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Supercurrents as Eulerian fluids

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Summary. — Macroscopic equations of hydrodynamics can be associated with rigid superconductors through a time-dependent Schrödinger-Ginzburg-Landau equation, as suggested by Fröhlich. Here, this issue is re-proposed in a slightly revised different approach and derived from a Lagrangian, through a variational procedure. Interesting novel aspects of the hydrodynamic equations for supercurrents are placed in evidence. Also, Lagrangian variables are introduced and the equations of interest are written in these variables in view of an extension to deformable bodies.

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1. – Introduction

A phenomenological approach to superconductors is generally based on the Ginzburg-Landau macroscopic theory, according to which superconductivity occurs if the material undergoes a thermodynamical phase transition. In this view, the free-energy density of the superconductor is assumed to depend, among the other thermodynamical quantities, on an internal variable, which is expressed by a complex-valued quantity. This quantity plays the role of an order parameter, whose real-valued norm is intended to represent the supercharge density and accounts for a second-order thermodynamical phase transition from the normal to the superconducting state.

In view of focusing the attention on the transition point, the free energy is typically expressed as a polynomial expansion of the order parameter around this point. It is

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worth remarking that the coefficients of the polynomial expansion depend on the temperature and in deformable superconductors possibly on the deformation tensor. Variational methods lead to the Ginzburg-Landau equation that governs the order parameter [1-7].

This equation is here extended into the dynamical framework through a variational procedure. This procedure also provides an equation for the supercurrents, which, in the presence of electromagnetic fields, re-proposes an extended complex-valued form of one of London's equation. An interesting remarkable aspect of this quasi–quantum-mechanical approach is the gauge invariance of all equations of interest.

In a very concise though suggestive paper, Fröhlich [3] shows that a time-dependent Ginzburg-Landau equation can be derived from two equations, namely the continuity equation and the balance of momentum for a specific Eulerian compressible fluid. The velocity field of this fluid is related to the aforementioned London equation. Prove of this equivalence with the aforementioned ideal fluid is not carried out explicitly by Fröhlich. Here a Lagrangian density is introduced, from which the time-dependent Ginzburg-Landau (GL) equation is derived as an Euler-Lagrange equation. From this equation the Euler hydrodynamic equations of interest are derived through a simple procedure. One of the results of this approach is that the Eulerian fluid is naturally governed by a hydrodynamic enthalpy, differently from Fröhlich, whose equations explicitly introduce the pressure acting on the fluid. The present derivation also places in evidence that the Eulerian fluid can be associated with a classical Schrödinger equation or with a large class of nonlinear Schrödinger equations. In all these cases the structure of the associated Euler equations for ideal fluids remains unaltered, as the non-linear terms only contribute to the hydrodynamic enthalpy. It is also worth noting that in all these cases the flow is an irrotational flow in the absence of electromagnetic fields. In the presence of electromagnetic fields the Euler equations for the associated fluid are coupled with the Maxwell equations through the *curl* of the velocity field.

Finite deformations in superconductors affect not only the mechanical behaviour of the body, but also the electromagnetic fields and thus the electromagnetic vector potential, which plays a dominant role in the phenomenological description of superconductivity. In this framework, the vector potential and the whole set of the Maxwell equations are expressed in Lagrangian form. These equations preserve their form provided that the electromagnetic fields are suitably transformed through the deformation gradient and the velocity. The question may arise as whether the form of the GL equation and London's laws should remain invariant in the Lagrangian description.

2. – Maxwell and London's equations

In the following, all fields are understood as functions of the spatial position $\mathbf{x} \in E_3 = \{\text{Euclidean space}\}$ and the time $t \in \mathbb{R}$. Equations and physical quantities are written in SI units. London's conjecture for superconductors is expressed by the equation [7-10]

$$-\Lambda \operatorname{curl} \boldsymbol{j}_s = \boldsymbol{B}, \tag{2.1}$$

where $\boldsymbol{B}$ denotes the magnetic induction, $\boldsymbol{j}_s$ the supercurrent density. The phenomenological constant Λ is usually written as $\Lambda \equiv (\mu_0 \lambda^2)^{-1}$, where μ_0 denotes the magnetic permeability of a vacuum and λ London's penetration depth [7,9]. Hereafter, the relative permeability is assumed to be 1. The Maxwell equations for bodies at rest read

$$\operatorname{div} \boldsymbol{B} = 0, \tag{2.2}$$

$$\operatorname{curl} \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0, \tag{2.3}$$

$$\operatorname{div} \boldsymbol{D} = \rho_f, \tag{2.4}$$

$$\operatorname{curl} \boldsymbol{H} = \boldsymbol{j} + \frac{\partial \boldsymbol{D}}{\partial t}\,, \tag{2.5}$$

where $\boldsymbol{E}$ and $\boldsymbol{H}$ respectively represent the electric and magnetic fields, $\boldsymbol{D}$ the electric displacement, ρ_f the electric free charge density and $\boldsymbol{j}$ is the total free current [11, 12]. Equations (2.2) and (2.3) lead to the notion of electromagnetic scalar and vector potentials φ and $\boldsymbol{A}$, respectively, as follows:

$$\boldsymbol{B} = \operatorname{curl} \boldsymbol{A}, \tag{2.6}$$

$$\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} - \boldsymbol{\nabla} \varphi. \tag{2.7}$$

Equations (2.1), (2.6) and (2.7) lead in turn to the two celebrated London laws

$$\Lambda \boldsymbol{j}_s = -\boldsymbol{A} + \boldsymbol{\nabla} g, \tag{2.8}$$

$$\Lambda \partial \boldsymbol{j}_s / \partial t = -\boldsymbol{E} + \boldsymbol{\nabla} f. \tag{2.9}$$

The symbol $\boldsymbol{\nabla}$ denotes the gradient operator. g and f are arbitrary functions of $\mathbf{x}$ and t, which need further specifications. The identification $f \equiv \varphi$ is due to Fritz and Heinz London [9] and a remarkable result of this identification is that eq. (2.9) predicts no acceleration of the supercurrent in the presence of an electrostatic field, for which $\boldsymbol{E} = -\boldsymbol{\nabla}\varphi$. The arguments for these assumptions, though interesting, are omitted here.

Additional assumptions, which are suggested by the form of the Maxwell equations for a moving charged particle and are based on the Lorenz-Lorentz gauge conditions, are

$$\boldsymbol{A} \equiv c^{-2} \varphi \mathbf{v}, \tag{2.10}$$

$$\Lambda \rho_s \equiv -c^{-2} \varphi, \tag{2.11}$$

ρ_s being the supercharge density and c the speed of light. The following equations can be derived from the aforementioned equations and assumptions:

$$\boldsymbol{\nabla}^2 \boldsymbol{A} - c^{-2} \frac{\partial^2 \boldsymbol{A}}{\partial\, t^2} = \lambda^{-2} \boldsymbol{A}, \tag{2.12}$$

$$\boldsymbol{\nabla}^2 \varphi - c^{-2} \frac{\partial^2 \varphi}{\partial\, t^2} = \lambda^{-2} \varphi, \tag{2.13}$$

$$\boldsymbol{\nabla}^2 \boldsymbol{H} - c^{-2} \frac{\partial^2 \boldsymbol{H}}{\partial\, t^2} = \lambda^{-2} \boldsymbol{H}. \tag{2.14}$$

It is worth remarking that Maxwell equations only entail the conservation of the total charge. The conservation of the supercharge is an additional assumption, which reads [10]

$$\operatorname{div} \boldsymbol{j}_s + \frac{\partial \rho_s}{\partial t} = 0. \tag{2.15}$$

For stationary fields, the Lorenz-Lorentz gauge condition reduces to the Coulomb gauge condition div $\boldsymbol{A} = 0$ and eq. (2.14) reduces to the well-known equation

$$\nabla^2 \boldsymbol{H} - \lambda^{-2} \boldsymbol{H} = 0, \tag{2.16}$$

which accounts for the Meissner-Ochsenfeld effect in superconductors [2-10]. Another interesting result stems from eq. (2.15), which for stationary fields reduces to

$$\operatorname{div} \boldsymbol{j}_s = 0. \tag{2.17}$$

Substitution of eq. (2.8) into (2.17) addresses a harmonic function for g, which identically vanishes in a simply connected domain for homogeneous Dirichlet boundary conditions. Under these assumptions, eq. (2.8) is deprived of the term ∇g and expresses London's law such as currently found in the literature.

3. – A moving point-wise charged particle: preliminary recalls

Consider a point-wise moving charge that is acted upon the Lorentz force. The equation of motion in terms of the electromagnetic potentials reads

$$m\,\dot{\mathbf{v}} = q(\boldsymbol{E} + \mathbf{v} \times \boldsymbol{B}) = q\left(-\frac{\partial \boldsymbol{A}}{\partial t} - \nabla\varphi + \mathbf{v} \times \operatorname{curl} \boldsymbol{A}\right), \tag{3.1}$$

where m and q are the mass and the electric charge, respectively, of the particle of interest. $\mathbf{v} \equiv \dot{\mathbf{x}}(\mathrm{t})$. The following Lagrangian and momentum are associated, respectively, with eq. (3.1):

$$L = \frac{m\mathrm{v}^2}{2} - q(\varphi - \boldsymbol{A} \cdot \mathbf{v}), \tag{3.2}$$

$$\mathbf{p} = (\partial L/\partial \mathbf{v}) = m\mathbf{v} + q\boldsymbol{A}. \tag{3.3}$$

The related Hamiltonian is

$$\mathcal{H} = \frac{1}{2m}(\mathbf{p} - q\boldsymbol{A})^2 + q\varphi. \tag{3.4}$$

In the quantum-mechanical treatment, $\mathbf{p}$ and $\mathcal{H}$ have to be understood as specific differential operators that act upon a complex-valued (wave) function $\psi(\mathbf{x}, t)$, which is also associated with the particle. More specifically, $\mathbf{p} \equiv -\iota\,\hbar\nabla$ and $\mathcal{H} \equiv -\frac{\hbar^2}{2m}(\nabla - \frac{\iota q}{\hbar}\boldsymbol{A})^2 + q\varphi$, $\hbar$ representing the reduced Planck constant and ι the imaginary unit. In this context, an additional equation is introduced, the Schrödinger equation, which governs $\psi(\mathbf{x}, t)$ [11,13]

$$\left\{\left[\frac{\hbar^2}{2m}\left(\nabla - \frac{\iota q}{\hbar}\boldsymbol{A}\right)^2 + q\varphi\right]\right\}\psi(\mathbf{x}, t) = -\iota\hbar\frac{\partial \psi}{\partial t}\,. \tag{3.5}$$

An analogous equation, which differs from eq. (3.5) for the positive sign on the r.h.s., governs the complex-valued function $\overline{\psi}$, conjugate of ψ.

4. – The Eulerian fluid associated with the moving particle

Equation (3.5) explicitly reads

$$\frac{\hbar^2}{2m}\{\boldsymbol{\nabla}^2\psi-(q/\hbar)^2\boldsymbol{A}^2\psi-\iota(q/\hbar)[2\boldsymbol{A}\cdot(\boldsymbol{\nabla}\psi)+\boldsymbol{\nabla}\cdot\boldsymbol{A}]\psi\}+(2m/\hbar^2)q\varphi\psi+\iota\hbar\frac{\partial\psi}{\partial t}=0. \quad (4.1)$$

The possible solution for this equation can be written in polar form as follows:

$$\psi(\mathbf{x},t)=\rho^{1/2}(\mathbf{x},t)\exp\left[\iota\frac{m}{\hbar}\theta(x,t)\right], \quad (4.2)$$

so that $\overline{\psi}\psi=\psi\overline{\psi}=\rho$.

Replacement of eq. (4.2) into (4.1) leads to the equation

$$\iota\hbar b+\frac{\hbar^2}{2m}[\boldsymbol{a}\cdot\boldsymbol{a}+\operatorname{div}\boldsymbol{a}-(q/\hbar)^2\boldsymbol{A}^2+\iota(q/\hbar)(2\boldsymbol{A}\cdot\boldsymbol{a}+\operatorname{div}\boldsymbol{A})]+q\varphi=0, \quad (4.3)$$

where

$$\boldsymbol{a}\equiv\left(\frac{\boldsymbol{\nabla}\rho}{2\rho}+\iota\frac{\boldsymbol{\nabla}\theta}{\hbar}m\right) \qquad \text{and} \qquad b\equiv\left(\frac{1}{2\rho}\frac{\partial\rho}{\partial t}+\iota\frac{m}{\hbar}\frac{\partial\theta}{\partial t}\right).$$

Two equations stem from eq. (4.3) by splitting it in its real (*Re*) and imaginary (Im) parts. Im (eq. (4.3)) corresponds to the equation

$$\frac{\partial\rho}{\partial t}+(\boldsymbol{\nabla}\rho)\cdot(\boldsymbol{\nabla}\theta)+\rho\operatorname{div}(\boldsymbol{\nabla}\theta)-(\boldsymbol{\nabla}\rho)\cdot(q/m)\boldsymbol{A}-\rho\operatorname{div}[(q/m)\boldsymbol{A}]=0 \quad (4.4)$$

or, equivalently, to

$$\frac{\partial\rho}{\partial t}+\operatorname{div}\rho[\boldsymbol{\nabla}\theta-(q/m)\boldsymbol{A}]=0. \quad (4.5)$$

From *Re* (eq. (4.4)) the following equation is derived:

$$\frac{\partial\theta}{\partial t}+\frac{(\boldsymbol{\nabla}\theta)^2}{2}+2[(q/2m)\boldsymbol{A}]^2-(q/m)\boldsymbol{A}\cdot\boldsymbol{\nabla}\theta-\left(\frac{\hbar}{2m}\right)^2 \times\left[\frac{1}{2}\left(\frac{\boldsymbol{\nabla}\rho}{\rho}\right)^2+\operatorname{div}\left(\frac{\boldsymbol{\nabla}\rho}{\rho}\right)\right]+(q/m)\varphi=0. \quad (4.6)$$

This equation can be more concisely written as

$$\frac{\partial\theta}{\partial t}+\frac{1}{2}\boldsymbol{v}^2+\left(\frac{\hbar}{2m}\right)^2\eta+(q/m)\varphi=0, \quad (4.7)$$

where

$$\boldsymbol{v}=[\boldsymbol{\nabla}\theta-(q/m)\boldsymbol{A}] \quad (4.8)$$

and

$$\eta \equiv -\left[\frac{1}{2}\left(\frac{\boldsymbol{\nabla}\rho}{\rho}\right)^2 + \operatorname{div}\left(\frac{\boldsymbol{\nabla}\rho}{\rho}\right)\right] \equiv \frac{1}{2}\left(\rho\boldsymbol{\nabla}\left(\frac{1}{\rho}\right)\right)^2 + \operatorname{div}\left(\rho\boldsymbol{\nabla}\left(\frac{1}{\rho}\right)\right). \tag{4.9}$$

Note that $\boldsymbol{v}(\mathbf{x},t)$ is the velocity field that is associated with the particle through the momentum operator as follows:

$$m\boldsymbol{v} = \frac{\overline{\psi}\mathbf{p}\psi - \psi\overline{\mathbf{p}}\overline{\psi}}{2\psi\overline{\psi}} - q\boldsymbol{A} = -\iota\frac{\hbar}{2}(\boldsymbol{a} - \overline{\boldsymbol{a}}) = m[\boldsymbol{\nabla}\theta - (q/m)\boldsymbol{A}], \tag{4.10}$$

where $\overline{\mathbf{p}} \equiv \iota\,\hbar\boldsymbol{\nabla}$ and $\overline{\boldsymbol{a}}$ denotes the conjugate of $\boldsymbol{a}$.

Perform the gradient of eq. (4.7) and take into account that spatial and time derivative commute with one another. The result is the following:

$$\frac{\partial}{\partial t}(\boldsymbol{\nabla}\theta) + \frac{1}{2}\boldsymbol{\nabla}(\boldsymbol{v})^2 + \left(\frac{\hbar}{2m}\right)^2 \boldsymbol{\nabla}\,\eta + (q/m)\boldsymbol{\nabla}\varphi = 0. \tag{4.11}$$

By taking into account eq. (4.8), eqs. (4.5) and (4.11) can be written in terms of the field $\boldsymbol{v}$ as

$$\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\boldsymbol{v}) = 0, \tag{4.12}$$

$$\frac{\partial\boldsymbol{v}}{\partial t} + (\boldsymbol{\nabla}\boldsymbol{v})\boldsymbol{v} = -\left(\frac{\hbar}{2m}\right)^2 \boldsymbol{\nabla}\,\eta + \boldsymbol{e}, \tag{4.13}$$

where

$$\boldsymbol{e} \equiv -(q/m)\left[(\boldsymbol{\nabla}\varphi) + \frac{\partial A}{\partial t} + (\operatorname{curl}\boldsymbol{A}) \times \boldsymbol{v}\right] = (q/m)(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}). \tag{4.14}$$

Equations (4.12) and (4.13) represent the Euler equations for a compressible rotational fluid of density ρ, which are similar to those introduced by Fröhlich [3]. Equation (4.13) is governed by the enthalpy η and by the electromagnetic force per unit mass $\boldsymbol{e}$, which establishes a coupling of the Euler equations with the Maxwell equations. In is worth noting that, in the absence of electromagnetic fields, θ plays the role of a velocity potential and the flow is an irrotational flow. Differently, the flow is a rotational flow in the presence of electromagnetic fields and the following relationship holds true, with reference to eqs. (4.8) and (2.6):

$$\operatorname{curl}\boldsymbol{v} = (q/m)\boldsymbol{B}. \tag{4.15}$$

The phenomenological relevance of these results emerges in the case that a large number of particles share the same complex-valued wave function. Of course, this is not the general case. However, it can be the case of bosons in a condensed state. It is the case of Cooper "electron pairs" at very low temperature, namely of supercharges and supercurrents [2-7].

5. – Time-dependent Ginzburg-Landau equation: the supercurrent as an Eulerian fluid

A macroscopic wave function enters the Ginzburg-Landau thermodynamical theory of phase transition for superconductors, according to which superconductivity represents a specific thermodynamical phase of a material. The macroscopic time-independent wave function, which in the following is still denoted as $\psi(\mathbf{x})$ by abusing notation, governs the second-order thermodynamical phase transition from the normal to the superconducting state, through the free-energy density

$$F(\psi,\overline{\psi},(\boldsymbol{\nabla}_{\boldsymbol{A}}\,\psi),\mathrm{curl}\,\boldsymbol{A},T)=\frac{\gamma^2}{2}|\boldsymbol{\nabla}_{\boldsymbol{A}}\,\psi|^2-c(\psi\overline{\psi})^2+\frac{d}{2}(\psi\overline{\psi})^4+\frac{1}{2\mu_0}(\mathrm{curl}\,\boldsymbol{A})^2, \tag{5.1}$$

where $\boldsymbol{\nabla}_{\boldsymbol{A}}\equiv(\boldsymbol{\nabla}-\iota\frac{k}{\gamma}\boldsymbol{A})$. c and d are possible functions of temperature T. Reference to the microscopic description suggests the following identifications: $k\equiv(\frac{q^*}{m})$ and $\gamma=(\frac{\hbar}{m})$, q^* representing the charge of Cooper's "electron pairs" and m their mass.

The Euler-Lagrange equations that are associated with the free energy F are

$$\frac{1}{\mu_0}\,\mathrm{curl}(\mathrm{curl}\,\boldsymbol{A})=\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{A}}\equiv\frac{k\gamma}{2\iota}(\overline{\psi}\boldsymbol{\nabla}\psi-\psi\boldsymbol{\nabla}\overline{\psi})-k^2(\psi\overline{\psi})\boldsymbol{A}, \tag{5.2}$$

$$\left\{\frac{\gamma^2}{2}|\boldsymbol{\nabla}_{\boldsymbol{A}}|^2+c-d(\psi\overline{\psi})\right\}\psi=0. \tag{5.3}$$

An analogous equation can be derived for $\overline{\psi}$.

With reference to eqs. (4.2) and (4.8), the r.h.s. of eq. (5.2) also reads

$$\frac{k\gamma}{2i}(\overline{\psi}\boldsymbol{\nabla}\psi-\psi\boldsymbol{\nabla}\overline{\psi})-k^2(\psi\overline{\psi})\boldsymbol{A}=k\rho(\boldsymbol{\nabla}\theta-k\boldsymbol{A})=k\rho\boldsymbol{v}\equiv\boldsymbol{j}_s, \tag{5.4}$$

$\boldsymbol{j}_s$ being the flow of the supercurrent.

In order to derive a time-dependent GL equation for the macroscopic wave function $\psi(\mathbf{x},t)$, the Lagrangian density L is preliminarily introduced in the absence of electromagnetic fields as

$$L=\frac{\gamma^2}{2}|\boldsymbol{\nabla}\psi|^2+\frac{\gamma}{2\iota}\left(\psi\frac{\partial\overline{\psi}}{\partial t}-\overline{\psi}\frac{\partial\psi}{\partial t}\right)-c(\psi\overline{\psi})^2+\frac{d}{2}(\psi\overline{\psi})^4, \tag{5.5}$$

whence the following equation is derived:

$$\frac{\gamma^2}{2}\boldsymbol{\nabla}^2\psi+[c-d(\psi\overline{\psi})]\psi-\frac{\gamma}{\iota}\frac{\partial\psi}{\partial t}=0. \tag{5.6}$$

It is worth noting that eq. (5.6) is a non-linear time-dependent equation that shares the structure of the Schrödinger equation [6].

In the presence of electromagnetic fields, the proposed Lagrangian density slightly modifies with respect to (5.5) as follows:

$$L_{\mathrm{em}}=\frac{\gamma^2}{2}(\boldsymbol{\nabla}_{\boldsymbol{A}}\psi)\cdot(\overline{\boldsymbol{\nabla}_{\boldsymbol{A}}\psi})+\frac{\gamma}{2\iota}\left[\psi\left(\frac{\partial\overline{\psi}}{\partial t}\right)_{\varphi}-\left(\frac{\partial\psi}{\partial t}\right)_{\varphi}\overline{\psi}\right]+\hat{F}\,[\psi,\overline{\psi},(\mathrm{curl}\,\boldsymbol{A})], \tag{5.7}$$

where

$$(\overline{\boldsymbol{\nabla}_{\boldsymbol{A}}\psi}) \equiv \left(\boldsymbol{\nabla} + \iota\frac{k}{\gamma}\boldsymbol{A}\right)\overline{\psi}, \tag{5.8}$$

$$\left(\frac{\partial\psi}{\partial t}\right)_{\varphi} \equiv \left(\frac{\partial}{\partial t} + \iota\frac{k}{\gamma}\varphi\right)\psi \tag{5.9}$$

and

$$\left(\overline{\frac{\partial\psi}{\partial t}}\right)_{\varphi} \equiv \left(\frac{\partial}{\partial t} - \iota\frac{k}{\gamma}\varphi\right)\overline{\psi}. \tag{5.10}$$

The Lagrangian (5.7) entails the following time-dependent Ginzburg-Landau-Schrödinger equation:

$$\frac{\gamma^2}{2}(\boldsymbol{\nabla}_{\boldsymbol{A}})^2\psi - \frac{\partial\hat{F}}{\partial\overline{\psi}} = \frac{\gamma}{\iota}\left(\frac{\partial\psi}{\partial t}\right)_{\varphi}, \tag{5.11}$$

where $(\boldsymbol{\nabla}_{\boldsymbol{A}})^2\psi \equiv (\boldsymbol{\nabla}_{\boldsymbol{A}}) \cdot [(\boldsymbol{\nabla}_{\boldsymbol{A}}\psi)]$.

If $\hat{F} \equiv -c(\psi\overline{\psi})^2 + \frac{d}{2}(\psi\overline{\psi})^4 + \frac{1}{2\mu_0}(\mathrm{curl}\,\boldsymbol{A})^2$, eq. (5.11) reduces to eq. (5.3) for stationary fields.

By re-proposing the arguments and the treatment expounded in sect. **4**, the equations of an Eulerian compressible fluid can be straightforwardly associated with eq. (5.11). Thus, the supercurrent is described by the flow of an Eulerian compressible rotational fluid, in the context of GL theory [3]. Note that the structure of the associated Euler equations, such as derived in sect. **4**, is unaltered by the presence of the non-linear terms that are possibly included in $(\frac{\partial\hat{F}}{\partial\overline{\psi}})$. In fact, these terms would only affect the form of the hydrodynamic enthalpy.

Natural boundary conditions for eq. (5.11), which are based on the operator $\boldsymbol{\nabla}_{\boldsymbol{A}}$, read

$$\boldsymbol{\nabla}_{\boldsymbol{A}}\psi \cdot \mathbf{n} = 0, \tag{5.12}$$

where $\mathbf{n}$ denotes the outward unit normal to the boundary of the domain of interest. It is not difficult to check that eq. (5.12) splits into the two following boundary conditions, which are clearly consistent with the classical conditions for the associated Eulerian fluid:

$$\frac{\partial\rho}{\partial n} = 0, \tag{5.13}$$

$$\boldsymbol{v} \cdot \mathbf{n} = 0. \tag{5.14}$$

It is worth remarking that eq. (5.11) is gauge invariant both with respect to the electromagnetic gauge transformations and with respect with the following gauge transformations [6, 14, 15]:

$$\left(\frac{\partial}{\partial t}\right) \to \left(\frac{\partial}{\partial t}\right)_{\varphi}; \ \boldsymbol{\nabla} \to \boldsymbol{\nabla}_{\boldsymbol{A}}. \tag{5.15}$$

6. – Deformable superconductors

Deformation may remarkably affect the behaviour of a superconductor. In some circumstances, the onset of supercurrents benefits from a deformation, which can drive the transition to the superconductive state to a noticeably higher temperature with respect to the standard range. Some recent results in superconducting films that are subject to infinitesimal deformations are provided and expounded in [16]. These results are within the classical GL theory. Finite deformations, however, do not only affect the constitutive response of a superconductor. They also affect the Maxwellian fields in the undeformed reference configurations and in turn the electromagnetic potentials. The question may arise then as whether the deformation should also enter the microscopic description or influence the macroscopic wave function.

Denote by V_R and V the undeformed and the current configuration, respectively, of a superconductor. $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$ represents the motion, where $\mathbf{x} \in V$ and $\mathbf{X} \in V_R$. $\mathbf{F}$ denotes the deformation gradient, whose transpose is $\mathbf{F}^{\mathrm{T}}$ and $\mathbf{v}$ the velocity. $\mathbb{C} \equiv \mathbf{F}^{\mathrm{T}}\,\mathbf{F}$ and $\mathbb{B} \equiv \mathbf{F}\,\mathbf{F}^{\mathrm{T}}$ are, respectively, the right and left Cauchy-Green deformation tensors. As usual, $\det \mathbf{F} \equiv J > 0$ [17].

The Maxwell equations in V_R read

$$\mathrm{Div}\,\boldsymbol{B}_R = 0, \tag{6.1}$$

$$\mathrm{Curl}\,\boldsymbol{E}_R + \dot{\boldsymbol{B}}_R = 0, \tag{6.2}$$

$$\mathrm{Div}\,\boldsymbol{D}_R = J\rho_f, \tag{6.3}$$

$$\mathrm{Curl}\,\boldsymbol{H}_R = \boldsymbol{j}_R + \dot{\boldsymbol{D}}_R, \tag{6.4}$$

where the differential operators Div, Curl and $\boldsymbol{\nabla}_R$ are defined in V_R [18-22]. Superposed dot denotes the Lagrangian or material time derivative. Specifically, $\dot{B}_R \equiv \frac{\mathrm{d}}{\mathrm{d}t}[\boldsymbol{B}_R(\boldsymbol{x}(\mathbf{X}, t), t)]_{\mathrm{X}}$. The correspondences among the Lagrangian and the Eulerian fields are

$$\begin{aligned} &\boldsymbol{D}_R = J\mathbf{F}^{-1}\boldsymbol{D}; \quad \boldsymbol{B}_R = J\mathbf{F}^{-1}\boldsymbol{B}; \quad \boldsymbol{E}_R = \mathbf{F}^{\mathrm{T}}(\boldsymbol{E} + \mathbf{v} \times \boldsymbol{B});\\ &\boldsymbol{H}_R = \mathbf{F}^{\mathrm{T}}(\boldsymbol{H} - \mathbf{v} \times \boldsymbol{D}); \quad \boldsymbol{j}_R = J\mathbf{F}^{-1}(\boldsymbol{j} - \rho_f \mathbf{v}). \end{aligned} \tag{6.5}$$

The electromagnetic potentials in Lagrangian form ϕ and $\boldsymbol{A_R}$ are derived as in sect. **2** through eqs. (6.1) and (6.2) as follows:

$$\boldsymbol{B}_R = \mathrm{Curl}\,\boldsymbol{A}_R, \tag{6.6}$$

$$\boldsymbol{E}_R = -\dot{\boldsymbol{A}}_R - \boldsymbol{\nabla}_R\phi. \tag{6.7}$$

The following relationships hold true:

$$\boldsymbol{A}_R = \mathbf{F}^{\mathrm{T}}\boldsymbol{A} \qquad \text{and} \qquad \phi(\mathbf{X}, t) = \varphi - \mathbf{v} \cdot \boldsymbol{A}. \tag{6.8}$$

In the reference configuration of a superconductor, London's law (2.8) is possibly written as

$$\Lambda(\boldsymbol{j}_s)_R = -\boldsymbol{A}_R + \boldsymbol{\nabla}_R \mathrm{G}. \tag{6.9}$$

Then, by taking into account eqs. $(6.5)_5$, (6.8) and eq. (6.9), the supercurrent flow in V reads

$$\Lambda \boldsymbol{j}_s = J^{-1}\mathbb{B}(-\boldsymbol{A} + \boldsymbol{\nabla}\hat{\mathrm{G}}). \tag{6.10}$$

Note that the presence of deformation tensor $\mathbb{B}$ introduces a sort of anisotropy for the behaviour of the supercurrent in the current configuration.

It is worth remarking that the classical gauge conditions, which play a fundamental role in deriving the Maxwell-London equations, are poorly significant in this context. In fact, the Lorenz-Lorentz gauge conditions for the Lagrangian potentials do not entail, as in the classical case, the remarkable result of providing uncoupled wave equations for the Maxwell fields in Lagrangian form [15, 22]. Thus, the arguments and the reasoning of sect. **2** cannot be straightforwardly applied in this framework.

A possible macroscopic wave function Ψ could be introduced in the reference configuration of the deformable superconductor, having in mind the Ginzburg-Landau theory [14]. In this framework, the related free-energy density per unit volume of the reference configuration should depend on the Lagrangian fields and specifically on $(\mathrm{Curl}\,\boldsymbol{A}_R)$ and on $[(\boldsymbol{\nabla}_R)_{\boldsymbol{A}}\Psi] \equiv [\boldsymbol{\nabla}_R - i\frac{k}{\gamma}\boldsymbol{A}_R]\Psi$ and the related possible Lagrangian density on $\dot{\Psi}$ as well. If such dependence is quadratic in $[(\boldsymbol{\nabla}_R)_{\boldsymbol{A}}\Psi]$ and specifically is of the form $\{J\frac{\gamma^2}{2}[(\boldsymbol{\nabla}_R)_A\Psi]\cdot\overline{[(\boldsymbol{\nabla}_R)_A\Psi]}\}$, an expression for the supercurrent flow in V_R, that is similar to eq. (5.4), is derived in terms of Ψ from a standard variational procedure. The consistency with eq. (6.9) of this expression for the supercurrent stems in a natural way. Consistency for the mass density transformation from the reference to the natural configuration suggests also writing

$$\Psi = J^{1/2}\widetilde{\Psi}, \tag{6.11}$$

so that $\Psi\overline{\Psi} \equiv \rho_R = J\tilde{\rho} \equiv J\psi\overline{\widetilde{\psi}}$. Ψ possibly satisfies an equation that is the analogue of eq. (5.3) or of eq. (5.6), though stated in the undeformed configuration of the superconductor. By re-proposing once more the arguments of sects. **4** and **5** in this context, an Eulerian compressible fluid can be still associated with the supercurrent in this configuration.

REFERENCES

[1] Ginzburg V. L. and Landau L. D., *Zh. Eksp. Teor. Fiz.*, **20** (1950) 1064.
[2] Bogoliubov N. N., *The Theory of Superconductivity* (Gordon & Breach) 1962.
[3] Fröhlich H., *Proc. Phys. Soc.*, **87** (1966) 330.
[4] Blatt J. M., *Theory of Superconductivity* (Academic Press) 1964.
[5] Jones W. and March N. H., *Theoretical Solid State Physics*, Vol. **2** (Dover Publication) 1985.
[6] Giaquinta G. and Mancini N. A., *Riv. Nuovo Cimento*, **1**, no. 9 (1978) 1.
[7] Kittel C., *Introduction to Solid State Physics* (John Wiley & Sons) 1986.
[8] Becker R., Heller G. and Sauter F., *Z. Phys.*, **85** (1933) 772.
[9] London F. and London H., *Proc. R. Soc. London, Ser. A*, **149** (1935) 71.
[10] Laue Von M., *Theory of Superconductivity* (Academic Press) 1952.
[11] Becker R. and Sauter F., *Electromagnetic Interactions*, Vol. **2** (Blackie & Sons) 1964.
[12] Jackson J. D., *Classical Electrodynamics* (J. Wiley & Sons) 1962.
[13] Landau L. V. and Lifschitz E., *Méchanique Quantique* (MIR ed.) 1974.

[14] MAUGIN G. A., *C.R. Acad. Sci. Paris: Série II*, **314** (1992) 889.
[15] TRIMARCO C., *Proc. Estonian Acad. Phys. Math.*, **56** (2007) 116.
[16] ANGILELLA G. C. N., BALESTRINO G., CERMELLI P., PODIO-GUIDUGLI P. and VARLAMOV A. A., *Eur. Phys. J. B*, **26** (2002) 67.
[17] TRUSDELL C. A. and NOLL W., *Handbuch der Physik, Bd.III/3* (Springer-Verlag) 1965.
[18] NELSON D. F., *Electric, Optic and Acoustic Interactions in Dielectrics* (John Wiley) 1979.
[19] MAUGIN G. A., *Material Inhomogeneities in Elasticity* (Chapman and Hall) 1993
[20] TRIMARCO C. and MAUGIN G. A., *Configurational Mechanics of Materials*, edited by KIENZLER R. and MAUGIN G. A. (Springer-Verlag) 2001, pp. 129-172.
[21] TRIMARCO C., *Tech. Mech.*, **22** (2001) 175.
[22] TRIMARCO C., *Int. J. Fracture*, **147** (2007) 13.