differentiating eq. (10) with respect to z, multiplying throughout by τ^* and then substituting for $\tau^* \frac{\mathrm{d}W}{\mathrm{d}z}$ using eq. (9) results in the second-order differential equation

$$\tau^* \frac{\mathrm{d}^2\tau}{\mathrm{d}z^2} = \left(if\frac{B_0}{R_f}\right)\left(1 - \frac{z}{h}\right) - \tau^* \frac{\mathrm{d}}{\mathrm{d}z}(\mathcal{F}_u + i\mathcal{F}_v), \tag{11}$$

assuming that $\mathrm{d}W_g/\mathrm{d}z = 0$.

Now, the corresponding equation in the absence of a slope, *i.e.*

$$\tau^* \frac{\mathrm{d}^2\tau}{\mathrm{d}z^2} = \left(if\frac{B_0}{R_f}\right)\left(1 - \frac{z}{h}\right), \tag{12}$$

is satisfied by a similarity function in which

$$\tau = u_*^2 \left(1 - \frac{z}{h}\right)^{\alpha}. \tag{13}$$

Substituting this into eq. (12) and equating powers of $(1 - \frac{z}{h})$ gives

$$\alpha^* + \alpha - 2 = 1. \tag{14}$$

Thus $\mathrm{Re}(\alpha) = 3/2$, where Re denotes the real part of α. Equating the numerical factors leads to $\mathrm{Im}(\alpha) \equiv \alpha_i = \frac{\sqrt{3}}{2}\mathrm{sgn}(f)$, where Im implies the imaginary part of the coefficient of α [9]. For now it will be assumed that the SBL is in the northern hemisphere thus a positive value of f may be assumed. Extending these ideas to seek a solution to eq. (11), we may assume that this differential equation is satisfied by a similarity function of the form $\tau = u_*^2(1 - \frac{z}{h})^{\frac{3}{2} + \alpha_i i}$, where α_i is a constant. This follows as once again, equating the powers of $(1 - \frac{z}{h})$ leads to $\mathrm{Re}(\alpha) = 3/2$, whilst α_i would be affected by the inclusion of the drainage forcing terms. Further consideration of α_i will be given later in this section.

Since $\tau^* \frac{\mathrm{d}^2\tau}{\mathrm{d}z^2}$ is proportional to $(1-\frac{z}{h})$, a solvable solution can be found with

$$\left(1-\frac{z}{h}\right)^{\alpha^*} \frac{\mathrm{d}}{\mathrm{d}z}(\mathcal{F}_u + i\mathcal{F}_v) \sim \left(1-\frac{z}{h}\right),$$

whence we get

$$\mathcal{F}_u + i\mathcal{F}_v = -\frac{\mathcal{F}_0 h}{\frac{1}{2}+i\alpha_i}\left(1-\frac{z}{h}\right)^{\frac{1}{2}+i\alpha_i}, \tag{15}$$

where $\mathcal{F}_0$ is a constant of integration. As the direction of $\mathcal{F}$ rotates anti-clockwise as we move away from the surface we can deduce that there is a region where the sign of $\mathcal{F}_u$ changes. This implies that, for example, if the force opposes the flow at the surface, then in some region above the surface, the effect of the drainage-force is to reinforce the flow. Thus the forcing effect of the slope can influence the wind shear far from the surface.

The magnitude of $\mathcal{F}$ is

$$|\mathcal{F}| = \left(\mathcal{F}_u^2 + \mathcal{F}_v^2\right)^{1/2} = \frac{\mathcal{F}_0 h}{\left(\frac{1}{4}+\alpha_i^2\right)^{1/2}}\left(1-\frac{z}{h}\right)^{1/2}.$$

Since $\mathcal{F}$ represents the effects of the underlying slope and is influenced by the relative buoyancy, $g\frac{\theta_{\mathrm{diff}}}{\theta_{\mathrm{ref}}}$, where θ_{diff} is the difference in potential temperature between the top of the SBL and the surface and θ_{ref} is a reference potential temperature (often taken to be 300 K), a typical functional form for the potential temperature could be used to give an estimate of the magnitude of $\mathcal{F}$. Several examples of functions that have been used are cited in [30]. Our function for $\mathcal{F}$ gives a realistic form for the potential temperature by following the square root function. This functional relationship implies that the potential temperature would increase gradually with height within the boundary layer, but as $z \to h$ the potential temperature would increase rapidly over a relatively shallow region. This behaviour is very similar to the linearly-mixed idealised model of the SBL described in [30], pp. 504-506. Stull's model allows the potential temperature to increase with height, but retains a strong increase in potential temperature at the top of the SBL. Such a regime is typically associated with moderately strong winds and turbulence. Thus we conclude that our analysis has led to a realistic functional form for $\mathcal{F}$.

Equation (11) can be written in the form

$$\tau^* \frac{\mathrm{d}^2\tau}{\mathrm{d}z^2} = \left(if\frac{B_0}{R_f}\right)\left(1-\frac{z}{h}\right) - \tau^*\gamma e^{i\phi}\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}z}. \tag{16}$$

Using eq. (15) to substitute for $\mathcal{F}$ into eq. (16) gives

$$\tau^*\tau'' + \tau^*\mathcal{F}_0\gamma e^{i\phi}\left(1-\frac{z}{h}\right)^{-\frac{1}{2}+\alpha_i i} = \left(if\frac{B_0}{R_f}\right)\left(1-\frac{z}{h}\right). \tag{17}$$

Substituting $\tau = u_*^2(1-\frac{z}{h})^{\frac{3}{2}+\alpha_i i}$ leads to the relationship

$$\frac{u_*^4}{h^2}\left(\frac{3}{4}-\alpha_i^2+2\alpha_i i\right) + u_*^2\mathcal{F}_0\gamma e^{i\phi} = if\frac{B_0}{R_f}. \tag{18}$$

We can equate the real and imaginary parts of eq. (18), respectively, to obtain

$$\frac{3}{4} + \frac{h^2}{u_*^2}\gamma \mathcal{F}_0 \cos\phi = \alpha_i^2, \tag{19}$$

and

$$2\frac{u_*^4}{h^2}\alpha_i + u_*^2 \mathcal{F}_0 \gamma \sin\phi = f\frac{B_0}{R_f}\,. \tag{20}$$

The next step is to solve eq. (19) for α_i. The positive root is taken since without the slope term the value of α_i is $\frac{\sqrt{3}}{2}$. (In the northern hemisphere the term $f\frac{B_0}{R_f}$ is positive because of the definitions of B_0 and f so α_i must be positive.) This gives

$$\alpha_i = \frac{\sqrt{3}}{2}\left(1 + \frac{4}{3}\mathcal{F}_0 \frac{h^2}{u_*^2}\gamma\cos\phi\right)^{1/2}, \tag{21}$$

from which we can obtain the following approximation:

$$\alpha_i \simeq \frac{\sqrt{3}}{2}\left(1 + \frac{2}{3}\mathcal{F}_0 \frac{h^2}{u_*^2}\gamma\cos\phi\right) + O(\gamma^2) \tag{22}$$

provided that $\frac{4}{3}\mathcal{F}_0\frac{h^2}{u_*^2}\gamma\cos\phi$ is less than one, and higher powers of γ are negligible. This is justified as we are considering slopes with gradient of the order of 1:1000.

Now using eq. (22) to substitute for α_i in eq. (20) results in

$$\sqrt{3}\frac{u_*^4}{h^2} + u_*^2\mathcal{F}_0\gamma\left(\frac{2\sqrt{3}}{3}\cos\phi + \sin\phi\right) = f\frac{B_0}{R_f}\,. \tag{23}$$

Using $B_0 = u_*^3/kL$, the following expression for h^2 can be derived:

$$h^2 = \sqrt{3}kR_f\frac{u_*L}{f}\left(1 - \frac{\mathcal{F}_0 L}{u_* f}kR_f\gamma\left(\frac{2\sqrt{3}}{3}\cos\phi + \sin\phi\right)\right)^{-1}. \tag{24}$$

Using eq. (24), to substitute for h^2 in Zilitinkevich's ratio gives

$$Z_i^2 = \frac{h^2 f}{u_* L} \tag{25}$$

$$= \frac{f}{u_*L}\sqrt{3}kR_f\left(\frac{f}{u_*L} - \frac{\mathcal{F}_0}{u_*^2}kR_f\gamma\left(\frac{2\sqrt{3}}{3}\cos\phi + \sin\phi\right)\right)^{-1},$$

$$\simeq \frac{f}{u_*L}\sqrt{3}kR_f\left(\frac{f}{u_*L} + \frac{\mathcal{F}_0}{u_*^2}kR_f\gamma\left(\frac{2\sqrt{3}}{3}\cos\phi + \sin\phi\right)\right), \tag{26}$$

neglecting higher powers of γ.

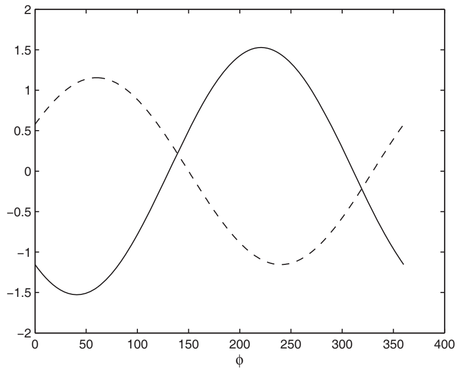

Fig. 2. – Plot showing variation of $-(\frac{2\sqrt{3}}{3}\cos\phi + \sin\phi)$ (continuous line) and $\sin\phi + \frac{1}{\sqrt{3}}\cos\phi$ (dashed line) with ϕ, where ϕ is the angle in degrees between the surface wind and the slope alignment.

The first term on the right-hand side of eq. (26) corresponds to Zilitinkevich's formula i) over horizontal terrain [29] and ii) over a uniformly sloping surface with constant drainage forcing [25]. The second and third terms on the right-hand side represent modifications depending upon the magnitude of the slope and alignment of the slope (with respect to the direction of the drainage force). Since $\mathcal{F}_0$ is negative, Z_i will be increased if the sign of $\frac{2\sqrt{3}}{3}\cos\phi + \sin\phi$ is negative. Therefore $-(\frac{2\sqrt{3}}{3}\cos\phi + \sin\phi)$ is plotted against ϕ (continuous line) in fig. 2. The effect of the slope on Z_i^2 is proportional to this quantity which is a function of the angle between the surface wind and slope alignment. The greatest increase in Z_i^2 occurs when $\phi \approx 221°$, and the greatest reduction occurs when $\phi \approx 41°$.

Since α is no longer a known constant, it is necessary to re-evaluate the drag law. Integrating eq. (9) between $z = 0$ and $z = h$, using $\alpha = \frac{3}{2} + \alpha_i i$, and taking the modulus gives

$$\frac{|W_g|}{u_*} = \frac{1}{kR_f}\frac{h}{L}\left|\frac{1}{\frac{1}{2} - \alpha_i i}\right|. \tag{27}$$

Upon rearrangement we obtain the following expression for u_*:

$$u_* = |W_g| k R_f \frac{L}{h}\left|\frac{1}{2} - \alpha_i i\right|. \tag{28}$$

Using eqs. (26) and (28), the heat flux, B_0, can be expressed as

$$B_0 = \frac{u_*^3}{kL} = |W_g|^2 k^2 R_f^2 \frac{L^2}{h^2}\left|\frac{1}{2} - \alpha_i i\right|^2 \frac{u_*}{kL}. \tag{29}$$

Now using eq. (19):

$$\begin{aligned}\left|\frac{1}{2} - \alpha_i i\right|^2 &= \frac{1}{4} + \alpha_i^2 \\ &= 1 + \frac{h^2}{u_*^2}\gamma\mathcal{F}_0\cos\phi.\end{aligned} \tag{30}$$

Hence using eqs. (24) and (30), the expression given in eq. (29) becomes

$$
\begin{aligned}
B_{\max} \equiv B_0 &= |W_g|^2 k R_f^2 u_* \frac{L}{h^2}\left(1 + \frac{h^2}{u_*^2}\gamma \mathcal{F}_0 \cos\phi\right) \\
&= |W_g|^2 \frac{R_f f}{\sqrt{3}}\left(1 - \frac{\mathcal{F}_0 L}{u_* f} k R_f \gamma \left(\frac{2\sqrt{3}}{3}\cos\phi + \sin\phi\right)\right) \\
&\quad + |W_g|^2 k R_f^2 \frac{L}{u_*}\gamma \mathcal{F}_0 \cos\phi. \\
&= \frac{|W_g|^2 R_f |f|}{\sqrt{3}} \\
&\quad - \frac{|W_g|^2 R_f^2}{\sqrt{3}} \frac{\mathcal{F}_0 L}{u_*} k\gamma \left(\sin\phi + \frac{1}{\sqrt{3}}\cos\phi\right).
\end{aligned}
\tag{31}
$$

Since the Coriolis parameter is negative in the southern hemisphere, replacing f by $|f|$ in eq. (31) leads to the general result applicable in both hemispheres. It can be seen from eq. (31) that, if $\gamma = 0$, then Derbyshire's result predicting a maximum sustainable downward surface buoyancy flux is recovered.

The extra term introduced by the inclusion of the height-dependent drainage force varies in magnitude according to the degree of inclination of the slope relative to the horizontal, and also with the direction of the slope with respect to the geostrophic wind direction. Since $\mathcal{F}_0$ is negative, $B_{\max}$ will be increased if the sign of $\frac{1}{\sqrt{3}}\cos\phi + \sin\phi$ is positive. $\frac{1}{\sqrt{3}}\cos\phi + \sin\phi$ is plotted against ϕ (dashed line) in fig. 2. The effect of the slope on $B_{\max}$ is proportional to this quantity which is a function of the angle between the surface wind and slope alignment. The greatest increase in $B_{\max}$ occurs when $\phi \approx 60°$, and the greatest reduction occurs when $\phi \approx 240°$.

4. – Conclusions

Derbyshire [9] analysed Nieuwstadt's theory [8] of the SBL over a uniform horizontal surface and deduced that a maximum sustainable surface buoyancy flux $B_{\max} \simeq \frac{R_{fc}}{\sqrt{3}} G^2 |f|$ could be predicted. Maguire and co-workers [25] introduced a shallow slope into Derbyshire's theory and modelled the effect of the slope by incorporating a drainage force which was constant with height. They found that a maximum sustainable surface buoyancy flux still existed, but that its magnitude was influenced by the presence of the sloping surface.

In this study the assumption that the drainage force is simply constant over the boundary layer was relaxed. By allowing the drainage force to vary with height, a function was found that is both physically plausible and also allows an analytical solution to the governing equations of the theoretical model, viz. $\mathcal{F} \propto (1 - \frac{z}{h})^{1/2}$.

A modified expression for the maximum surface buoyancy flux is found which was consistent with Derbyshire's results as the slope $\gamma \to 0$. In contrast to the previous study [25] where the drainage forcing was assumed to be constant across the boundary layer, the inclusion of height-dependent drainage forcing led to a modification of Zilitinkevich's ratio, Zi, where the additional term varied according to the angle between the surface wind direction and the orientation of the slope to the wind. Since Zi is a parameter that operates over relatively long time scales of the order of boundary layer development, recommendations for further work in this area include the numerical modelling of the SBL over a shallow uniform slope in order to investigate how the development of the SBL over time is influenced by the presence of a sloping surface. Results from such a study could be used to verify the validity of the assumptions made regarding the applicability of steady-state conditions.

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