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Considerations on incompressibility in linear elasticity

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Summary. — The classical way to treat incompressible linear elastic materials is to use the inverse constitutive relationship (strain as a function of the stress), based on the compliance tensor, in place of the direct constitutive equation (stress as a function of strain), based on the elasticity (stiffness) tensor. This is because the elasticity tensor is affected by a diverging bulk modulus, required in order to allow the material to sustain any hydrostatic load, and is therefore not defined. In this work we show that there is a part of the elasticity tensor that can be "saved" also for incompressible materials, by "filtering" the components that deal with hydrostatic loads. The procedure is based on the treatment of incompressibility by means of the constraint of isochoric motion, *i.e.* of conservation of volume, and fourth-order tensor algebra.

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1. – Introduction

Materials for which the stiffness under hydrostatic loads is very high compared to that under distortional loads are often assumed to be incompressible. In the realm of linear elasticity, the bulk elastic modulus, κ, of incompressible materials is thought to diverge, so that the elasticity (stiffness) tensor $\mathbb{L}$ is not defined. Therefore, "incompressible" materials are typically represented by means of the compliance tensor $\mathbb{Z}$ (see, *e.g.* [1]), so that the diverging modulus is reduced to a compliance approaching zero.

In this work, we aim at giving a representation of the (stiffness) elasticity tensor $\mathbb{L}$ that is suitable for the description of incompressible materials, with *no* diverging terms. In order to achieve this, we treat incompressibility by imposing the kinematical constraint of isochoric motion (*e.g.*, [2]). In linear elasticity, the linearised kinematical constraint

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is expressed by prescribing the infinitesimal strain $\boldsymbol{\varepsilon}$ to be purely deviatoric, *i.e.* such that $\mathrm{tr}(\boldsymbol{\varepsilon}) = 0$, and by adding the term $-p\boldsymbol{I}$ to the stress, with the hydrostatic pressure p having the meaning of Lagrange multiplier associated with the kinematical constraint $\mathrm{tr}(\boldsymbol{\varepsilon}) = 0$.

With no loss of generality, we show that, for the case of isotropic materials, this procedure automatically *filters* the diverging terms in the elasticity tensor $\mathbb{L}$, and provides the correct components of the *deviatoric part* of the elasticity tensor.

2. – General framework

Let us denote by $\mathbb{E}_2$ and $\mathbb{E}_4$ the spaces of second- and fourth-order tensors, respectively, and let $Q : \mathbb{E}_2 \to \mathbb{R}$ be a quadratic form defined by

$$Q(\boldsymbol{A}) = \frac{1}{2}\boldsymbol{A} : \mathbb{Q} : \boldsymbol{A}, \tag{2.1}$$

where $\boldsymbol{A} \in \mathbb{E}_2$ is a symmetric second-order tensor, and $\mathbb{Q} \in \mathbb{E}_4$ is a fourth-order tensor equipped with both diagonal and pair symmetry [3, 4]. The full-symmetric identity in the space of fourth-order tensors, $\mathbb{I} \in \mathbb{E}_4$, can be decomposed as [3]

$$\mathbb{I} = \mathbb{K} + \mathbb{M}, \tag{2.2}$$

where $\mathbb{K}$ and $\mathbb{M}$ are fourth-order tensors defined by

$$\mathbb{K} = \frac{1}{3}\boldsymbol{I} \otimes \boldsymbol{I}, \tag{2.3}$$

$$\mathbb{M} = \mathbb{I} - \frac{1}{3}\boldsymbol{I} \otimes \boldsymbol{I}, \tag{2.4}$$

respectively, and $\boldsymbol{I}$ is the identity in the space of second-order tensors $\mathbb{E}_2$. It can be shown [3] that tensors $\mathbb{K}$ and $\mathbb{M}$ are *orthogonal*, in the sense that $\mathbb{K} : \mathbb{M} = \mathbb{M} : \mathbb{K} = \mathbb{O}$ ($\mathbb{O} \in \mathbb{E}_4$ being the null fourth-order tensor), and *idempotent*, *i.e.* $\mathbb{K} : \mathbb{K} = \mathbb{K}$, and $\mathbb{M} : \mathbb{M} = \mathbb{M}$.

For any symmetric second-order tensor $\boldsymbol{A} \in \mathbb{E}_2$, the application of tensors $\mathbb{K}$ and $\mathbb{M}$ onto $\boldsymbol{A}$ gives the spherical and deviatoric part of $\boldsymbol{A}$, respectively, *i.e.*

$$\mathbb{K} : \boldsymbol{A} = A^{\mathrm{s}}\boldsymbol{I} = \frac{1}{3}\,\mathrm{tr}(\boldsymbol{A})\boldsymbol{I}, \qquad \text{and} \qquad \mathbb{M} : \boldsymbol{A} = \boldsymbol{A}^{\mathrm{d}} = \boldsymbol{A} - \frac{1}{3}\,\mathrm{tr}(\boldsymbol{A})\boldsymbol{I}. \tag{2.5}$$

It can be proven (cf., for example, [3]) that tensors $\mathbb{K}$ and $\mathbb{M}$ constitute a basis spanning the subspace of $\mathbb{E}_4$ of symmetric, *isotropic* fourth-order tensors. Therefore, if $\mathbb{Q}$ is assumed to be isotropic, it is possible to write

$$\mathbb{Q} = q_K\mathbb{K} + q_M\mathbb{M}, \tag{2.6}$$

where q_K and q_M are the scalar coefficients of the representation of $\mathbb{Q}$ in the basis $\{\mathbb{K}, \mathbb{M}\}$. Because of the orthogonality of this representation, it is easy to show that the inverse of $\mathbb{Q}$ is given by

$$\mathbb{Q}^{-1} = \frac{1}{q_K}\mathbb{K} + \frac{1}{q_M}\mathbb{M}. \tag{2.7}$$

Use of the decomposition of $\mathbb{I}$ given in eq. (2.2) leads to the identities

$$\boldsymbol{A} = \mathbb{I} : \boldsymbol{A} = \mathbb{K} : \boldsymbol{A} + \mathbb{M} : \boldsymbol{A}, \tag{2.8}$$
$$\mathbb{Q} = \mathbb{I} : \mathbb{Q} : \mathbb{I}. \tag{2.9}$$

By use of eqs. (2.2) and (2.9), and the symmetry of $\boldsymbol{A}$, the quadratic form $Q(\boldsymbol{A})$ in eq. (2.1) can be rewritten as

$$Q(\boldsymbol{A}) = \frac{1}{2}\,\boldsymbol{A} : [\mathbb{K} : \mathbb{Q} : \mathbb{K} + \mathbb{K} : \mathbb{Q} : \mathbb{M} + \mathbb{M} : \mathbb{Q} : \mathbb{K} + \mathbb{M} : \mathbb{Q} : \mathbb{M}] : \boldsymbol{A}. \tag{2.10}$$

This result can be recast in a more compact notation by introducing the following decomposition of $Q(\boldsymbol{A})$:

$$Q_K(\boldsymbol{A}) = \frac{1}{2}\,\boldsymbol{A} : [\mathbb{K} : \mathbb{Q} : \mathbb{K}] : \boldsymbol{A}, \tag{2.11}$$
$$Q_{\text{mixed}}(\boldsymbol{A}) = \frac{1}{2}\,\boldsymbol{A} : [\mathbb{K} : \mathbb{Q} : \mathbb{M} + \mathbb{M} : \mathbb{Q} : \mathbb{K}] : \boldsymbol{A}, \tag{2.12}$$
$$Q_M(\boldsymbol{A}) = \frac{1}{2}\,\boldsymbol{A} : [\mathbb{M} : \mathbb{Q} : \mathbb{M}] : \boldsymbol{A}. \tag{2.13}$$

Indeed, use of definitions (2.11)–(2.13) allows for rewriting eq. (2.10) as

$$Q(\boldsymbol{A}) = Q_K(\boldsymbol{A}) + Q_{\text{mixed}}(\boldsymbol{A}) + Q_M(\boldsymbol{A}). \tag{2.14}$$

Under the assumption of isotropy of $\mathbb{Q}$ (cf. eq. (2.6)), the quantity $Q_{\text{mixed}}(\boldsymbol{A})$ vanishes identically because tensors $\mathbb{K}$ and $\mathbb{M}$ are orthogonal to each other. The orthogonality of tensors $\mathbb{K}$ and $\mathbb{M}$ also implies that the products $\mathbb{K} : \mathbb{Q} : \mathbb{K}$ and $\mathbb{M} : \mathbb{Q} : \mathbb{M}$, which feature in the definitions of $Q_K(\boldsymbol{A})$ and $Q_M(\boldsymbol{A})$ (cf., eqs. (2.11) and (2.13), respectively), become

$$\mathbb{K} : \mathbb{Q} : \mathbb{K} = q_K\mathbb{K}, \qquad \text{and} \qquad \mathbb{M} : \mathbb{Q} : \mathbb{M} = q_M\mathbb{M}. \tag{2.15}$$

Finally, the quantity $Q(\boldsymbol{A})$ in eq. (2.14) becomes

$$Q(\boldsymbol{A}) = \frac{1}{2} 3 q_K (A^{\text{s}})^2 + \frac{1}{2} q_M \boldsymbol{A}^{\text{d}} : \boldsymbol{A}^{\text{d}}, \tag{2.16}$$

where $A^{\text{s}} = \frac{1}{3}\,\text{tr}(\boldsymbol{A})$ is the spherical component of tensor $\boldsymbol{A}$, and $\boldsymbol{A}^{\text{d}} : \boldsymbol{A}^{\text{d}} = \text{tr}[(\boldsymbol{A}^{\text{d}})^T \boldsymbol{A}^{\text{d}}]$.

If we give Q, $\mathbb{Q}$ and $\boldsymbol{A}$ the meaning of strain energy potential W, linear elasticity tensor $\mathbb{L}$ and infinitesimal strain $\boldsymbol{\varepsilon}$, respectively, the quadratic form (2.1) reads

$$W(\boldsymbol{\varepsilon}) = \frac{1}{2}\,\boldsymbol{\varepsilon} : \mathbb{L} : \boldsymbol{\varepsilon}. \tag{2.17}$$

By assuming that $\mathbb{L}$ is isotropic, it is possible to decompose it into

$$\mathbb{L} = L_K\mathbb{K} + L_M\mathbb{M} = 3\kappa\mathbb{K} + 2\mu\mathbb{M}, \tag{2.18}$$

where κ and μ in $L_K = 3\kappa$ and $L_M = 2\mu$ are the *bulk modulus* and the *shear modulus*, respectively. Following the procedure outlined above, we can write eq. (2.17) in the same form as (2.16), *i.e.*

$$W(\boldsymbol{\varepsilon}) = \frac{1}{2}[9\kappa(\varepsilon^{\mathrm{v}})^2 + 2\mu\, \boldsymbol{\varepsilon}^{\mathrm{d}} : \boldsymbol{\varepsilon}^{\mathrm{d}}], \tag{2.19}$$

where $\varepsilon^{\mathrm{v}} = \varepsilon^{\mathrm{s}}$ is the volumetric (spherical) component of the strain. If we instead give Q, $\mathbb{Q}$ and $\boldsymbol{A}$ the meaning of complementary potential Ψ (the Legendre transform of W), linear compliance tensor $\mathbb{Z}$ and Cauchy stress $\boldsymbol{\sigma}$, respectively, the quadratic form (2.1) reads

$$\Psi(\boldsymbol{\sigma}) = \frac{1}{2}\, \boldsymbol{\sigma} : \mathbb{Z} : \boldsymbol{\sigma}, \tag{2.20}$$

and an isotropic compliance tensor $\mathbb{Z}$ can be decomposed into

$$\mathbb{Z} = Z_K \mathbb{K} + Z_M \mathbb{M} = \frac{1}{3\kappa}\mathbb{K} + \frac{1}{2\mu}\mathbb{M}, \tag{2.21}$$

where the components $Z_K = \frac{1}{3\kappa}$ and $Z_M = \frac{1}{2\mu}$ are the algebraic inverses of $L_K = 3\kappa$ and $L_M = 2\mu$, according to (2.7). Following again the same procedure, eq. (2.20) can be written as

$$\Psi(\boldsymbol{\sigma}) = \frac{1}{2}\left[\frac{1}{\kappa}(\sigma^{\mathrm{h}})^2 + \frac{1}{2\mu}\boldsymbol{\sigma}^{\mathrm{d}} : \boldsymbol{\sigma}^{\mathrm{d}}\right], \tag{2.22}$$

where $\sigma^{\mathrm{h}} = \sigma^{\mathrm{s}} = \frac{1}{3}\,\mathrm{tr}(\boldsymbol{\sigma})$ represents the hydrostatic (spherical) contribution of the stress.

In eqs. (2.19) and (2.22), the volumetric contribution to $W(\boldsymbol{\varepsilon})$ and the hydrostatic contribution $\Psi(\boldsymbol{\sigma})$ are completely decoupled from their deviatoric counterparts. Equation (2.22) becomes useful, for example, for a treatment of *incompressibility* based on the algebraic properties of fourth-order tensors. For the sake of brevity, this paper has been limited to the case of isotropic elastic materials.

3. – Constrained strain energy function

According to eq. (2.5), the strain tensor $\boldsymbol{\varepsilon} \in \mathbb{E}_2$ admits the additive decomposition given by tensors $\mathbb{K}$ and $\mathbb{M}$:

$$\boldsymbol{\varepsilon} = \mathbb{K} : \boldsymbol{\varepsilon} + \mathbb{M} : \boldsymbol{\varepsilon} = \varepsilon^{\mathrm{v}}\boldsymbol{I} + \boldsymbol{\varepsilon}^{\mathrm{d}}. \tag{3.1}$$

It is therefore possible to introduce two functions, denoted by $\mathcal{K} : \mathbb{E}_2 \to \mathbb{R}$ and $\boldsymbol{\mathcal{M}} : \mathbb{E}_2 \to \mathbb{E}_2$, respectively, such that $\mathcal{K}(\boldsymbol{A}) = A^{\mathrm{s}}$, and $\boldsymbol{\mathcal{M}}(\boldsymbol{A}) = \mathbb{M} : \boldsymbol{A} = \boldsymbol{A}^{\mathrm{d}}$.

If we let $W : \mathbb{E}_2 \to \mathbb{R}$ denote the strain energy, then there exists a function $U : \mathbb{R} \times \mathbb{E}_2 \to \mathbb{R}$ such that the following identity holds:

$$W(\boldsymbol{\varepsilon}) = U(\mathcal{K}(\boldsymbol{\varepsilon}), \boldsymbol{\mathcal{M}}(\boldsymbol{\varepsilon})). \tag{3.2}$$

3˙1. *Incompressible case.* – In the theoretical framework of linear elasticity, incompressibility is modelled by imposing the kinematic constraint $\mathrm{tr}(\boldsymbol{\varepsilon}) = 0$. This constraint can be accounted for by introducing a new function $V : \mathbb{E}_2 \to \mathbb{R}$ such that eq. (3.2) is re-defined as

$$W(\boldsymbol{\varepsilon}, \pi) = U(\mathcal{K}(\boldsymbol{\varepsilon}), \boldsymbol{\mathcal{M}}(\boldsymbol{\varepsilon}), \pi) = V(\boldsymbol{\mathcal{M}}(\boldsymbol{\varepsilon})) - \pi\,\mathcal{K}(\boldsymbol{\varepsilon}), \tag{3.3}$$

where π is the unknown scalar Lagrange multiplier associated with the constraint $\mathcal{K}(\boldsymbol{\varepsilon}) = 0$.

Within the quadratic approximation of the strain energy density, and under the assumption of isotropic material, the procedure shown in sect. **2** leads to express $V(\boldsymbol{\mathcal{M}}(\boldsymbol{\varepsilon}))$ in eq. (3.3) as

$$V(\boldsymbol{\mathcal{M}}(\boldsymbol{\varepsilon})) = \frac{1}{2} L_M \boldsymbol{\mathcal{M}}(\boldsymbol{\varepsilon}) : \boldsymbol{\mathcal{M}}(\boldsymbol{\varepsilon}) = \frac{1}{2} L_M \boldsymbol{\varepsilon}^{\mathrm{d}} : \boldsymbol{\varepsilon}^{\mathrm{d}}, \tag{3.4}$$

where, according to the isotropic decomposition of the elasticity tensor given in eq. (2.6), $L_M = 2\mu$ denotes the coefficient of $\mathbb{M}$, *i.e.* twice the shear modulus.

By substituting eq. (3.4) back into eq. (3.3), the strain energy density reads

$$W(\boldsymbol{\varepsilon}, \pi) = \frac{1}{2}(2\mu)\boldsymbol{\mathcal{M}}(\boldsymbol{\varepsilon}) : \boldsymbol{\mathcal{M}}(\boldsymbol{\varepsilon}) - \pi\,\mathcal{K}(\boldsymbol{\varepsilon}), \tag{3.5}$$

and the Cauchy stress is given by

$$\boldsymbol{\sigma} = \frac{\partial W}{\partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}, \pi) = -\frac{1}{3}\pi \boldsymbol{I} + 2\mu \boldsymbol{\varepsilon}^{\mathrm{d}}. \tag{3.6}$$

If we now set $\pi/3 = p$, eq. (3.6) becomes

$$\boldsymbol{\sigma} = -p\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}^{\mathrm{d}}, \tag{3.7}$$

where the deviatoric part of stress is given by $\boldsymbol{\sigma}^{\mathrm{d}} = 2\mu\boldsymbol{\varepsilon}^{\mathrm{d}}$. This result can be used in order to perform the Legendre transformation of $V(\mathcal{M}(\boldsymbol{\varepsilon})) = V(\boldsymbol{\varepsilon}^{\mathrm{d}})$ into a function of the stress, $\Psi(\boldsymbol{\sigma}^{\mathrm{d}})$. Indeed, by introducing the compliance Z_M, and writing $\boldsymbol{\varepsilon}^{\mathrm{d}} = Z_M \boldsymbol{\sigma}^{\mathrm{d}}$, we obtain

$$\Psi(\boldsymbol{\sigma}^{\mathrm{d}}) = \frac{1}{2} Z_M \boldsymbol{\sigma}^{\mathrm{d}} : \boldsymbol{\sigma}^{\mathrm{d}}. \tag{3.8}$$

For eq. (3.8) to be consistent with the expression of the elastic energy as a function of the Cauchy stress, given in eq. (2.22), it is necessary that $Z_M = \frac{1}{2\mu}$, and that the material constant Z_K vanishes identically, *i.e.* $Z_K = 0$ (cf., for example, [1]). Thus, for the case of an incompressible linear elastic material, there can be *no* hydrostatic contribution to the compliance. This result can be seen as the outcome of a "*filtering procedure*" that, applied to $\Psi(\boldsymbol{\sigma})$ in eq. (2.22), gives the correct expression of the elastic energy to be used in the case of an incompressible linear elastic material. Finally, the inverse Legendre transformation of the "*filtered*" energy, *i.e.* $\Psi(\boldsymbol{\sigma}^{\mathrm{d}})$, gives the strain energy for incompressibility, *i.e.* $V(\boldsymbol{\varepsilon}^{\mathrm{d}})$.

3˙2. *Comparison with the case of an incompressible fluid.* – We assumed that function V depends only on the deviatoric part of the strain tensor, $\boldsymbol{\mathcal{M}}(\boldsymbol{\varepsilon})$, on the basis of considerations similar to those made by Eringen in [5], when treating the case of an incompressible fluid. Indeed, by writing the Helmholtz free-energy density of an incompressible fluid in a form analogous to eq. (3.3), the energy term denoted by V cannot depend on the fluid mass density. This result, however, can be generalised to the case of a fluid whose mass density, rather than being constant (this would be the case of a *truly* incompressible fluid), is prescribed through a constitutive (or state) function of a set of thermodynamic state variables other than the pressure. If the mass density is independent of fluid pressure, the Helmholtz free-energy density *cannot* be transformed into the Gibbs free-energy density by performing a Legendre transformation between mass density and pressure. On the other hand, by writing

$$W(\varrho,\varphi) = V(\varphi) - \pi\left[\frac{1}{\varrho} - \frac{1}{\varrho_0(\varphi)}\right], \tag{3.9}$$

where φ denotes a generic thermodynamic variable (other than pressure) assigned to the fluid, and ϱ_0 a given function of φ, it is possible to define the Gibbs-like potential

$$G(\pi,\varphi) := V(\varphi) + \frac{\pi}{\varrho_0(\varphi)}, \tag{3.10}$$

where the Lagrange multiplier π is identified with the fluid pressure. The case of incompressible fluid is recovered if the constitutive law $\varrho(\varphi)$ reduces to requiring that the fluid mass density equals a reference constant, *i.e.* $\varrho = \bar{\varrho}_0$.

4. – Filtering procedure

In subsect. **3**˙1, we assumed that function V depends only on the deviatoric part $\boldsymbol{\varepsilon}^{\mathrm{d}}$ of the strain tensor. In general, however, we may let V depend on the whole strain tensor, $\boldsymbol{\varepsilon}$, under the assumption that a "*fictitious*" elasticity tensor, $\mathbb{L}^*$, exists. Under this assumption, function V becomes

$$V(\boldsymbol{\varepsilon}) = \frac{1}{2}\boldsymbol{\varepsilon} : \mathbb{L}^* : \boldsymbol{\varepsilon}. \tag{4.1}$$

However, this expression can be also written as

$$V(\boldsymbol{\varepsilon}) = \frac{1}{2}3L_K^*(\varepsilon^{\mathrm{v}})^2 + \frac{1}{2}L_M^*\boldsymbol{\varepsilon}^{\mathrm{d}} : \boldsymbol{\varepsilon}^{\mathrm{d}}. \tag{4.2}$$

Now, for consistency with the theory shown in subsect. **3**˙1, the material constant L_K^* has to be chosen equal to zero, *i.e.* $L_K^* = 0$, while the material constant L_M^* has to coincide with the "*true*" one, and is therefore defined as $L_M^* = L_M = 2\mu$.

The result $L_K^* = 0$ does not contradict the fact that, for an incompressible material, the bulk modulus diverges. Indeed, consistency with previous theories is retrieved by admitting that the "*true*" bulk modulus is given by

$$\kappa = \frac{1}{3}L_K^* + \kappa_0. \tag{4.3}$$

If this is the case, requiring $L^*_K = 0$ and that κ_0 diverges, implies that the true bulk modulus, κ, diverges too, as the theory predicts.

5. – Conclusion

The procedure shown in sect. **4** applies in a straightforward way when it is used directly on Ψ, for it leads to setting $Z_K = 0$. When this procedure is applied to the strain energy density function, it should be used in order to identify function Ψ. Consistency with previous theories is retrieved by admitting that the "true" volumetric elastic coefficient, L_K, diverges (whereas the compliance Z_K is zero). This result is not in contradiction with the fact that L_K contains additively the fictitious coefficient L^*_K which, in turn, vanishes identically.

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