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The ECHO code for astrophysical plasmas: Special and General Relativistic MHD

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Summary. — The main features of a novel numerical code for astrophysical fluids and magneto-fluids named ECHO (*Eulerian conservative high-order*) are presented. Here the module for special and general relativistic MHD is discussed and applications of the code to astrophysical problems are summarized.

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1. – Introduction

In this contribution we summarize the main features of the new code ECHO for 3-D general relativistic MHD, or GRMHD [1], based on a *Eulerian conservative high-order* scheme that completes and refines our previous works for classical MHD [2,3] and special relativity, or RMHD [4, 5]. The reader is referred to these papers for the details of the numerical scheme and tests, and to [6] for an updated review on numerical GRMHD.

The issue of high numerical accuracy in conservative schemes becomes very important when not only shocks and discontinuities, but also fine smooth structures like turbulent fields and waves, are of primary interest. These small-scale structures can be smeared out by the excessive numerical diffusion typical of low-order schemes. Furthermore, higher than second-order accuracy is desirable when moving to 3-D, where numerical grids are necessarily limited in size. This especially applies to GR, due to the gradients of the metric terms that must be treated with appropriate resolution. High-order schemes are commonly used in classical gas dynamics, while general recipes for applying these methods to MHD were given in [2, 3], where the solenoidal constraint for the magnetic field was enforced as a built-in condition (*upwind constrained transport* method, UCT). Here we extend this framework to GRMHD by taking advantage of the formalism for the $3+1$ splitting of space-time. Specifically, we write all terms entering the conservative form of the GRMHD equations as quantities measured by the so-called *Eulerian* observer associated with the 3-D metric (not necessarily diagonal), highlighting the closest possible comparison with the equations of MHD and RMHD by using 3-D vectors and tensors alone. As a consequence, we are able to write the source terms in such a way that they do not contain 4-D Christoffel symbols explicitly, and the overall structure can be used also for flat space in a generic curvilinear coordinates system. ECHO thus allows for different sets of equations (from classic to general relativistic MHD) and different algorithms within a single framework. It is written in Fortran90 in a modular fashion and it is parallelized with MPI directives. The GRMHD version works in a generic user-defined space-time metric and may be easily extended to deal with a time-dependent metric once coupled to any solver for Einstein's equations. The classical MHD module is described by S. Landi *et al.* elsewhere in these proceedings.

2. – The ECHO scheme for GRMHD

The code ECHO solves any MHD-like set of hyperbolic equations containing a fluid sub-set, in divergence form plus source terms, and a magnetic sub-set, which is basically the induction equation in curl form with the associated divergence-free condition. The starting point is the discretization of the equations. Here we assume a finite-difference approach and thus we adopt the corresponding version of UCT. This is known to be more convenient than finite-volume methods for high-order treatments of multi-dimensional problems, since only 1-D reconstruction algorithms are needed. Given a computational cell of size h_i along each direction i, the fluid conservative variables $\mathcal{U}_j$ are defined at cell centers C with a *point value* representation (as well as the source terms $\mathcal{S}_j$). The other conservative variables are the $\mathcal{B}^i$ components, which are here discretized as point values at cell interfaces S_i^+, normal to direction i (the *staggering* technique). In a conservative approach, the spatial differential operators of divergence and curl are translated numerically by making use of the Gauss and Stokes theorems, respectively. Fluid fluxes $\mathcal{F}_j^i$ are to be calculated at cell faces S_i^+, while magnetic fluxes $\mathcal{E}_k$ must be calculated at cell edges L_k^+, parallel to the direction k. The spatially discretized equations are then written in the following way:

$$\frac{\mathrm{d}}{\mathrm{d}t}[\mathcal{U}_j]_C+\sum_i \frac{1}{h_i}\left([\hat{\mathcal{F}}_j^i]_{S_i^+}-[\hat{\mathcal{F}}_j^i]_{S_i^-}\right)=[\mathcal{S}_j]_C, \quad \frac{\mathrm{d}}{\mathrm{d}t}[\mathcal{B}^i]_{S_i^+}+\sum_{j,k}[ijk]\frac{1}{h_j}\left([\hat{\mathcal{E}}_k]_{L_k^+}-[\hat{\mathcal{E}}_k]_{L_k^-}\right)=0,$$

known as *semi-discrete* form, since the time derivatives are left analytical. Here the hat indicates high-order approximation of the numerical flux function, and we have indicated with $\pm$ the opposite faces, or edges, with respect to the direction of derivation. Time evolution is achieved here by means of Runge-Kutta integration schemes (typically of second or third order). In the same framework, the non-evolutionary solenoidal constraint becomes

$$\sum_i \frac{1}{h_i}\left([\hat{\mathcal{B}}^i]_{S_i^+}-[\hat{\mathcal{B}}^i]_{S_i^-}\right)=0.$$

A couple of test problems are shown in fig. 1.

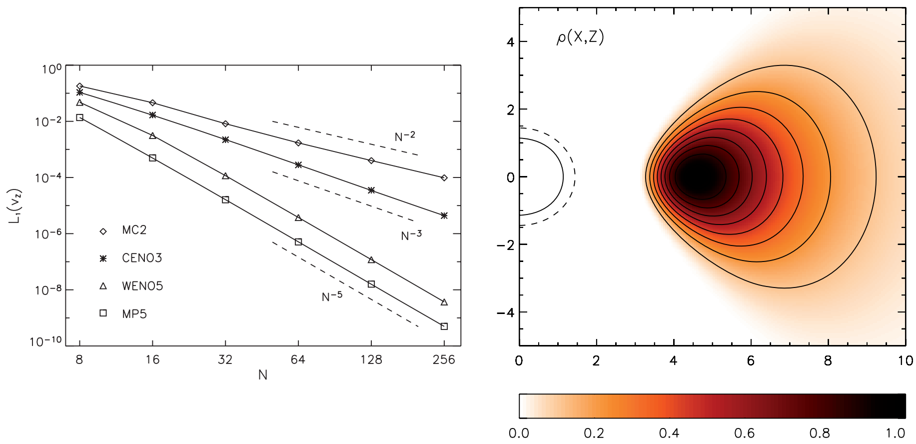

Fig. 1. – Left-hand panel: convergence test for 2-D circularly polarized relativistic Alfvén waves. The formal order of accuracy for various schemes (up to 5) is measured by plotting the average error on the transverse velocity *versus* the number N of grid points. Right-hand panel: thick magnetized torus orbiting a maximally rotating Kerr black hole. The initial solution (here the density ρ) is maintained after a few orbital periods.

The 10 steps needed to evolve numerically the above system in time are described in details in [1], as well as the high-order reconstruction routines available in the code. Recently, also compact spectral-like methods have been implemented in ECHO (see Landi's contribution in these proceedings). Notice that in our scheme the solenoidal condition is preserved algebraically in time for *any* (formal) order of spatial accuracy.

The above form of the equations hold for both classical and relativistic MHD. In numerical GRMHD the spacetime metric is first split into the $3+1$ form

$$\mathrm{d}s^2 = -\alpha^2 \mathrm{d}t^2 + \gamma_{ij}\,(\mathrm{d}x^i + \beta^i \mathrm{d}t)(\mathrm{d}x^j + \beta^j \mathrm{d}t),$$

where α is called the *lapse function*, β^i the *shift vector*, and γ_{ij} is the usual spatial three-metric. Then the four-velocity, the momentum-energy tensor and the electromagnetic field tensor are decomposed according to the temporal and spatial parts. The conservative variables, fluxes, and sources in the fluid equations are (for a *stationary* metric):

$$\vec{\mathcal{U}} = \gamma^{1/2} \begin{bmatrix} D \\ S_j \\ U \end{bmatrix}, \qquad \vec{\mathcal{F}}^i = \gamma^{1/2} \begin{bmatrix} \alpha v^i D - \beta^i D \\ \alpha W^i_j - \beta^i S_j \\ \alpha S^i - \beta^i U \end{bmatrix},$$

$$\vec{\mathcal{S}} = \gamma^{1/2} \begin{bmatrix} 0 \\ \frac{1}{2}\alpha W^{ik}\partial_j \gamma_{ik} + S_i \partial_j \beta^i - U \partial_j \alpha \\ \frac{1}{2} W^{ik} \beta^j \partial_j \gamma_{ik} + W_i{}^j \partial_j \beta^i - S^j \partial_j \alpha \end{bmatrix},$$

where γ is the determinant of the three-metric. The expression for the stress tensor, momentum density, and energy density in terms of the fluid (ρ, $\vec{v}$, and p) and electromagnetic

$(\vec{B}, \vec{E})$ quantities are, respectively,

$$\vec{W} = \rho h \Gamma^2 \vec{v}\,\vec{v} - \vec{E}\,\vec{E} - \vec{B}\,\vec{B} + \left[p + \frac{1}{2}(E^2 + B^2) \right] \vec{\gamma},$$
$$\vec{S} = \rho h \Gamma^2 \vec{v} + \vec{E} \times \vec{B},$$
$$U = \rho h \Gamma^2 - p + \frac{1}{2}(E^2 + B^2),$$

where $h = 1+\varepsilon+p/\rho$ is the specific enthalpy (ε the specific internal energy), given in terms of ρ and p according to the equation of state $p = p(\rho, \varepsilon)$ chosen, $\Gamma = (1 - v^2)^{-1/2}$ is the Lorentz factor, while we have indicated the spatial three-metric tensor with the symbol $\vec{\gamma}$. In the $3 + 1$ split, the Ohm relation for ideal GRMHD needed to close the system is still the usual *freeze-in* condition of vanishing electric field in the frame comoving with the fluid $\vec{E} = -\vec{v} \times \vec{B}$. As far as the induction equation is concerned, in GRMHD we define

$$\mathcal{B}^i = \gamma^{1/2} B^i, \qquad \mathcal{E}_i = \alpha E_i + \epsilon_{ijk} \beta^j B^k = -[ijk] \mathcal{V}^j \mathcal{B}^k,$$

where $\mathcal{V}^j = \alpha v^j - \beta^j$ is called transport velocity.

The special relativistic Minkowskian limit (RMHD) is recovered by letting $\alpha = 1$ and $\vec{\beta} = 0$ in all the above relations. In the case of flat metric in Cartesian coordinates we also have $\gamma_{ij} = \mathrm{diag}(1, 1, 1)$, and covariant and contravariant components coincide.

3. – Astrophysical applications

ECHO has been used in a variety of astrophysical scenarios under different implementations and numerical settings. As far as the relativistic version is concerned, the main application has been the interaction of a pulsar's highly relativistic wind with the external expanding supernova remnant. This creates a hot bubble of ultra-relativistic particles and hot plasma that shines as a Pulsar Wind Nebula (*e.g.*, the Crab Nebula) through non-thermal emission, namely synchrotron and Inverse Compton scattering, from radio to gamma-rays up to the TeV band. Axisymmetric models have been proposed for the nebular dynamics and emission properties [7-9], where the jet-torus structure has been reproduced and compared to observations through synthetic surface brightness maps; for bow-shock nebulae [10]; for the study of particular aspects thought to be important in the PWN context, like relativistic MHD Rayleigh-Taylor and Kelvin-Helmholtz instabilities [11,12]. Moreover, the physics of GRMHD winds from proto-neutron stars and magnetars have also been recently investigated via axisymmetric simulations in Schwarzschild metric, with application to the core-collapse/GRB scenario [13, 14].

REFERENCES

[1] Del Zanna L., Zanotti O., Bucciantini N. and Londrillo P., *Astron. Astrophys.*, **473** (2007) 11.
[2] Londrillo P. and Del Zanna L., *Astrophys. J.*, **530** (2000) 508.
[3] Londrillo P. and Del Zanna L., *J. Comput. Phys.*, **195** (2004) 17.
[4] Del Zanna L. and Bucciantini N., *Astron. Astrophys.*, **390** (2002) 1177.
[5] Del Zanna L., Bucciantini N. and Londrillo P., *Astron. Astrophys.*, **400** (2003) 397.
[6] Font J. A., *Living Rev. Relat.*, **11** (2008) 7.

[7] Del Zanna L., Amato E. and Bucciantini N., *Astron. Astrophys.*, **421** (2004) 1063.
[8] Del Zanna L., Volpi D., Amato E. and Bucciantini N., *Astron. Astrophys.*, **453** (2006) 621.
[9] Volpi D., Del Zanna L., Amato E. and Bucciantini N., *Astron. Astrophys.*, **485** (2008) 337.
[10] Bucciantini N., Amato E. and Del Zanna L., *Astron. Astrophys.*, **434** (2005) 189.
[11] Bucciantini N., Amato E., Bandiera R., Blondin J. M. and Del Zanna L., *Astron. Astrophys.*, **423** (2004) 253.
[12] Bucciantini N. and Del Zanna L., *Astron. Astrophys.*, **454** (2006) 393.
[13] Bucciantini N., Thompson T. A., Arons J., Quataert E. and Del Zanna L., *Mon. Not. R. Astron. Soc.*, **368** (2006) 1717.
[14] Bucciantini N., Quataert E., Arons J., Metzger B. D. and Thompson T. A., *Mon. Not. R. Astron. Soc.*, **383** (2008) L25.