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Finite deformation non-isotropic elasto-plasticity with evolving structural tensors. A framework

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Summary. — The paper discusses a constitutive model for finite deformation anisotropic elasto-plasticity, within the framework of the multiplicative decomposition of the deformation gradient and of the theory of the structural tensors. It is shown that the driving force for the evolution of the anelastic deformation is not the classic stress, rather a thermodynamic force where the stress is corrected for a term caused by the transformation of the material directions due to plastic deformations. Two scenarios can arise, one in which the material directions do not evolve, and the other in which the material directions can evolve due to anelastic phenomena. The general framework model presented here can be applied to various phenomena in material science.

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1. – Introduction

Finite deformation anisotropic elasto-plasticity raises several questions, among them, the role of the plastic spin, the evolution of the material directions, and the form of the dissipation. Historically, the earliest models referred to crystal plasticity, and it was assumed that during deformation material directions remain attached to the crystal structure, so they are not affected by irreversible processes [1, 2]. However, this is not always true: materials with a fibril structure, such as amorphous polymers, can modify their anisotropic structure when subjected to irreversible stretching, so that the yield locus changes its shape and rotates. Similarly, in the phenomenon of cold drawing, at low extension rate the molecular chains align in the direction of the applied stress, inducing anisotropy, and turning opaque an originally transparent sample. Biotissues, for example blood vessels, are made by collagen fibrils arranged in a cross lamellar fashion [3], and their directions can evolve for two phenomena: an applied stress causes reorientation,

but, in addition biological growth modify fibrils orientation [4]. It is well known that the mechanics of tissue growth can be analysed in a framework similar to that of finite deformation plasticity interpreting the anelastic gradient of deformation as the growth tensor. The main goal of the paper is to analyse the extension of finite deformation plasticity to the case that structural directions can evolve due to mechanical coupled phenomena.

It has already been observed that the expression of the plastic dissipation takes different forms according to the process being (globally) isotropic or not. The reader can refer for example to the paper of Lu and Papadopoulos and of Steinmann [5,6] where an expression of the dissipation for anisotropic materials is presented. The paper will show that these results are equivalent forms of the plastic dissipation for non-isotropic materials.

The schematization of the constitutive behaviour of anisotropic elastoplastic materials in the field of finite deformations is intensely debated in the literature. Some constitutive models use the total and the anelastic deformations as primary variables [6, 7] and an additive decomposition of the total deformation. The additive decomposition of the kinematic process, that retains the structure of the classical infinitesimal theory of plasticity, has been used by many authors [8, 7]. In any case it has been noted that at least for simple shear tests of rigid-plastic materials the additive strain model leads to a spurious shear stress increase or drop, so that its applicability to this kind of cases may be questioned [9].

We adopt the multiplicative decomposition of the gradient of deformation tensor, initially introduced by Lee [1], that is based on the decomposition of the deformation process in an elastic and an anelastic part introducing an intermediate, eventually fictitious, stress-free configuration of the continuum. It is well known that the multiplicative decomposition of the deformation gradient is not univocal since it is insensitive to a rigid-body motion. Even if in some cases, such as the isoclinic configuration of crystals originally introduced by Mandel, the intermediate configuration can be univocally determined, in general it is necessary to introduce some additional conditions in the constitutive model that rules the plastic spin in order to overcome this indeterminacy [10, 11]. Moreover, by definition, the existence of the stress-free intermediate configuration requires that the zero stress state be an admissible state for the material.

The free energy is assumed to depend only on the elastic part of the deformation process [5, 12, 13]. In the seventies and eighties, substantially two different approaches were followed: in the former, to which belong, among others, the models developed by Green, Naghdi, Casey, [14-18], the constitutive relations are developed using a free-energy function that, according to the principle of the determinism of stress [19], is explicitly invariant for an isomorphism applied to all the physically admissible configurations of the continuum. This requirement leads to a free-energy function depending on the total and the anelastic deformations that are so treated as primary variables. The second approach, that includes the models developed for instance by Lee [1], Mandel [20], Lubliner [21], Lubarda and Lee [22], Dashner [10], considers the full invariance requirement too restrictive in the development of the constitutive models so that it becomes impossible to model materials that show a persistent anisotropic lattice like crystals [10]. For this reason, in this approach it is assumed that the free energy be objective with respect only to the final configuration of the deformation process. In this case the potential must depend only on the elastic part of the deformation through the elastic right Cauchy-Green deformation tensor.

The main differences between the two approaches lie in the invariance property imposed to the free-energy function. The models using the total and the anelastic deformations as primary variables [14-16], according to the principle of the determin-

ism of stress [19], are requested to be invariant for an isomorphism applied to all the configurations introduced in the schematization of the kinematic process and so also to the intermediate configuration that is treated just as any other admissible configuration for the body. This derives from the assumption that the body is considered to be able to reach the intermediate configuration by an elastic unloading from the final one. On the other hand, this property appears redundant on the point of view of the constitutive model developed using the multiplicative decomposition scheme and depending only on the elastic right Cauchy-Green deformation tensor that ensures the invariance of the free-energy function for an isomorphism applied to the final configuration. In this way, as pointed out by Lubarda and Lee in the reply to the criticism of Casey in [17], the invariance requirement is guaranteed also for the intermediate configuration because it is the final configuration of an ideal deformation path obtained combining the deformation process and the elastic unloading.

The anisotropic characteristics of the material are schematized using the structural tensors that are added to the principal variables of the model [23-25]. The structural tensors are defined as a set of projectors on the preferential directions of the material. In crystal plasticity they can rotate but do not modify their orientation with respect to the crystal directions, so their evolution is ruled by the rotational part of the deformation gradient, obtained after a polar decomposition. An interesting discussion of crystal plasticity can be found in the paper of Mićunović in this same issue [26]. In this paper we refer to materials with a fibre structure, so that each fibre can stretch and slide. The projection of the total deformation on the fibre directions then rules the evolution of the material structure. Therefore we assume that the structural tensors evolve according to the entire deformation gradient.

Starting with the same idea, Steinmann and Menzel have developed a thermodynamic model for anisotropic elasto-plasticity that they claim to be quite general [5]. Starting from the reduced dissipation inequality, they introduced a non-standard form of dissipative material. The model developed in this paper is based on the same assumptions and, being thermodynamically based, obtains similar results to Steinmann and Menzel. However, they focused on the spatial configuration, while, in this paper, we focus on the intermediate configuration, which is more convenient for computational purposes. We obtain a form of the dissipation that is prone to a more direct physical interpretation, and that can be shown to be equivalent to the one presented in [5]. A new symmetric form of the dissipation is proposed, that avoids the use of Mandel stress tensor, more convenient in the numerical applications.

Successively, with reference to fibre materials, we examine a model characterised by the same structural tensors for both the elastic and the plastic potentials, focusing on the irreversible evolution of the structural tensors; some simple examples show how the model can be applied to fiber materials.

The model illustrated in the paper is purely phenomenological and thermodynamically consistent: in this sense it can be regarded as a framework for developing specific models applicable to materials with evolving material structure.

2. – Kinematics

The classical multiplicative decomposition of the deformation gradient, developed in the context of crystal plasticity [1], is used (see fig. 1). The deformation process is ideally decomposed in its elastic and anelastic parts introducing an intermediate, eventually fictitious, stress free configuration B_a, so that the deformation gradient tensor and the

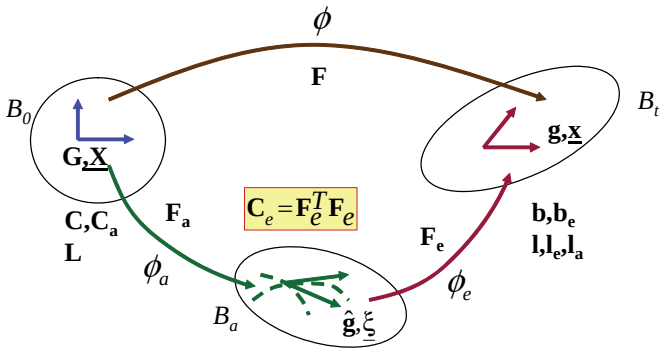

Fig. 1. – Process schematization.

Jacobian of the deformation process are decomposed as

$$\mathbf{F} = \mathbf{F}_e \mathbf{F}_a, \qquad J = J_e\, J_a. \tag{1}$$

In the paper, objects defined in the reference configuration B_0 will be preferentially indicated with capital letters, small letters will denote objects defined in the current configuration B_t, while a hat will generally distinguish objects defined in the intermediate configuration B_a, unless differently specified, generally for consistency with classical notations. To each configuration is associated a reference system, whose covariant base vectors are indicated as G_A, $\hat{\mathsf{g}}_\alpha$, g_a on the reference, intermediate and current configuration, respectively. The base vectors span locally the tangent spaces $T_P B_0, T_{\hat{p}} B_a, T_p B_t$.

Introducing the covariant metric tensors G, $\hat{\mathsf{g}}$, g of the reference, intermediate and current configuration frameworks, respectively, it is possible to define the family of deformation tensors reported in table I using the transport operators (Pull-back, Push-forward), denoted by ϕ^*, ϕ_*, respectively. The measures of deformation are defined as difference of the convective metrics projected onto the relevant configurations. One has, in the reference configuration

$$2\mathbf{E} = \phi^*(\mathsf{g}) - \mathsf{G}, \qquad 2\mathbf{E}_e = \phi^*(\mathsf{g}) - \phi_a^*(\hat{\mathsf{g}}), \qquad 2\mathbf{E}_a = \phi_a^*(\hat{\mathsf{g}}) - \mathsf{G}, \tag{2}$$

in the intermediate configuration

$$2\hat{\mathbf{E}} = \phi_e^*(\mathsf{g}) - \phi_{*a}(\mathsf{G}), \qquad 2\hat{\mathbf{E}}_e = \phi_e^*(\mathsf{g}) - \hat{\mathsf{g}}, \qquad 2\hat{\mathbf{E}}_a = \hat{\mathsf{g}} - \phi_{*a}(\mathsf{G}), \tag{3}$$

in the spatial configuration

$$2\mathbf{e} = \mathsf{g} - \phi_*(\mathsf{G}), \qquad 2\mathbf{e}_e = \mathsf{g} - \phi_{*e}(\hat{\mathsf{g}}), \qquad 2\mathbf{e}_a = \phi_{*e}(\hat{\mathsf{g}}) - \phi_*(\mathsf{G}). \tag{4}$$

Note that these measures of deformation are additive.

The gradient of velocity defined in the current configuration is denoted as $\boldsymbol{l} : T_x B_t \to T_x B_t$ and is given by

$$\boldsymbol{l} = \nabla_{\vec{x}} \dot{\vec{x}} = \dot{\mathbf{F}} \mathbf{F}^{-1}. \tag{5}$$

TABLE I. – *Convective metrics.*

Tangent space projections

B_0		B_a		B_t
	$\xrightarrow[\mathbf{F}_a^{-T}(\bullet)\mathbf{F}_a^{-1}]{\phi_{*a}(\bullet)}$		$\xrightarrow[\mathbf{F}_e^{-T}(\bullet)\mathbf{F}_e^{-1}]{\phi_{*e}(\bullet)}$	
G	$\xrightarrow{\phi_{*a}(\mathsf{G})}$	$\mathbf{b}_a^{-1}$	$\xrightarrow{\phi_{*e}\left(\mathbf{b}_a^{-1}\right)=\phi_*(\mathsf{G})}$	$\mathbf{b}^{-1}$
$\mathbf{C}_a=\mathbf{F}_a^T\mathbf{F}_a$	$\xleftarrow{\phi_a^*(\hat{\mathsf{g}})}$	$\hat{\mathsf{g}}$	$\xrightarrow{\phi_{*e}(\hat{\mathsf{g}})}$	$\mathbf{c}_a=\mathbf{b}_e^{-1}=\mathbf{F}_e^{-T}\mathbf{F}_e^{-1}$
$\mathbf{C}=\mathbf{F}^T\mathbf{F}$	$\xleftarrow{\phi_a^*(\mathbf{C}_e)=\phi^*(\mathsf{g})}$	$\mathbf{C}_e=\mathbf{F}_e^T\mathbf{F}_e$	$\xleftarrow{\phi^*{}_e(\mathsf{g})}$	g
	$\xleftarrow[\mathbf{F}_a^T(\bullet)\mathbf{F}_a]{\phi^*{}_a(\bullet)}$		$\xleftarrow[\mathbf{F}_e^T(\bullet)\mathbf{F}_e]{\phi^*{}_e(\bullet)}$	

The tensor $\boldsymbol{l}$ so defined is a mixed-variant tensor, acting on a vector of T_xB_t and yielding the derivative of the velocity in the tangent direction.

Analogously to what done in table I, it is possible to define the family of gradient of velocity tensors reported in table II and that are related among themselves by the pull-back push-forward operators. The pull back of $\boldsymbol{l}$ along the diagonal of the table yields the

TABLE II. – *Velocity tensors.*

B_0		B_a		B_t
	$\xrightarrow[\mathbf{F}_a^{-T}(\bullet)\mathbf{F}_a^{-1}]{\phi_{*a}(\bullet)}$		$\xrightarrow[\mathbf{F}_e^{-T}(\bullet)\mathbf{F}_e^{-1}]{\phi_{*e}(\bullet)}$	
$\mathbf{L}=\mathbf{F}^{-1}\dot{\mathbf{F}}$	$\longrightarrow$	$\mathbf{b}_a^{-1}\hat{\boldsymbol{l}}$	$\longrightarrow$	$\mathbf{b}^{-1}\boldsymbol{l}$
$\downarrow$	$\nwarrow$			$\uparrow$
$\hat{\mathbf{L}}=\mathbf{C}_a\mathbf{L}$	$\longleftarrow$	$\hat{\boldsymbol{l}}$	$\longrightarrow$	$\mathbf{b}_e^{-1}\boldsymbol{l}$
$\downarrow$			$\nwarrow$	$\uparrow$
$\hat{\hat{\mathbf{L}}}=\mathbf{C}\mathbf{L}=\mathbf{F}^T\dot{\mathbf{F}}$	$\longleftarrow$	$\hat{\hat{\boldsymbol{l}}}=\mathbf{C}_e\hat{\boldsymbol{l}}$	$\longleftarrow$	$\boldsymbol{l}$
	$\xleftarrow[\mathbf{F}_a^T(\bullet)\mathbf{F}_a]{\phi^*{}_a(\bullet)}$		$\xleftarrow[\mathbf{F}_e^T(\bullet)\mathbf{F}_e]{\phi^*{}_e(\bullet)}$	

mixed forms of the gradient of velocity in the intermediate and reference configurations. The remaining tensors contained in table II are covariant forms, obtained multiplying the mixed gradient by the relevant metrics. These forms are known as convective gradients of velocity.

It is possible to decompose each of the gradient of velocity tensors of table II in its elastic and anelastic parts as follows:

$$\boldsymbol{l} = \dot{\mathbf{F}}\mathbf{F}^{-1} = \boldsymbol{l}_e + \boldsymbol{l}_a \stackrel{\text{def}}{=} \dot{\mathbf{F}}_e\mathbf{F}_e^{-1} + \mathbf{F}_e\dot{\mathbf{F}}_a\mathbf{F}_a^{-1}\mathbf{F}_e^{-1}, \tag{6}$$

$$\hat{\boldsymbol{l}} = \hat{\boldsymbol{l}}_e + \hat{\boldsymbol{l}}_a \stackrel{\text{def}}{=} \mathbf{F}_e^{-1}\dot{\mathbf{F}}_e + \dot{\mathbf{F}}_a\mathbf{F}_a^{-1}, \tag{7}$$

$$\mathbf{L} = \mathbf{L}_e + \mathbf{L}_a \stackrel{\text{def}}{=} \mathbf{F}_a^{-1}\mathbf{F}_e^{-1}\dot{\mathbf{F}}_e\mathbf{F}_a + \mathbf{F}_a^{-1}\dot{\mathbf{F}}_a, \tag{8}$$

$$\hat{\hat{\boldsymbol{l}}} = \mathbf{F}_e^T\boldsymbol{l}\mathbf{F}_e = \hat{\hat{\boldsymbol{l}}}_e + \hat{\hat{\boldsymbol{l}}}_a \stackrel{\text{def}}{=} \mathbf{F}_e^T\dot{\mathbf{F}}_e + \mathbf{F}_e^T\mathbf{F}_e\dot{\mathbf{F}}_a\mathbf{F}_a^{-1}, \tag{9}$$

$$\hat{\hat{\mathbf{L}}} = \mathbf{F}^T\boldsymbol{l}\mathbf{F} = \hat{\hat{\mathbf{L}}}_e + \hat{\hat{\mathbf{L}}}_a \stackrel{\text{def}}{=} \mathbf{F}_a^T\mathbf{F}_e^T\dot{\mathbf{F}}_e\mathbf{F}_a + \mathbf{F}_a^T\mathbf{F}_e^T\mathbf{F}_e\dot{\mathbf{F}}_a, \tag{10}$$

$$\hat{\mathbf{L}} = \hat{\mathbf{L}}_e + \hat{\mathbf{L}}_a \stackrel{\text{def}}{=} \mathbf{F}_a^T\mathbf{F}_e^{-1}\dot{\mathbf{F}}_e\mathbf{F}_a + \mathbf{F}_a^T\dot{\mathbf{F}}_a. \tag{11}$$

It can be seen that the anelastic part of the velocity gradient of the intermediate configuration $\hat{\boldsymbol{l}}_a = \dot{\mathbf{F}}_a\mathbf{F}_a^{-1}$ has the important physical meaning of evolution of the intermediate configuration, $\hat{\boldsymbol{l}}_a = \frac{\partial \dot{\mathsf{x}}_a}{\partial \mathsf{x}_a}$.

Measures of the velocity of deformation are directly obtained from the symmetric part of the gradient of velocity tensors of table II, and can be split in their elastic and anelastic components as sketched in table III:

$$\mathbf{d} = \mathbf{d}_e + \mathbf{d}_a = \operatorname{sym}(\boldsymbol{l}_e) + \operatorname{sym}(\boldsymbol{l}_a). \tag{12}$$

An insight in the meaning of these tensors is given by the observation that they can be obtained performing objective time derivatives of the metrics reported in table I. As shown in table III the spatial velocity of deformation $\mathbf{d}$ is equal to half the Lie derivative of the spatial metric, and its pull-back on the reference configuration is the convective material velocity of deformation $\hat{\hat{\mathbf{D}}} = \dot{\mathbf{C}}/2$. Correspondingly, the material velocity of deformation is given by the Lie derivative along the flux $-\mathbf{v}$ of the metric G, whose push-forward on the spatial configuration is equal to $-\overline{\dot{\mathbf{b}^{-1}}}/2$. The velocity of deformation of the intermediate configuration is given by the backward anelastic Lie derivative L_v^a of the metric $\hat{\mathbf{g}}$ for its anelastic part plus the forward elastic Lie derivative L_v^e of $\hat{\mathbf{g}}$ for its elastic part, as indicated in table III. The definition of Lie derivative and of the Lie derivative along the flux $-\mathbf{v}$ is operationally defined as (see [27])

$$L_v(\bullet) = = \phi_* \left\{ \frac{\mathrm{d}}{\mathrm{d}t}\phi^*(\bullet)(t) \right\}, \tag{13}$$

$$L_{-v}(\bullet) = \phi^* \left\{ \frac{\mathrm{d}}{\mathrm{d}\tau}\phi_*\left(\bullet(t-\tau)\right) \right\}. \tag{14}$$

The stress tensors are introduced by means of the virtual power identity as follows:

$$P_{vi} = \int_{B_t} \boldsymbol{\sigma}\cdot\mathbf{d}\,\mathrm{d}v = \int_{B_0} \boldsymbol{\tau}\cdot(\mathbf{g}\boldsymbol{l})^{\text{sym}}\,\mathrm{d}V = \int_{B_0} \mathbf{g}\boldsymbol{\tau}\cdot\boldsymbol{l}\,\mathrm{d}V \qquad \boldsymbol{\tau} = J\boldsymbol{\sigma}. \tag{15}$$

TABLE III. – *Rate of deformation measures.*

B_0		B_a		B_t
	$\xrightarrow[\mathbf{F}_a^{-T}(\bullet)\mathbf{F}_a^{-1}]{\phi_{*a}(\bullet)}$		$\xrightarrow[\mathbf{F}_e^{-T}(\bullet)\mathbf{F}_e^{-1}]{\phi_{*e}(\bullet)}$	
$\frac{1}{2}L_{-v}(\mathsf{G}) = \mathbf{D} = \mathbf{D}_e + \mathbf{D}_a$	$\xrightarrow{\phi_{*a}}$	$\frac{1}{2}L^e_{-v}\left(\mathbf{b}_a^{-1}\right) = \left(\mathbf{b}_a^{-1}\hat{\mathbf{l}}\right)^{sym} = \left(\mathbf{b}_a^{-1}\hat{\mathbf{l}}_e\right)^{sym} - \frac{1}{2}\dot{\overline{\mathbf{b}_a^{-1}}}$	$\xrightarrow{\phi_{*e}}$	$-\frac{\dot{\overline{\mathbf{b}^{-1}}}}{2}$
$\frac{1}{2}L_{-v}(\mathbf{C}_a) + \frac{1}{2}\dot{\mathbf{C}}_a$	$\xleftarrow{\phi^{*}{}_a}$	$\frac{1}{2}L^e_{-v}\left(\hat{\mathbf{g}}\right) + \frac{1}{2}L^a_v\left(\hat{\mathbf{g}}\right) = \hat{\mathbf{d}}_e + \hat{\mathbf{d}}_a$	$\xrightarrow{\phi_{*e}}$	$-\frac{\dot{\overline{\mathbf{b}_e^{-1}}}}{2} + \frac{1}{2}L_v\left(\mathbf{b}_e^{-1}\right) = \left(\mathbf{b}_e^{-1}\mathbf{l}\right)^{sym}$
$\frac{1}{2}\dot{\mathbf{C}} = \hat{\hat{\mathbf{D}}}$	$\xleftarrow{\phi_a^*\left(\hat{\hat{\mathbf{d}}}\right)}$	$\frac{1}{2}L^a_v\left(\mathbf{C}_e\right) = \hat{\hat{\mathbf{d}}} = \frac{1}{2}\dot{\mathbf{C}}_e + sym\left(\mathbf{C}_e\hat{l}_a\right)$	$\xleftarrow{\phi^{*}{}_e(\mathbf{d})}$	$\frac{1}{2}L_v\mathbf{g} = \mathbf{d} = \mathbf{d}_e + \mathbf{d}_a = \frac{1}{2}L^e_v\mathbf{g} + \mathbf{d}_a$
	$\xleftarrow[\mathbf{F}_a^{T}(\bullet)\mathbf{F}_a]{\phi^{*}{}_a(\bullet)}$		$\xleftarrow[\mathbf{F}_e^{T}(\bullet)\mathbf{F}_e]{\phi^{*}{}_e(\bullet)}$	

Using the different gradient of velocity tensors defined in table II, according to the relevant metric, the virtual power can be given any of the equivalent forms summarized in table IV. The stress tensors are summarized in table V. In it on the diagonal are reported the mixed forms of the stress tensors known as Mandel stresses, obtained multiplying the contravariant form times the appropriate metric. The other contravariant forms are obtained by push-forward or pull-back, adding the relevant contravariant metric (not indicated in the table).

The definitions of the tensors reported in table V are

(16a) $\mathbf{S}_e = \phi_e^*(\boldsymbol{\tau}) = \mathbf{F}_e^{-1}\boldsymbol{\tau}\mathbf{F}_e^{-T}$ Elastic second Piola-Kirchhoff stress tensor,

(16b) $\mathbf{S} = \phi^*(\boldsymbol{\tau}) = \phi_a^*(\mathbf{S}_e) = \mathbf{F}_a^{-1}\mathbf{S}_e\mathbf{F}_a^{-T}$ Second Piola-Kirchhoff stress tensor,

TABLE IV. – *Virtual power.*

		B_0	B_a	B_t	
$\int_{B_0}$		$\mathbf{T}\cdot\mathbf{L}$	$\mathbf{T}_e\cdot\mathbf{b}_a^{-1}\hat{l}$	$\mathbf{b}\boldsymbol{\tau}\cdot\mathbf{b}^{-1}\boldsymbol{l}$	$\mathrm{d}V$
		$\boldsymbol{\Sigma}\cdot\hat{\mathbf{L}}$	$\boldsymbol{\Sigma}_e\cdot\hat{l}$	$\mathbf{b}_e\boldsymbol{\tau}\cdot\mathbf{b}_e^{-1}\boldsymbol{l}$	
		$\mathbf{S}\cdot\hat{\hat{\mathbf{D}}}$	$\mathbf{S}_e\cdot\hat{\hat{\mathbf{d}}}$	$\mathbf{g}\boldsymbol{\tau}\cdot\boldsymbol{l}$	

TABLE V. – *The stress tensor family.*

B_0		B_a		B_t
$\mathbf{T}$		$\mathbf{T}_e$		$\mathbf{b}\boldsymbol{\tau}$
$\downarrow$	$\nwarrow$	$\uparrow$		$\uparrow$
$\boldsymbol{\Sigma}$		$\boldsymbol{\Sigma}_e$		$\mathbf{b}_e\boldsymbol{\tau}$
$\downarrow$		$\uparrow$	$\nwarrow$	$\uparrow$
$\mathbf{S}$		$\mathbf{S}_e$		$\mathbf{g}\boldsymbol{\tau}$
	$\xleftarrow[\mathbf{F}_a^{-1}(\bullet)\mathbf{F}_a^{-T}]{\phi^*{}_a(\bullet)}$		$\xleftarrow[\mathbf{F}_e^{-1}(\bullet)\mathbf{F}_e^{-T}]{\phi^*{}_e(\bullet)}$	

(16c) $\boldsymbol{\Sigma}_e = \mathbf{C}_e\mathbf{S}_e$ Mandel stress tensor,

(16d) $\boldsymbol{\Sigma} = \mathbf{C}_a^{-1}\mathbf{C}\mathbf{S}$ Mandel stress tensor on B_0,

(16e) $\mathbf{T}_e = \mathbf{b}_a\boldsymbol{\Sigma}_e$,

(16f) $\mathbf{T} = \mathbf{C}\mathbf{S}$.

3. – Constitutive model

Much attention has recently received the problem of setting a consistent model for anisotropic elasto-plasticity at large strains [6, 5, 12, 8, 28] and it seems that there are still several open questions. In this section a model of anisotropic elasto-plasticity is developed allowing for the evolution of the material directions. The constitutive model is developed in a thermodynamical consistent framework, defining a free energy and exploiting the dissipation inequality.

Some definitions and results are preliminary recalled (see [24, 29]). It is well known that [27, p. 220] any function of G, $\mathbf{C}$ and θ [the temperature] is necessarily isotropic. To describe non-isotropic materials then requires the introduction of additional variables.

Let $O(3)$ denote the group of orthogonal transformations, $O(3) = \{\mathbf{Q} \in Lin_R \mid \mathbf{Q}^T\mathbf{Q} = \mathbf{I}\}$; a scalar-valued function f of tensorial arguments $\mathbf{D}$ is said to be invariant relative to the symmetry group $G \subset O(3)$ if

$$f\left(\mathbf{Q}^{-T}\mathbf{D}\mathbf{Q}^{-1}\right) = f\left(\mathbf{D}\right) \quad \forall \mathbf{Q} \in G. \tag{17}$$

If $G \equiv O\left(3\right)$ the function is said to be isotropic.

If the material group is characterized by symmetry with respect to special directions, then the invariant group is given by those transformations that leave unchanged some structural tensors $\mathbf{M}_i$:

$$G_0 = \{\mathbf{Q} \in O\left(3\right) : \mathbf{Q}\mathbf{M}_i\mathbf{Q}^T = \mathbf{M}_i\}. \tag{18}$$

The structural tensors are the invariant projectors generated from the unit vectors of the principal material directions $\vec{A}_i$ as [29]

$$\mathbf{M}_i = \vec{A}_i \otimes \vec{A}_i, \qquad i = 1, 2, 3. \tag{19}$$

By virtue of Neumann's principle, the symmetry group of a given material must be included in the symmetry group of any tensor function in any constitutive laws of the material. In fact the principle of isotropy of space [24, 25] states that

Theorem 1. Let f be an anisotropic scalar valued function of a set of variables $\{\mathbf{D}_i\}$, then f is invariant relative to the symmetry group G_0 (18) if and only if it can be expressed as an isotropic function of $\{\mathbf{D}_i\}$ and of the set of structural tensors $\{\mathbf{M}_j\}$ as follows:

$$f(\{\mathbf{D}_i\}) = \hat{f}(\{\mathbf{D}_i\}, \{\mathbf{M}_j\}) = \hat{f}\left(\{\mathbf{Q}\mathbf{D}_i\mathbf{Q}^T\}, \{\mathbf{Q}\mathbf{M}_j\mathbf{Q}^T\}\right) \ \forall \mathbf{Q} \in O(3). \tag{20}$$

To the function $\hat{f}$ can be applied the general representation theorem for scalar-valued isotropic functions [23-25].

Theorem 2. A scalar-valued isotropic function f of a finite number of tensor agencies and relative to any compact point symmetry group can be represented as a function of a complete and irreducible set of invariants of the tensorial arguments.

Lists of sets of invariants for one, two, three or more variables has been furnished in the literature for various types of material symmetry [30, 23, 31].

Theorem 1 and Theorem 2 enable us to express $\hat{f}$ in terms of a list of invariants of the tensor $\{\mathbf{D}_i\}$ and of the structure tensors $\{\mathbf{M}_j\}$.

Let $\vec{A}_i$ be tangent to a set of material directions, so that $\mathbf{M}_i = \vec{A}_i \otimes \vec{A}_i$ is the associated projector. These vectors transform according to

$$\hat{a}_i = \phi_{*a}(\vec{A}_i) = \mathbf{F}_a \vec{A}_i, \tag{21}$$

$$a_i = \phi_*(\vec{A}_i) = \mathbf{F}\vec{A}_i \tag{22}$$

and the structural tensors in the intermediate and spatial configurations are

$$\hat{\mathbf{m}}_i = \hat{a}_i \otimes \hat{a}_i = \mathbf{F}_a\vec{A}_i \otimes \mathbf{F}_a\vec{A}_i = \mathbf{F}_a\mathbf{M}_i\mathbf{F}_a^T, \tag{23}$$

$$\mathbf{m}_i = a_i \otimes a_i = \mathbf{F}\vec{A}_i \otimes \mathbf{F}\vec{A}_i = \mathbf{F}\mathbf{M}_i\mathbf{F}^T. \tag{24}$$

The tensors defined in this way are contravariant. They yield the component of a covector on the material direction $\vec{A}_i$ projected in the transformed direction of $\vec{A}_i$, *i.e.*

$$\hat{\mathbf{m}}_i\mathbf{G}^A = \left(\mathbf{G}^A \cdot \vec{A}_i\right)\hat{\vec{a}}_i, \tag{25}$$

$$\mathbf{m}_i\mathbf{g}^a = (\mathbf{g}^a \cdot \vec{a}_i)\,\vec{a}_i = \left(\mathbf{G}^A \cdot \vec{A}_i\right)\vec{a}_i. \tag{26}$$

Since $\mathbf{M}_i$ is a projector, one has

$$\mathbf{M}_i^n = \mathbf{M}_i \qquad \mathbf{m}_i^n = \mathbf{m}_i\|\mathbf{a}_i\|^{2(n-1)}. \tag{27}$$

The transformations of the symmetry group defined by (18) are mixed variant tensors whose push-forward is

$$\phi_*\mathbf{Q} = \mathbf{F}\mathbf{Q}\mathbf{F}^{-1}. \tag{28}$$

The symmetry groups relative to two different configurations differing for a superimposed diffeomorphism are related by Noll's rule:

$$G_{0a} = \phi_{*a} G_0 = \{\phi_{*a}\mathbf{Q}\}, \tag{29}$$

$$G_{0t} = \phi_{*} G_0 = \{\phi_{*}\mathbf{Q}\}, \tag{30}$$

so that one has

$$(\phi_*\mathbf{Q})\,\mathbf{m}_i\,(\phi_*\mathbf{Q})^T = \mathbf{F}\mathbf{Q}\mathbf{F}^{-1}\mathbf{m}_i\mathbf{F}^{-T}\mathbf{Q}^t\mathbf{F}^T = \mathbf{F}\mathbf{M}_i\mathbf{F}^T = \mathbf{m}_i. \tag{31}$$

3˙1. *Helmoltz free energy.* – Let G_0 be the (elastic) material symmetry group of the continuous. By virtue of the results reported in the previous section, the anisotropic Helmoltz free energy can be expressed as a scalar-valued isotropic function of the elastic deformation gradient and of the structural tensors $\hat{\mathbf{m}}_i$, that define the symmetry properties of the material in the intermediate configuration, as follows:

$$\hat{\psi} = \psi\left(\mathbf{F}_e, \hat{\mathbf{m}}_i\right). \tag{32}$$

The requirement of objectivity (rigid rotation of the reference framework) implies that the free energy remains unchanged for a rigid-body motion applied to the deformed configuration [36, p. 252], [27, p. 194]:

$$\mathbf{F}_e^+ \overset{\text{def}}{=} \mathbf{Q}\mathbf{F}_e; \hat{\mathbf{m}}_i^+ \overset{\text{def}}{=} \mathbf{Q}\hat{\mathbf{m}}_i | \mathbf{Q} \in \{SO3\} \rightarrow \psi\left(\mathbf{F}_e^+, \hat{\mathbf{m}}_i^+\right) = \psi\left(\mathbf{F}_e, \hat{\mathbf{m}}_i\right). \tag{33}$$

Relation (33) implies (see [27]) that the free energy depends on the elastic deformation through the elastic right Cauhcy-Green deformation tensor defined in table I, so that

$$\psi = \psi\left(\mathbf{C}_e, \hat{\mathbf{m}}_i\right). \tag{34}$$

By virtue of the representation theorems 1 and 2 [30, 23] the free energy can be represented as a function of a complete set of invariants of its arguments. Since $\mathbf{C}_e$ is a covariant tensor, in order to perform its trace, and the trace of its iterates, the contravariant metric $\hat{g}^\sharp$ has to be used. A list of the invariants can be obtained following the method indicated by Itskov [32]; it is given by the power up to the order 3 of the symmetric arguments. Accounting for (27) the independent invariants reduce to

$$\begin{array}{cccc} \mathrm{tr}(\mathbf{C}_e), & \mathrm{tr}(\mathbf{C}_e^2), & \mathrm{tr}(\mathbf{C}_e^3), & \mathrm{tr}(\hat{\mathbf{m}}_i) \\ \mathrm{tr}(\mathbf{C}_e\hat{\mathbf{m}}_i) & \mathrm{tr}(\mathbf{C}_e^2\hat{\mathbf{m}}_i), & \mathrm{tr}(\hat{\mathbf{m}}_i\hat{\mathbf{m}}_j) & i \neq j \neq k \\ , & \mathrm{tr}(\hat{\mathbf{m}}_i\hat{\mathbf{m}}_j\hat{\mathbf{m}}_k), & \mathrm{tr}(\mathbf{C}_e\hat{\mathbf{m}}_i\hat{\mathbf{m}}_j) & \end{array} \tag{35}$$

The trace of $\mathbf{m}_i$ can be replaced by $\|\hat{\mathbf{a}}_i\|^2$.

3˙2. *Mechanical dissipation.* – Assuming, for a general deformation process, that all the balance equations hold and supposing that an elastic free-energy potential ψ exists, it is well known that the second law of thermodynamics can be written for isothermal processes in the form [33]:

$$\mathcal{D} = \boldsymbol{\tau}\cdot\boldsymbol{l} - \rho_0\dot{\psi} \geq 0, \tag{36}$$

with ρ_0 = mass density in the reference configuration.

Alternative forms of the dissipation can be obtained using any one of the expressions of the internal power listed in table IV.

This relation represents the particularization to isothermal processes of the Clausius-Plank inequality. According to the maximum dissipation principle we can obtain a full set of constitutive relations for a general (objective) material.

The free energy refers to the intermediate configuration, although the material directions are defined in the material framework; introducing their push-forward into the elastic configuration (23) one has

$$\mathcal{D}_a = \mathbf{S}_e \cdot \hat{\dot{\mathbf{l}}} - \rho_0 \dot{\psi}(\mathbf{C}_e, \hat{\mathbf{m}}_i). \tag{37}$$

Splitting the dot product of the external power in its elastic and anelastic parts, taking account of the symmetry of $\mathbf{S}_e$ and performing the material derivative of the free energy (37) gives

$$\mathcal{D}_a = \mathbf{S}_e \cdot \hat{\dot{\mathbf{d}}}_e + \mathbf{S}_e \cdot \hat{\dot{\mathbf{d}}}_a - \rho_0 \nabla_{\mathbf{C}_e} \hat{\psi} \cdot \dot{\mathbf{C}}_e - \rho_0 \sum_{i=1,n} \nabla_{\hat{\mathbf{m}}_i} \hat{\psi} \cdot \dot{\hat{\mathbf{m}}}_i. \tag{38}$$

The expression of the material derivative of ψ can be obtained writing $\hat{\psi}_a(\mathbf{C}_e, \hat{\mathbf{m}}_i)$ as $\hat{\psi}_a(\mathbf{F}_a^{-T}\mathbf{C}\mathbf{F}_a^{-1}, \mathbf{F}_a\mathbf{M}_i\mathbf{F}_a^T)$ and performing the material derivatives in the reference configuration. For a purely elastic increment of deformation ($\dot{\mathbf{F}}_a = 0 \rightarrow \mathbf{d}_a = 0; D_a = 0$) (38) becomes

$$(\mathbf{S}_e - 2\rho_0 \nabla_{\mathbf{C}_e} \psi) \cdot \frac{\dot{\mathbf{C}}_e}{2} - \sum_{i=1,n} \nabla_{\hat{\mathbf{m}}_i} \hat{\psi} \cdot \left(\phi_{*a}(\dot{\mathbf{M}}_i) \right) = 0 \tag{39}$$

that yields the elastic constitutive equation $\mathbf{S}_e = 2\rho_0 \nabla_{\mathbf{C}_e} \hat{\psi}$ and the condition that for an elastic (non-dissipative) step must be $\dot{\mathbf{M}}_i = 0 \, \forall i$.

For a general deformation process one gets, from (38) and the elastic constitutive equation, the reduced dissipation inequality

$$\mathcal{D}_a = \mathbf{S}_e \cdot \hat{\dot{\mathbf{d}}}_a - \sum_{i=1,n} \rho_0 \nabla_{\hat{\mathbf{m}}_i} \hat{\psi} \cdot \dot{\hat{\mathbf{m}}}_i \geq 0. \tag{40}$$

Expression (40) of the dissipation contains symmetric tensors, but the time derivative of $\hat{\mathbf{m}}_i$ is not objective. The previous expression is similar to the one obtained by Steinmann and Menzel [5]. Recalling that

$$L_v^a(\hat{\mathbf{m}}_i) = \dot{\hat{\mathbf{m}}}_i - 2[\hat{\mathbf{l}}_i \hat{\mathbf{m}}_i]^{\mathrm{sym}}, \tag{41}$$

the dissipation (40) takes the form

$$\mathcal{D}_a = \mathbf{S}_e \cdot \hat{\dot{\mathbf{d}}}_a - \sum_{i=1,n} \rho_0 \nabla_{\hat{\mathbf{m}}_i} \psi \cdot L_v^a(\hat{\mathbf{m}}_i) - 2 \sum_{i=1,n} \rho_0 \nabla_{\hat{\mathbf{m}}_i} \psi \cdot \left[\hat{\mathbf{l}}_a \hat{\mathbf{m}}_i \right]^{\mathrm{sym}}. \tag{42}$$

A pull-back on the material frame yields

$$\mathcal{D}_0 = \mathbf{S} \cdot \hat{\dot{\mathbf{D}}}_a - \sum_{i=1,n} \rho_0 \nabla_{\mathbf{M}_i} \psi_0 \cdot \dot{M}_i - 2 \sum_{i=1,n} \rho_0 \nabla_{\mathbf{M}_i} \psi_0 \cdot [\mathbf{L}_a \mathbf{M}_i]^{\mathrm{sym}} . \tag{43}$$

Two possible scenarios are then included in the model: the material directions can undergo mechanical evolution, at the expenses of energy dissipation due to the thermodynamical forces $-\rho_0 \nabla_{\mathbf{M}_i} \psi_0$; if material directions do not evolve, in any case they are dragged due to plastic deformations; the last term in (43) is therefore associated to the anelastic velocity of deformation. In order to get the flow rule for it we go back to (38), rewriting the dissipation in the intermediate configuration introducing the mixed variant stress tensor $\mathbf{\Sigma}_e = \mathbf{C}_e \mathbf{S}_e$, the so-called Mandel stress:

$$\mathcal{D}_a = \mathbf{\Sigma}_e \cdot \hat{\mathbf{l}}_a - 2 \sum_{i=1,n} \rho_0 \nabla_{\hat{\mathbf{m}}_i} \psi \cdot \left[\hat{\mathbf{l}}_a \hat{\mathbf{m}}_i \right]^{\mathrm{sym}} - \sum_{i=1,n} \rho_0 \nabla_{\hat{\mathbf{m}}_i} \psi \cdot L_v^a(\hat{\mathbf{m}}_i). \tag{44}$$

Observing that

$$\nabla_{\hat{\mathbf{m}}_i} \psi \cdot \left[\hat{\mathbf{l}}_a \hat{\mathbf{m}}_i \right]^{\mathrm{sym}} = \nabla_{\hat{\mathbf{m}}_i} \psi \hat{\mathbf{m}}_i \cdot \hat{\mathbf{l}}_a \tag{45}$$

(where the symmetry of $\hat{\mathbf{m}}_i$ has been used), eq. (44) becomes

$$\mathcal{D}_a = \left(\mathbf{\Sigma}_e - 2 \sum_{i=1,n} \rho_0 \nabla_{\hat{\mathbf{m}}_i} \psi \hat{\mathbf{m}}_i \right) \cdot \hat{\mathbf{l}}_a - \sum_{i=1,n} \rho_0 \nabla_{\hat{\mathbf{m}}_i} \psi \cdot L_v^a(\hat{\mathbf{m}}_i) \tag{46}$$

Next we introduce, on the intermediate configuration, the dual thermodynamic forces

$$\begin{aligned} \hat{\mathbf{z}}_i &= \rho_0 \nabla_{\hat{\mathbf{m}}_i} \psi, \\ \hat{\mathbf{y}} &= 2\rho_0 \left(\mathbf{C}_e \nabla_{\mathbf{C}_e} \psi - \sum_{i=1,n} \nabla_{\hat{\mathbf{m}}_i} \psi \hat{\mathbf{m}}_i \right), \end{aligned} \tag{47}$$

so that the dissipation can be written as

$$\mathcal{D}_a = \sum_{i=1,n} \left(\hat{\mathbf{y}} \cdot \hat{\mathbf{l}}_a - \hat{\mathbf{z}}_i \cdot L_v^a(\hat{\mathbf{m}}_i) \right). \tag{48}$$

It is remarkable that the tensor $\hat{\mathbf{y}}$ is symmetric, as can be checked by direct evaluation. This is due to the fact that ψ is an isotropic function of $\mathbf{C}_e$ and $\hat{\mathbf{m}}_i$ An alternative expression of the dissipation is obtained multiplying expression (46) for the convective

metric $\mathbf{C}_e$ and recalling eq. (9):

$$\begin{aligned}\mathcal{D}_a &= \sum_{i=1,n} 2\rho_0 \left(\nabla_{\mathbf{C}_e}\hat{\psi} - \mathbf{C}_e^{-1}\nabla_{\hat{\mathbf{m}}_i}\hat{\psi}\hat{\mathbf{m}}_i\right)\cdot\hat{\hat{\boldsymbol{l}}}_a - \hat{\mathbf{z}}_i\cdot L_v^a(\hat{\mathbf{m}}_i)\\ &= \left(\mathbf{S}_e - 2\sum_{i=1,n}\mathbf{C}_e^{-1}\hat{\mathbf{z}}_i\hat{\mathbf{m}}_i\right)\cdot\hat{\hat{\boldsymbol{l}}}_a - \hat{\mathbf{z}}_i\cdot L_v^a(\hat{\mathbf{m}}_i)\\ &= \hat{\hat{\mathbf{y}}}_i\cdot\hat{\hat{\boldsymbol{l}}}_a - \hat{\mathbf{z}}_i\cdot L_v^a(\hat{\mathbf{m}}_i).\end{aligned} \tag{49}$$

Anelastic constitutive equations compatible with the dissipation inequality are developed within the framework of the generalized standard dissipative material [34], that is, we postulate the existence of an admissible elastic domain $\mathbb{E}(\hat{\mathbf{y}}_i, \hat{\mathbf{z}}_i)$ and make use of the maximum dissipation principle, stating that the actual value of the thermodynamic forces maximize expression (49) among all their admissible values.

Let

$$\mathbb{E} = \{(\hat{\mathbf{y}}, \hat{\mathbf{z}}) : \hat{\chi}\,(\hat{\mathbf{y}}, \hat{\mathbf{z}}_i) \leq 0\}. \tag{50}$$

The maximum dissipation principle and the stress admissibility condition can be enforced using the Lagrangian multiplier method, so that we can write the following constrained minimization problem:

$$\min_{\hat{\mathbf{y}}^*, \hat{\mathbf{z}}_i^*} \max_{\dot{\lambda}^*} \hat{\mathcal{L}}\left(\hat{\mathbf{y}}^*, \hat{\mathbf{z}}_i^*, \dot{\lambda}^*\right) = -\mathcal{D}_a\left(\hat{\hat{\mathbf{y}}}^*, \hat{\mathbf{z}}_i^*\right) + \dot{\lambda}^*\hat{\chi}\,(\hat{\mathbf{y}}, \hat{\mathbf{z}}_i)\,,$$

whose optimality conditions are

$$\hat{\mathbf{d}}_a = \dot{\lambda}\nabla_{\hat{\mathbf{y}}}\hat{\chi}\,(\hat{\mathbf{y}}, \hat{\mathbf{z}}_i) \tag{51a}$$

$$L_v^a(\hat{\mathbf{m}}_i) = \dot{\lambda}\nabla_{\mathbf{z}_i}\hat{\chi}\,(\hat{\mathbf{y}}, \hat{\mathbf{z}}_i) \tag{51b}$$

$$\hat{\chi}\,(\hat{\mathbf{y}}, \hat{\mathbf{z}}_i) = 0 \tag{51c}$$

completed by the Kuhn-Tucker conditions

$$\dot{\lambda} \geq 0 \qquad \hat{\chi}\,(\hat{\mathbf{y}}, \hat{\mathbf{z}}_i) \leq 0 \qquad \dot{\lambda}\hat{\chi}\,(\hat{\mathbf{y}}, \hat{\mathbf{z}}_i) = 0. \tag{51d}$$

Summarizing, the full set of constitutive relations on the intermediate configuration is given by eqs. (47), (51a)-(51d).

Only the symmetric part of the anelastic velocity gradient is thus defined through associated evolution flow rules. This is in agreement with the statement that the anelastic spin requires specific constitutive equations.

4. – Some consequences of the model

4·1. *Yield function.* – The equations of the model have been set in the intermediate configuration, since it is the natural frame for defining the free energy. However, the elastic domain is more naturally defined in the spatial configuration; in order to account

for anisotropy, applying the representation theorem, the function χ_t must be an isotropic function of its arguments and of the structural tensors $\mathbf{m}_i$, *i.e.*

$$\mathbb{E} = \{(\mathbf{y}, \mathbf{z}_i) : \chi\left(\mathbf{y}, \mathbf{z}, \mathbf{m}_i, \mathbf{g}, \mathbf{g}^\sharp\right) \leq 0\}, \tag{52}$$

where the dependence on the covariant and contravariant metrics, necessary for performing the trace operations, have been explicitly added. Indeed, the covariant metric applied to the tensors $\mathbf{y}, \mathbf{m}_i$ produces their mixed form, and the same does the contravariant metric with $\mathbf{z}_i$, *e.g.*, the trace of $\mathbf{y}^2$ is obtained through the explicit operation $\mathbf{y} \cdot \mathbf{g} \cdot \mathbf{y} : \mathbf{g}$. The yield function can then be represented in the intermediate configuration by means of a pull back operation, that is tantamount with performing a change of variables,

$$\hat{\chi}\left(\hat{\mathbf{y}}, \hat{\mathbf{z}}_i, \hat{\mathbf{m}}_i\right) = \chi\left(\hat{\mathbf{y}}, \hat{\mathbf{z}}_i, \hat{\mathbf{m}}_i, \mathbf{C}_e, \mathbf{C}_e^{-1}\right). \tag{53}$$

An alternate strategy is to set the constitutive equations in the spatial framework, thus projecting forward the free energy. The spatial formulation has been thoroughly investigated by Steinmann and Menzel [5], and it is able to shed light on the meaning of the thermodynamic variable $\mathbf{y}$. A detailed discussion is beyond the limit of the present work and will be the object of a forthcoming paper, more focused on computational developments.

4˙2. *The isotropic case.* – In an isotropic material all the directions are equivalent, or, alternatively, all the material tensors vanish, $\mathbf{M}_i = \mathbf{0}, \forall i$. The standard expression of the free energy is thus recovered, and the dissipation (42) takes the form

$$\mathcal{D}_{|ogg} = \mathbf{S}_e \cdot \hat{\dot{\mathbf{d}}}_a \geq 0. \tag{54}$$

The previous expression refers to stress and rate of deformation measures defined in the intermediate configuration. It can be equivalently expressed using stress and rate of deformation measures defined in the different configurations, using the results of table IV and table II. In particular, after some algebra, one has

$$B_t \qquad \mathcal{D}_t = \boldsymbol{\tau} \cdot \boldsymbol{l}_a = \boldsymbol{\tau} \cdot \mathbf{d}_a, \tag{55a}$$

$$B_a \qquad \mathcal{D}_a = \mathbf{S}_e \cdot \hat{\dot{\boldsymbol{l}}}_a = \mathbf{S}_e \cdot \hat{\dot{\mathbf{d}}}_a \tag{55b}$$

$$= \boldsymbol{\Sigma}_e \cdot \hat{\boldsymbol{l}}_a^T, \tag{55c}$$

$$B_0 \qquad \mathcal{D}_0 = \mathbf{S} \cdot \hat{\dot{\mathbf{L}}}_a \tag{55d}$$

$$= \boldsymbol{\Sigma} \cdot \hat{\mathbf{L}}_a^T = \mathbf{T} \cdot \mathbf{L}_a^T. \tag{55e}$$

The form (55e) of the dissipation has also been found by Lu and Papadopoulos [6].

It is underlined that in order to obtain the previous forms of dissipation it is sufficient that $\dot{\hat{\mathbf{m}}}_i = \mathbf{0}, \forall i$. The thermodynamic force $\hat{\mathbf{y}}$, since $\hat{\mathbf{z}}_i = \mathbf{0}$, melts down to the Mandel stress $\boldsymbol{\Sigma}_e$ and $\hat{\mathbf{y}} \to \mathbf{S}_e$. The associated flow rule becomes then

$$\hat{\dot{\boldsymbol{l}}}_a = \dot{\lambda} \nabla_{\mathbf{S}_e} \chi\left(\mathbf{S}_e\right). \tag{56}$$

4˙3. *Evolution of the material directions*. – A detailed examination of all the possible cases that can be described by the evolution of the material directions is not the goal of the present work, neither is the illustration of specific forms of the constitutive equations. The model presented here has been cast in an associated setting. In general, therefore, plastic deformation and evolution of the structural tensors occur together, and the calibration of the plastic potential requires accurate experimental check. In order to discuss some simple examples, the yield function is split in two parts, one, χ_1, depending on the internal stress $\mathbf{y}$ and the structural tensors, the other, χ_2, depending on $\mathbf{z}_i$ and the structural tensors, so that the classical form of plasticity is preserved. When $\chi_2(\mathbf{z}_i, \mathbf{m}_i) = 0$ the material direction $\mathbf{A}_i$ can evolve, and this can happen in either one of two ways: either the vector rotates, keeping the unitary norm, or its norm can be modified, without rotation, or a combination of the two. The second situation does not modify the anisotropic properties of the material, but modifies its response to external actions, since the internal energy depends on $\mathbf{m}_i$. This effect can, thus, produce hardening in a specific direction, not necessarily produced by plastic deformations (that occur only if also $\chi_1 = 0$). The increase in the stiffness of polymer materials due to stretching of the oriented chains can be modeled in this way.

In order to illustrate this effect, the simple stretching of a specimen having a family of fibres in the 1-direction is examined. Therefore the material is transversely isotropic, and the only relevant structural tensor is $\mathbf{M}_1 = \vec{G}_1 \otimes \vec{G}_1$. The specimen is subjected to a series of stretching processes, such that they maintain the same shape, but the axis of stretching rotates in the 12 plane. The deformation process is ruled by the following gradient of deformation tensor

$$\mathbf{F} = \mathbf{F}_1\mathbf{Q} \qquad \mathbf{F}_1 = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{\lambda}} \end{bmatrix}, \tag{57}$$

where $\mathbf{Q}$ is a rotation matrix in the plane 12, characterised by the angle α that the 1-axis forms with the direction of stretch. The deformation process chosen corresponds to an uniaxial stress state when the direction of stretch coincides with the 1-axis. For all the other stretch directions in the plane 12, because of the anisotropy of the specimen, the stress state is no longer uniaxial, but the 33 component of the stress is always zero.

We consider a simplified elastic potential proposed by Holzapfel and Gasser [3] for the media wall of human arteria, composed by two contributions, the first anisotropic and the second isotropic:

$$\hat{\psi} = \frac{k}{2}\left(\mathrm{tr}(\bar{\mathbf{C}}_e : \hat{\mathbf{m}}_i) - 1\right)^2 + c\left(\mathrm{tr}(\bar{\mathbf{C}}_e) - r\right). \tag{58}$$

The modified elastic Cauchy-Green tensor $\bar{\mathbf{C}}_e$ is obtained from a split of the deformation gradient in a spherical and a distortional part, according to the expression

$$\mathbf{F} = (J^{1/3}\mathbf{I})\bar{\mathbf{F}}. \tag{59}$$

In this way the elastic potential (58) rules only the deviatoric deformation process. The material is then assumed incompressible. This form of the elastic potential for arterial walls was first proposed by Fung [35]. The values of the material constants are $k = 0.8393, c = 3$, proposed in [3] for the media. The potential (58) is quite crude, containing

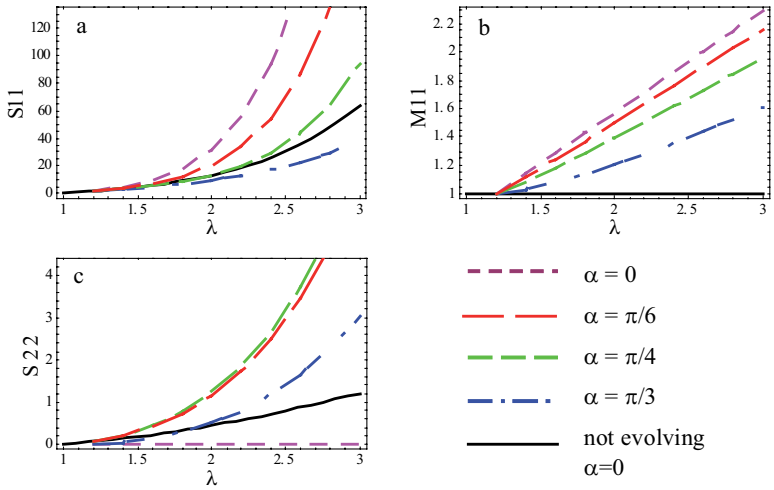

Fig. 2. – (Colour on-line) Simple stretching of a specimen having a set of fibres in direction 1. a) component of the second Piola-Kirchhoff tensor in the direction of stretching (kPa); b) 11 component of the structural tensor; c) component of the second Piola-Kirchhoff tensor in the direction normal to stretching (kPa).

only the first invariant of the deformation tensor, so that a not very accurate prediction of the stresses is expected, but it can effectively be used for our purposes.

Also in the yield function χ_2 only the first invariant of the thermodynamic force is considered,

$$\hat{\chi}_2(\hat{\mathbf{z}}_1) = \mathrm{tr}(\hat{\mathbf{z}}_1 : \hat{\mathbf{m}}_1) - \mu_0, \tag{60}$$

μ_0 being an activation energy for chains disentanglement. The hypothesis is made that disentanglement starts well before any plastic deformation can occur, that is reasonable for soft tissues and polymers in the glass transition region. In this way we can investigate the effects of the evolution of the structural tensor.

Figure 2 compares the response of the specimen for four directions of stretching ($\alpha = 0, \pi/6, \pi/4, \pi/3$) and with the response of the anisotropic specimen to a stretching in the 1-direction in the case the structural tensor is considered to remain constant. Comparing the latter (black curve) with the magenta curve relative to the case $\alpha = 0$ but with evolving structural tensor, the stiffening effect induced by the fibre orientation can be observed. Stretching the specimen in other directions also causes elongation in the chains, as can be seen from the diagram of the 11 component of the structural tensor $\mathbf{M}_1$, and stress arises also in the transverse direction. Given the simplicity of the anelastic potential (60), the fibre direction does not rotate in this example, and the evolution of the structural tensor is linear.

5. – Conclusions

The main findings of the study can be summarized as follows. Elastic anisotropic behaviour can be effectively described introducing suitable structural tensors in the free-energy functional, using the representation theorem of isotropic functions. The same is

needed in general for the yield function; in the case of isotropy it depends on an internal force that is given by the sum of the gradient of the free energy with respect to the elastic deformation tensor (the elastic stress tensor) plus the gradients of the free energy with respect to the structural tensors, since the rotation of the structural directions due to anelastic deformations generates energy dissipation. An eventual additional limit condition on the thermodynamic variables conjugated to the structural tensors rules the evolution of the material directions.

The equations of the model have been presented in the intermediate configuration, that is convenient for computational purposes. In forthcoming works the various forms of the limit conditions will be examined, and their consequences on simple deformation patterns, and the computational implementation will be discussed; indeed the exponential algorithm widely used in finite deformation plasticity is not applicable to non-isotropic models, and a specific integration algorithm has to be used.

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