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Gauge invariance and effective sources in classical electrodynamics—A methodological note

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Summary. — It is shown by direct integration how transverse currents can be looked at as a strict consequence of gauge invariants in the restricted Coulomb gauge. Some consequences as about "clothed charges" even at the classical level are also briefly exploited, jointly to a brief touch on the role played by gauge symmetry *vs.* Lorentz invariance into the electromagnetic properties of the physical vacuum.

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1. – Introduction

Looking at the immense, upperly unbounded, literature on the classical theory of electromagnetism since its historical beginning, it is worth noticing how the notion of "effective source" is intimately linked to the companion concept of "*screening*", described via the introduction of a "*dielectric function*" so to confer to this last a crucial, fundamental role both theoretically and computationally. Said in other words, this means that "effectiveness of the sources" is essentially viewed as a consequence of charge-charge interactions (even if mutuated by the field), *i.e.* it is nothing else than a consequence of some peculiar aspects of a specific many-body problem. Once more a comment is in order: electric charges are given for granted, pre-existing, physical entities and no reference to their being Noether's topological invariants of given fields, and as such carriers of their specific intrinsic properties, *i.e.* dicotomy, quantization, conservation and Lorentz invariance, is accounted for.

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Perhaps, this is at the very heart of some misunderstandings that are currently encountered. More striking is the occurrence that, at the level at which the wave equation is taken into consideration, the "true" current density, $\mathbf{J}$, appears as the source term only in the equation for the Maxwell field $\mathbf{A}$ in the Lorentz *gauge*, and $\mathbf{A}$ is generally looked at as a "ghost", unphysical field. When the so-called "truly physical" fields $\mathbf{E}$ and $\mathbf{B}$ are concerned, the source term is different than the "true" $\mathbf{J}$. Such an attitude holds true in the most authoritative and, rightly, widespread treatises like those of Landau and Liftshitz, Jackson, Portis, de Groot, and Barut, just to limit to the most renewed and frequently consulted and quoted [1-6]. In this paper I would like to show how the problem of the effective source can be directly connected to the deep physical origin of the electromagnetic theory term founded on the gauge symmetry of the Lagrangian term [7-14]

$$\mathcal{L} := \mathbf{J}(\mathbf{r}(t), t) \cdot \mathbf{A}(\mathbf{r}(t), t), \tag{1.1}$$

$\mathbf{J}(\mathbf{r}(t), t)$ being a vector source term representing the "true" current density vector field, and $\mathbf{A}(\mathbf{r}(t), t)$ the Maxwell field, *i.e.* the vector potential (note that explicit time dependence is introduced *ab inicio* jointly to the $t \to \mathbf{r}(t)$ Eulerian notation, where $\mathbf{r}(t)$ is piecewise regular curve). This is more convincing when we start from the first pair of Maxwell's Equations (ME),

$$\mathbf{curl}(\mathbf{E}) + \frac{1}{c}\partial_t \mathbf{B} = \mathbf{0}, \qquad \text{and} \qquad \mathrm{div}(\mathbf{B}) = 0, \tag{1.2}$$

and we know that this is a good starting point, may be the only one logically consistent and necessary, inasmuch as, due to them being homogeneous equations everywhere, they can be looked at as equations of constraint from which the potentials $\mathbf{A}$ and $\boldsymbol{\Phi}$ are deduced as Lagrangian "coordinates" jointly to their class of indeterminacy. It is superfluous to remind that they are still homogeneous even when (continuous) media are introduced so to impose the introduction of four-vector fields jointly to constitutive equations with related dielectric functions [15-17]. Then, we immediately and unavoidably are faced with the problem that the "Physical Vacuum" is not empty even at the classical level so we feel that "charge clothing" problems are laten *ab inicio*, *i.e.* they exist despite the formal fact that we are allowed to put the dielectric function equal to one all together. " In vacuum, and in the absence of matter, where no difference between *bare* and *clothed* source should be in order due to the absence of any charge to charge interaction.

So my aim, in this contribution, is to obtain well-known, old, familiar, fundamental results (without any novelty as about the implied contents) from another point of view.

2. – Transverse current in the Coulomb gauge

When the formal solution (up to a gauge transformation) of the first pair of ME, *i.e.*

$$\mathbf{E} = -\frac{1}{c}\partial_t \mathbf{A} - \mathrm{grad}(\Phi), \tag{2.1}$$

$$\mathbf{B} = \mathbf{curl}(\mathbf{A}), \tag{2.2}$$

are inserted into the second pair of ME, and the transversality condition $\mathrm{div}(\mathbf{A}) = 0$

is accounted for, the fields $\mathbf{A}$ and Φ appear to be still coupled, despite their being "Lagrangian" coordinates, due to the very procedure under which they have been obtained:

$$\Box\mathbf{A} = -\frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\mathrm{grad}(\partial_t\Phi), \tag{2.3}$$

$$\nabla^2\Phi = -4\pi\rho. \tag{2.4}$$

Two comments are in order: firstly, Φ is explicitly time dependent, so we are in the presence of a "generalized" Poisson equation (PE) (compare with the different situation in the stationary regime where the Lorentz gauge $\mathrm{div}(\mathbf{A})+\frac{1}{c}\partial_t\Phi = 0$ degenerates trivially into the Coulomb gauge), and secondly, in the expression for the field $\mathbf{E}$ as a function of $\mathbf{A}$ and Φ, whereas the sign "minus" referred to the term $\frac{1}{c}\partial_t\mathbf{A}$ is intrinsic, the same choice as about the sign of grad(Φ) is, at this level, quite arbitrary even if allotted. Now, let us show as dynamical decoupling can be obtained via an "effective source"—the "transverse vectorial source $\mathbf{J}_t$"—by inserting in a self-consistent way the PE into the r.h.s. of eq. (2.3). Accounting for the Green's function of the PE, we can write

$$\Phi(\mathbf{r},t) = \int \mathrm{d}^3r' \,\frac{\rho(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\,. \tag{2.5}$$

Thus, we can deduce that

$$\partial_t\Phi(\mathbf{r},t) = \int \mathrm{d}^3r' \,\frac{\partial_t\,\rho(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|} = -\int \mathrm{d}^3r' \frac{\mathrm{div}_{\mathbf{r}'}[\mathbf{J}(\mathbf{r}',t)]}{|\mathbf{r}-\mathbf{r}'|}\,, \tag{2.6}$$

insofar the continuity equation

$$\partial_t\rho + \mathrm{div}(\mathbf{J}) = 0 \tag{2.7}$$

is implied by the second pair of ME.

The second term on the RHS of eq. (2.3) now becomes

$$\begin{aligned}\frac{1}{c}\mathrm{grad}_{\mathbf{r}}[\partial_t\Phi(\mathbf{r},t)] &= -\frac{1}{c}\mathrm{grad}_{\mathbf{r}}\int \mathrm{d}^3r' \,\frac{\mathrm{div}_{\mathbf{r}'}[\mathbf{J}(\mathbf{r}',t)]}{|\mathbf{r}-\mathbf{r}'|}\\ &= -\frac{1}{c}\mathrm{grad}_{\mathbf{r}}\int \mathrm{d}^3r' \,\left\{\mathrm{div}_{\mathbf{r}'}\left(\frac{\mathbf{J}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\right) - \mathbf{J}(\mathbf{r}',t)\right.\\ &\qquad \left.\cdot\,\mathrm{grad}_{\mathbf{r}'}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\right\},\\ &= \mathbf{J}_2(\mathbf{r},t) + \mathbf{J}_1(\mathbf{r},t).\end{aligned} \tag{2.8}$$

Let us firstly analyze the term we called $\mathbf{J}_2$. Insofar the operator $\mathrm{grad}_{\mathbf{r}}$ can be factorized with respect to the integration over d^3r', after the identification $\mathbf{u} \equiv \mathbf{J}(\mathbf{r}',t)$

and $\mathbf{v} \equiv \mathrm{grad}_{\mathbf{r}'}(\frac{1}{|\mathbf{r}-\mathbf{r}'|})$, we can profit of the vectorial identity

$$\mathrm{grad}(\mathbf{u}\cdot\mathbf{v}) = (\mathbf{v}\cdot\mathrm{grad})\mathbf{u} + (\mathbf{u}\cdot\mathrm{grad})\mathbf{v} + \mathbf{v}\wedge[\mathbf{curl}(\mathbf{u})] + \mathbf{u}\wedge[\mathrm{curl}(\mathbf{v})], \tag{2.9}$$

to obtain

$$(\mathbf{v}\cdot\mathrm{grad})\mathbf{u} = \left(\mathrm{grad}_{\mathbf{r}'}\frac{1}{|\mathbf{r}-\mathbf{r}'|}\cdot\mathrm{grad}_{\mathbf{r}}\right)\mathbf{J}(\mathbf{r}',t) = \mathbf{0} \tag{2.10}$$

$$(\mathbf{J}(\mathbf{r}',t) \text{ does not depend on } \mathbf{r}),$$

$$\begin{aligned}(\mathbf{u}\cdot\mathrm{grad})\mathbf{v} &= (\mathbf{J}(\mathbf{r}',t)\cdot\mathrm{grad}_{\mathbf{r}})\mathrm{grad}_{\mathbf{r}'}\left(\frac{1}{\mathbf{r}-\mathbf{r}'}\right)\\ &= -\mathbf{J}(\mathbf{r}',t)\nabla^2\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) = 4\pi\mathbf{J}(\mathbf{r}',t)\delta(\mathbf{r}-\mathbf{r}'),\end{aligned} \tag{2.11}$$

$$\mathbf{v}\wedge[\mathbf{curl}(\mathbf{u})] = \mathrm{grad}_{\mathbf{r}'}\left(\frac{1}{\mathbf{r}-\mathbf{r}'}\right)\wedge[\mathbf{curl}_{\mathbf{r}}(\mathbf{J}(\mathbf{r}',t))] = \mathbf{0} \tag{2.12}$$

$$(\mathbf{J}(\mathbf{r}',t) \text{ does not depend on } \mathbf{r})$$

$$\mathbf{u}\wedge[\mathbf{curl}(\mathbf{v})] = \mathbf{J}(\mathbf{r}',t)\wedge\mathbf{curl}_{\mathbf{r}}\left[\mathrm{grad}_{\mathbf{r}'}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\right] = \mathbf{0}. \tag{2.13}$$

By combining eqs. (2.9)–(2.13), we obtain

$$\mathbf{J}_2 = \frac{4\pi}{c}\int \mathrm{d}^3r'\mathbf{J}(\mathbf{r}',t)\delta(\mathbf{r}-\mathbf{r}') = \frac{4\pi}{c}\mathbf{J}(\mathbf{r},t). \tag{2.14}$$

The RHS of eq. (2.14) compensates for the first term on the RHS of eq. (2.3).

Let us turn to the term $\mathbf{J}_1$. Inasmuch as it looks like the gradient of a divergence, we can profit of the identity $\mathrm{grad}(\mathrm{div}(\mathbf{v})) = \mathbf{curl}(\mathbf{curl}(\mathbf{v})) + \nabla^2\mathbf{v}$ in order to write down $\mathbf{J}_1$ as the sum of two terms, *i.e.*

$$\mathbf{J}_1 = \mathbf{J}_{11} + \mathbf{J}_{12} = -\frac{1}{c}\int \mathrm{d}^3r'\mathbf{curl}_{\mathbf{r}}\left[\mathbf{curl}_{\mathbf{r}'}\left(\frac{\mathbf{J}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\right)\right] \tag{2.15}$$

$$-\frac{1}{c}\int \mathrm{d}^3r'(\mathbf{curl}_{\mathbf{r}}\cdot\mathbf{curl}_{\mathbf{r}'})\frac{\mathbf{J}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\,. \tag{2.16}$$

To simplify notations, let us put $\mathbf{R} = |\mathbf{r}-\mathbf{r}'|$, and prove firstly that $\mathbf{J}_{12} = \mathbf{0}$. Indeed, the integrand in the second piece on the RHS of eq. (2.15) has the structure

$$
\begin{aligned}
(2.17)\qquad & (\partial_x\partial_{x'}+\partial_y\partial_{y'}+\partial_z\partial_{z'})\left[\frac{J_x(\mathbf{r}',t)}{R}\widehat{\mathbf{x}}+\frac{J_y(\mathbf{r}',t)}{R}\widehat{\mathbf{y}}+\frac{J_z(\mathbf{r}',t)}{R}\widehat{\mathbf{z}}\right]\\
&= \widehat{\mathbf{x}}(\partial_x\partial_{x'}+\partial_y\partial_{y'}+\partial_z\partial_{z'})\frac{J_x(\mathbf{r}',t)}{R}\\
&\quad +\widehat{\mathbf{y}}(\partial_x\partial_{x'}+\partial_y\partial_{y'}+\partial_z\partial_{z'})\frac{J_y(\mathbf{r}',t)}{R}\\
&\quad +\widehat{\mathbf{z}}(\partial_x\partial_{x'}+\partial_y\partial_{y'}+\partial_z\partial_{z'})\frac{J_z(\mathbf{r}',t)}{R}\\
&= \widehat{\mathbf{x}}\left\{\partial_x\left(R^{-1}\partial_{x'}J_x(\mathbf{r}',t)+J_x(\mathbf{r}',t)\partial_{x'}R^{-1}\right)\right.\\
&\qquad +\partial_y\left(R^{-1}\partial_{y'}J_x(\mathbf{r}',t)+J_x(\mathbf{r}',t)\partial_{y'}R^{-1}\right)\\
&\qquad \left.+\partial_z\left(R^{-1}\partial_{z'}J_x(\mathbf{r}',t)+J_x(\mathbf{r}',t)\partial_{z'}R^{-1}\right)\right\}\\
&\quad +\widehat{\mathbf{y}}\left\{\text{as before “\textit{mutatis mutandis}”}\right\}\\
&\quad +\widehat{\mathbf{z}}\left\{\text{as before “\textit{mutatis mutandis}”}\right\}.
\end{aligned}
$$

The coefficients of the unit vectors $\widehat{\mathbf{x}}$, $\widehat{\mathbf{y}}$, and $\widehat{\mathbf{z}}$ can be rewritten so to obtain

$$
\begin{aligned}
(2.18)\qquad & (\partial_x\partial_{x'}+\partial_y\partial_{y'}+\partial_z\partial_{z'})\left[\frac{J_x(\mathbf{r}',t)}{R}\widehat{\mathbf{x}}+\frac{J_y(\mathbf{r}',t)}{R}\widehat{\mathbf{y}}+\frac{J_z(\mathbf{r}',t)}{R}\widehat{\mathbf{z}}\right]\\
&= \widehat{\mathbf{x}}\left\{\left[(\partial_x R^{-1})(\partial_{x'}J_x)+\underbrace{R^{-1}(\partial_x\partial_{x'}J_x)}_{=0}+\underbrace{(\partial_x J_x)}_{=0}(\partial_{x'}R^{-1})+J_x\partial_x\partial_{x'}R^{-1}\right]\right.\\
&\quad \left.+\left[(\partial_y R^{-1})(\partial_{y'}J_y)+\underbrace{R^{-1}(\partial_y\partial_{y'}J_y)}_{=0}+\underbrace{(\partial_y J_y)}_{=0}(\partial_{y'}R^{-1})+J_y\partial_y\partial_{y'}R^{-1}\right]+\left[\ldots\right]\right\}\\
&\quad +\widehat{\mathbf{y}}\left\{\left[\text{as before “\textit{mutatis mutandis}”}\right]\right\}+\widehat{\mathbf{z}}\left\{\left[\text{as before “\textit{mutatis mutandis}”}\right]\right\}\\
&= \widehat{\mathbf{x}}\left\{\left(\mathrm{grad}_{\mathbf{r}}R^{-1}\right)\cdot(\mathrm{grad}_{\mathbf{r}}J_x)-J_x\nabla^2(R^{-1})\right\}\\
&\quad +\widehat{\mathbf{y}}\left\{\left(\mathrm{grad}_{\mathbf{r}}R^{-1}\right)\cdot(\mathrm{grad}_{\mathbf{r}}J_y)-J_y\nabla^2(R^{-1})\right\}\\
&\quad +\widehat{\mathbf{z}}\left\{\left(\mathrm{grad}_{\mathbf{r}}R^{-1}\right)\cdot(\mathrm{grad}_{\mathbf{r}}J_z)-J_z\nabla^2(R^{-1})\right\}.
\end{aligned}
$$

Thus, we arrive at the equation

$$
\begin{aligned}
(2.19)\qquad \mathbf{J}_{12} &= \widehat{\mathbf{x}}\int \mathrm{d}^3r'\left(\mathrm{grad}_{\mathbf{r}}R^{-1}\right)\cdot(\mathrm{grad}_{\mathbf{r}}J_x)+\widehat{\mathbf{y}}\int \mathrm{d}^3r'\left(\mathrm{grad}_{\mathbf{r}}R^{-1}\right)\cdot(\mathrm{grad}_{\mathbf{r}}J_y)\\
&\quad +\widehat{\mathbf{z}}\int \mathrm{d}^3r'\left(\mathrm{grad}_{\mathbf{r}}R^{-1}\right)\cdot(\mathrm{grad}_{\mathbf{r}}J_z)-\int \mathrm{d}^3r'\mathbf{J}(\mathbf{r}',t)\nabla^2(R^{-1}).
\end{aligned}
$$

The fourth term on the RHS of eq. (2.19) is immediately evaluated as being $-4\pi\mathbf{J}(\mathbf{r},t)$, whereas for each of the three other terms we can profit of the Green-Gauss-Ostrogardsky Theorem:

$$
(2.20)\qquad \int \mathrm{d}^3r'\left[\varphi\nabla^2\psi+\mathrm{grad}(\varphi)\cdot\mathrm{grad}(\psi)\right]=\int\left[\varphi\,\mathrm{grad}(\psi)\right]\cdot\mathbf{n}\mathrm{d}\Sigma.
$$

Moreover, by accounting for the fact that $\mathrm{grad}_{\mathbf{r}'}(R^{-1}) = -\mathrm{grad}_{\mathbf{r}}(R^{-1})$, we can write

$$-\int \mathrm{d}^3r' \left(\mathrm{grad}_{\mathbf{r}}(R^{-1})\right) \cdot (\mathrm{grad}_{\mathbf{r}}(J_i)) = \int \mathrm{d}^3r' J_i(\mathbf{r}', t) \cdot \nabla^2(R^{-1}) = -\int J_i(\mathbf{r}', t) \left(\mathrm{grad}_{\mathbf{r}}(R^{-1})\right) \mathbf{n}\mathrm{d}\Sigma, \tag{2.21}$$

where $i = x, y, z$, so to arrive, as previously stated, at the result

$$\mathbf{J}_{12} = -4\pi\mathbf{J}(\mathbf{r}, t) + 4\pi\mathbf{J}(\mathbf{r}, t) = \mathbf{0}. \tag{2.22}$$

Finally, let us explicitly calculate

$$\mathbf{J}_{11} = \int \mathrm{d}^3r' \mathbf{curl}_{\mathbf{r}} \left[\mathbf{curl}_{\mathbf{r}'} \left(R^{-1}\mathbf{J}(\mathbf{r}', t)\right)\right] \tag{2.23}$$

$$= \int \mathrm{d}^3r' \mathbf{curl}_{\mathbf{r}} \left[\left(\mathrm{grad}_{\mathbf{r}}(R^{-1})\right) \wedge \mathbf{J}(\mathbf{r}', t) + R^{-1} \left(\mathbf{curl}_{\mathbf{r}'} \left(\mathbf{J}(\mathbf{r}', t)\right)\right)\right], \tag{2.24}$$

where the vector identity

$$\mathbf{curl}(\psi\mathbf{v}) = \mathrm{grad}(\psi) \wedge \mathbf{v} + \psi\mathbf{curl}(\mathbf{v}) \tag{2.25}$$

has been taken into account. To solve, we can use the vector identity

$$\mathbf{curl}(\mathbf{a} \wedge \mathbf{b}) = \mathbf{a}\ \mathrm{div}(\mathbf{b}) + (\mathbf{b} \cdot \mathrm{grad})\mathbf{a} - (\mathbf{a} \cdot \mathrm{grad})\mathbf{b} - \mathbf{b}\ \mathrm{div}(\mathbf{a}). \tag{2.26}$$

Then, the identification $\mathbf{a} = \mathrm{grad}_{\mathbf{r}}(R^{-1})$, and $\mathbf{b} = \mathbf{J}(\mathbf{r}', t)$ allows for arriving at the final result for eq. (2.3), *i.e.*

$$\Box\mathbf{A} = -\frac{4\pi}{c}\mathbf{J}_t, \tag{2.27}$$

where the gauge-dependent *effective source* $\mathbf{J}_t$ is defined as

$$\mathbf{J}_t(\mathbf{r}, t) = \mathbf{curl}_r \int \mathrm{d}^3r' \frac{\mathbf{curl}_{\mathbf{r}'}[\mathbf{J}(\mathbf{r}', t)]}{|\mathbf{r} - \mathbf{r}'|}. \tag{2.28}$$

Insofar this “shielded” current field is divergence free, this could imply (at least locally) a modified or “screened” charge density to preserve (as it must be) the Noether invariance of the charge. Such an occurrence looks quite surprising due to the fact that we are in a vacuum; unless vacuum is not so empty as it looks and intuitively assumed or better it can be filled with “something else” the condensation energy of which is competitive to the vacuum energy.

3. – Conclusion and future perspectives

Few comments as conclusions to the ideas and relative calculations previously introduced and described. Firstly, inasmuch as in the Coulomb gauge the fourth Maxwell equation is not transformed into a wave equation but in an explicitly time-dependent Poisson equation, the question of causality could be raised for consistency with the relativity requirement. As shown by Brill and Goodman [18], the transversality of the solution allows the causality requirement to be completely fulfilled and in our opinion this fact is of the outmost relevance, particularly from the conceptual point of view. Indeed, transversality is a direct consequence of gauge symmetry alone and gauge invariance is the primarily class of symmetry of ME and, as is well known, a Lagrangian which is a homogeneous function of degree one with respect to the velocities (Lagrangian to which gauge symmetry can be traced back) implies that time cannot be absolute at all in principle [19, 20]. Note how this attitude shades light on the almost doubtful procedure to deduce the gauge invariance of the ME after starting their Lorentz symmetry due to the fact that the gauge group is much larger than the Lorentz group, the latter being a group of external symmetries whereas the gauge group can lead to both internal and external symmetries.

Secondly, one could object that the result we arrived at could be obtained, after following a much shorter way, by invoking the well-celebrated Helmholtz theorem referring to the fact that a general vector field can be decoupled into a transverse (irrotational) part and a longitudinal (solenoidal) part. The problem in our hands was not a general mathematical problem: instead it was a well-specified physical problem and we used the practice of a "self-consistent" search for the solution that is within the best tradition to find "dielectric functions" in the (linear) response theory.

In conclusion, I feel that if one is addressed to the question: "can we introduce a *minimal* coupling, *i.e.* Lagrangian (1.1) *ab inicio* and *then* deduce the first couple of the ME, the Lorentz force, charge conservation (and so on) as the "equation of motion" deducible from this Lagrangian?", we can hopefully look at the positive answer: It is worth remembering that such an attitude is an invitation to follow the path firstly put forward by Feynmann, as reported by Dyson [21] and Dirac (cf. ref. [19]) and more recently discussed by Hojman, Shepley [22] and Peres [23].

Moreover, let us consider how these themes could be looked at as more or less deeply connected with the items expounded within the context of the Seminar by M. Consoli [24] and C. Trimarco [25], particularly when superfluidity and superconductivity are viewed as spontaneously broken symmetry phenomena [26-29], particularly in common solids and/or fluids where relativistic effects are ignorable insofar largely negligible in most of the experimental occurrences.

Preliminary results obtained in deriving the first couple of the ME jointly the Lorentz force and continuity equation as Euler-Lagrange equation associated to the Lagrangian (1.1) alone and the contextual condition of "minimal coupling" regime (*i.e.* the corresponding energy being zero) encourage me and I postpone to a forthcoming paper their publication or, at least, I hope strongly.

* * *

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