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Initial-value problem of the quantum dual BBGKY hierarchy

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Summary. — We present a rigorous formalism to describe the evolution of observables of quantum many-particle systems. We construct a solution of the initial-value problem of the quantum dual BBGKY hierarchy of equations as an expansion over particle clusters, whose evolution is governed by the corresponding-order cumulant (semi-invariant) of the evolution operators of finitely many particles. For initial data from the space of sequences of bounded operators the existence and uniqueness theorem is proved.

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1. – Introduction

Evolution equations of quantum many-particle systems arise in many problems of modern statistical mechanics [1-3]. In the theory of such equations, during the last decade, many new results have been obtained, in particular concerning the fundamental problem of the rigorous derivation of quantum kinetic equations [4-9].

A description of quantum many-particle systems can be formulated in terms of two sets of objects: observables and states. The mean value defines a duality between observables and states and, as a consequence, there exist two approaches to the description of the evolution. Usually the evolution of many-particle systems is described, in the framework of the evolution of states, by the BBGKY hierarchy for marginal density operators, which is equivalent to the von Neumann (quantum Liouville) equation for the density operator in the case of finitely many particles. In the papers [5-9] a solution of the Cauchy problem to the quantum BBGKY hierarchy is constructed in the form of iteration series for initial data in the space of sequences of trace class operators. In [10, 11] for the quantum BBGKY hierarchy (for the classical many-particle systems in [12]) a solution is represented in the form of series over particle clusters, whose evolution is described by the corresponding order cumulant (semi-invariant) of evolution operators of finitely many particles. Using an analog of Duhamel formulas, such a solution expansion reduces to an iteration series, which is valid for a particular class of initial data and interaction potentials. An equivalent approach for the description of many-particle system evolution is given by the evolution of observables and by the dual BBGKY hierarchy. For classical systems this approach is studied in the paper [13].

In this paper we deduce the initial-value problem to the quantum dual BBGKY hierarchy describing the evolution of observables of many-particle quantum systems, obeying Maxwell-Boltzmann statistics and construct its solution in the form of an expansion over clusters of the decreasing number of particles, whose evolution is governed by the corresponding-order cumulant (semi-invariant) of groups of operators of finitely many particles (groups of operators of the Heisenberg equations). We also discuss the problem of the description of infinite-particle systems in the Heisenberg picture of evolution.

2. – The quantum dual BBGKY hierarchy

In order to describe the observables of quantum many-particle systems by the marginal observables (s-particle observables) we study the evolution of the system by means of the quantum dual BBGKY hierarchy.

2˙1. *The initial-value problem of the quantum dual BBGKY hierarchy.* – Let the space $\mathfrak{L}(\mathcal{F}_{\mathcal{H}})$ be the space of sequences $g = (I, g_1, \ldots, g_n, \ldots)$ of bounded operators g_n (I is a unit operator) defined on the Hilbert space $\mathcal{H}_n$ and satisfying symmetry property: $g_n(1, \ldots, n) = g_n(i_1, \ldots, i_n)$, if $\{i_1, \ldots, i_n\} \in \{1, \ldots, n\}$, with an operator norm [2, 14]. We will also consider a more general space $\mathfrak{L}_\gamma(\mathcal{F}_{\mathcal{H}})$ with a norm

$$\|g\|_{\mathfrak{L}_\gamma(\mathcal{F}_{\mathcal{H}})} = \max_{n \geq 0} \frac{\gamma^n}{n!} \|g_n\|_{\mathfrak{L}(\mathcal{H}_n)},$$

where $0 < \gamma < 1$ and $\|\cdot\|_{\mathfrak{L}(\mathcal{H}_n)}$ is an operator norm. An observable of many-particle quantum system is a sequence of self-adjoint operators from $\mathfrak{L}_\gamma(\mathcal{F}_{\mathcal{H}})$. The case of the unbounded observables can be reduced to the case under consideration [15]. For example, the Hamiltonian $H = \bigoplus_{n=0}^{\infty} H_n$ is defined on the subspace $L_0^2(\mathbb{R}^{3n}) \subset L^2(\mathbb{R}^{3n})$ of infinitely differentiable functions with compact support and the n-particle Hamiltonian H_n acts according to the formula

$$H_n \psi_n = -\frac{\hbar^2}{2} \sum_{i=1}^{n} \Delta_{q_i} \psi_n + \sum_{k=1}^{n} \sum_{i_1 < \ldots < i_k = 1}^{n} \Phi^{(k)}(q_{i_1}, \ldots, q_{i_k}) \psi_n, \tag{1}$$

where $h = 2\pi\hbar$ is the Planck constant and $\Phi^{(k)}$ is a k-body interaction potential satisfying Kato conditions [2].

The evolution of marginal observables $G(t) = (G_0, G_1(t, 1), \ldots, G_s(t, 1, \ldots, s), \ldots)$ is described by the initial-value problem for the following hierarchy of evolution equations:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} G_s(t, Y) &= \mathcal{N}_s(Y) G_s(t, Y) + \\ &+ \sum_{n=1}^{s} \frac{1}{n!} \sum_{k=n+1}^{s} \frac{1}{(k-n)!} \sum_{j_1 \neq \ldots \neq j_k = 1}^{s} \mathcal{N}_{\mathrm{int}}^{(k)}(j_1, \ldots, j_k) G_{s-n}(t, Y \backslash \{j_1, \ldots, j_n\}), \end{aligned} \tag{2}$$

$$G_s(t) \mid_{t=0} = G_s(0), \quad s \geq 1, \tag{3}$$

where the following abridged notations are used: $Y \equiv (1,\ldots,s)$, $Y\backslash\{j\} \equiv (1,\ldots,j-1,j+1,\ldots,s) = (1,\ldots,\overset{j}{\vee},\ldots,s)$, and, if $g \in \mathcal{D}(\mathcal{N}) \subset \mathfrak{L}_{\gamma}(\mathcal{F}_{\mathcal{H}})$, the von Neumann operator $\mathcal{N} = \bigoplus_{n=0}^{\infty} \mathcal{N}_n$ is defined by the formula

$$(\mathcal{N}g)_n = -\frac{i}{\hbar}(g_n H_n - H_n g_n),$$

and the operator $\mathcal{N}_{\mathrm{int}}^{(n)}$ is defined by

$$\mathcal{N}_{\mathrm{int}}^{(n)} g_n = -\frac{i}{\hbar}\big(g_n \Phi^{(n)} - \Phi^{(n)} g_n\big). \tag{4}$$

We refer to eqs. (2) as the quantum dual BBGKY hierarchy, since the canonical BBGKY hierarchy [1] for marginal density operators is the dual hierarchy of evolution equations to eqs. (2) with respect to bilinear form (positive continuous linear functional on the space of observables, whose value is interpreted as its average value),

$$\langle G(t)|F(0)\rangle = \sum_{s=0}^{\infty} \frac{1}{s!}\,\mathrm{Tr}_{1,\ldots,s}\; G_s(t,1,\ldots,s) F_s(0,1,\ldots,s), \tag{5}$$

where $F_s(0)$, $s \geq 1$, are marginal density operators (or s-particle density operators).

In the case of two-body interaction potential, hierarchy (2) has the form

$$\frac{\mathrm{d}}{\mathrm{d}t} G_s(t,Y) = \mathcal{N}_s(Y) G_s(t,Y) + \sum_{j_1 \neq j_2 = 1}^{s} \mathcal{N}_{\mathrm{int}}^{(2)}(j_1,j_2) G_{s-1}(t, Y\backslash\{j_1\}), \quad s \geq 1, \tag{6}$$

where the operator $\mathcal{N}_{\mathrm{int}}^{(2)}$ is defined by (4) for $n = 2$. For $\mathcal{H} = L^2(\mathbb{R}^{3n})$, the evolution of kernels of operators $G_s(t)$, $s \geq 1$, for eqs. (6), is given by

$$\begin{aligned}
i\hbar\frac{\partial}{\partial t} G_s(t,q_1,\ldots,q_s;q'_1,\ldots,q'_s) &= \Bigg(-\frac{\hbar^2}{2}\sum_{i=1}^{s}(-\Delta_{q_i} + \Delta_{q'_i}) + \\
&+ \sum_{1=i<j}^{s} \big(\Phi^{(2)}(q'_i - q'_j) - \Phi^{(2)}(q_i - q_j)\big)\Bigg) G_s(t,q_1,\ldots,q_s;q'_1,\ldots,q'_s) + \\
&+ \sum_{1=i\neq j}^{s} \big(\Phi^{(2)}(q'_i - q'_j) - \Phi^{(2)}(q_i - q_j)\big) G_{s-1}\Big(t,q_1,\ldots,\overset{j}{\vee},\ldots,q_s;q'_1,\ldots,\overset{j}{\vee},\ldots,q'_s\Big),
\end{aligned}$$

where $(q_1,\ldots,\overset{j}{\vee},\ldots,q_s) \equiv (q_1,\ldots,q_{j-1},q_{j+1},\ldots,q_s)$. The dual BBGKY hierarchy for a system of classical particles stated in [1, 13] is defined by similar recurrence evolution equations.

The quantum dual BBGKY hierarchy (2) can be derived from the sequence of the Heisenberg equations provided that observables of a system are described in terms of marginal operators (s-particle observables) [13].

A different way of looking to the derivation of the quantum dual BBGKY hierarchy consists in the construction of adjoint (dual) equations to the quantum BBGKY hierarchy [2] with respect to the bilinear form (5).

2˙2. *Remarks.* – In the paper [13], for classical systems of particles with a two-body interaction potential, an equivalent representation for the dual hierarchy generator was used. In the case under consideration, on the subspace $\mathcal{D}(\mathfrak{B}^+) \subset \mathfrak{L}_\gamma(\mathcal{F}_\mathcal{H})$, its generator has the following representation: $\mathfrak{B}^+ = \mathcal{N} + [\mathcal{N}, \mathfrak{a}^+]$, where $[\cdot,\cdot]$ is a commutator and the operator $\mathfrak{a}^+$ is defined on the space $\mathfrak{L}_\gamma(\mathcal{F}_\mathcal{H})$ (an analog of the creation operator) as follows:

$$(\mathfrak{a}^+ g)_s(Y) = \sum_{j=1}^{s} g_{s-1}(Y\backslash\{j\}). \tag{7}$$

In a general case the generator of the quantum dual BBGKY hierarchy (2) can be represented in the following form:

$$\mathfrak{B}^+ = \mathcal{N} + \sum_{n=1}^{\infty} \frac{1}{n!}[\ldots[\mathcal{N}, \underbrace{\mathfrak{a}^+], \ldots, \mathfrak{a}^+}_{n\text{-times}}] = e^{-\mathfrak{a}^+}\mathcal{N}e^{\mathfrak{a}^+}. \tag{8}$$

Representation (8) is correct in consequence of definition (7) of the operator $\mathfrak{a}^+$ and the validity of identity

$$\big([\ldots[\mathcal{N}, \underbrace{\mathfrak{a}^+], \ldots, \mathfrak{a}^+}_{n\text{-times}}]g\big)_s = \sum_{k=n+1}^{s} \frac{1}{(k-n)!} \sum_{j_1\neq\ldots\neq j_k=1}^{s} \mathcal{N}_{\mathrm{int}}^{(k)}(j_1,\ldots,j_k) g_{s-n},$$

which for a two-body interaction potential reduces to the following one: $([\mathcal{N}, \mathfrak{a}^+]g)_s(Y) = \sum_{j_1\neq j_2=1}^{s} \mathcal{N}_{\mathrm{int}}^{(2)}(j_1,j_2) g_{s-1}(Y\backslash\{j_1\})$ (see eq. (6)).

3. – A solution of the initial-value problem of the quantum dual BBGKY hierarchy

Further we will use some abridged notations: $Y \equiv (1,\ldots,s)$, $Y\backslash\{j_1,\ldots,j_{s-n}\} \equiv X$, the set $(Y\backslash X)_1$ consists of one element from $Y\backslash X = (j_1,\ldots,j_{s-n})$, *i.e.* the set $(j_1,\ldots,j_{s-n})$ is a connected subset of the partition P ($|\mathrm{P}| = 1$, $|\mathrm{P}|$ denotes the number of considered partitions).

3˙1. *A solution expansion.* – A solution of the initial-value problem of the quantum dual BBGKY hierarchy (2), (3) is determined by the expansion ($s \geq 1$)

$$G_s(t,Y) = \sum_{n=0}^{s} \frac{1}{(s-n)!} \sum_{j_1\neq\ldots\neq j_{s-n}=1}^{s} \mathfrak{A}_{1+n}\big(t,(Y\backslash X)_1, X\big)\, G_{s-n}(0, Y\backslash X), \tag{9}$$

where the $(1+n)$-th–order cumulant $\mathfrak{A}_{1+n}(t,(Y\backslash X)_1,X)$ is defined by the expression

$$\mathfrak{A}_{1+n}\big(t,(Y\backslash X)_1,X\big) = \sum_{\mathrm{P}:\{(Y\backslash X)_1, X\} = \bigcup_i X_i} (-1)^{|\mathrm{P}|-1}(|\mathrm{P}|-1)! \prod_{X_i\subset \mathrm{P}} \mathcal{G}_{|X_i|}(t,X_i), \tag{10}$$

where $\sum_\mathrm{P}$ is the sum over all possible partitions P of the set $\{(Y\backslash X)_1, j_1,\ldots,j_{s-n}\}$ into $|\mathrm{P}|$ nonempty mutually disjoint subsets $X_i \subset \{(Y\backslash X)_1, X\}$, and

$$\mathcal{G}_n(t)g_n = e^{\frac{i}{\hbar}tH_n} g_n e^{-\frac{i}{\hbar}tH_n} \tag{11}$$

is the group of operators of the Heisenberg equation.

3˙2. *A group of operators of the quantum dual BBGKY hierarchy.* – On the space $\mathfrak{L}_\gamma(\mathcal{F}_\mathcal{H})$ solution (9) of the initial-value problem of the dual BBGKY hierarchy (2), (3) is determined by a one-parameter mapping: $\mathbb{R}^1 \ni t \mapsto U^+(t)g$

$$(U^+(t)g)_s(Y) := \sum_{n=0}^{s} \frac{1}{(s-n)!} \sum_{j_1 \neq \ldots \neq j_{s-n}=1}^{s} \mathfrak{A}_{1+n}\big(t, (Y\backslash X)_1, X\big)\, g_{s-n}(Y\backslash X) \tag{12}$$

with the following properties.

If $g \in \mathfrak{L}_\gamma(\mathcal{F}_\mathcal{H})$ and $\gamma < e^{-1}$, then the one-parameter mapping: $\mathbb{R}^1 \ni t \mapsto U^+(t)g$ is a C_0^*-group. The infinitesimal generator $\mathfrak{B}^+ = \bigoplus_{n=0}^\infty \mathfrak{B}_n^+$ of this group of operators is a closed operator for the $*$-weak topology and on the domain of the definition $\mathcal{D}(\mathfrak{B}^+) \subset \mathfrak{L}_\gamma(\mathcal{F}_\mathcal{H})$ which is the everywhere dense set for the $*$-weak topology of the space $\mathfrak{L}_\gamma(\mathcal{F}_\mathcal{H})$ it is defined by the operator

$$\begin{aligned}(\mathfrak{B}^+ g)_s(Y) &= \mathcal{N}_s(Y) g_s(Y) + \\ &+ \sum_{n=1}^{s} \frac{1}{n!} \sum_{k=n+1}^{s} \frac{1}{(k-n)!} \sum_{j_1 \neq \ldots \neq j_k=1}^{s} \mathcal{N}_{\mathrm{int}}^{(k)}(j_1, \ldots, j_k) g_{s-n}(Y\backslash\{j_1, \ldots, j_n\}),\end{aligned} \tag{13}$$

where the operator $\mathcal{N}_{\mathrm{int}}^{(k)}$ is given by (4).

If $g \in \mathfrak{L}_\gamma(\mathcal{F}_\mathcal{H})$, mapping (12) is defined provided that $\gamma < e^{-1}$ and that the following estimate holds:

$$\big\|U^+(t)g\big\|_{\mathfrak{L}_\gamma(\mathcal{F}_\mathcal{H})} \leq e^2(1-\gamma e)^{-1}\|g\|_{\mathfrak{L}_\gamma(\mathcal{F}_\mathcal{H})}. \tag{14}$$

On the space $\mathfrak{L}_\gamma(\mathcal{F}_\mathcal{H})$ the $*$-weak continuity property of the group $U^+(t)$ over the parameter $t \in \mathbb{R}^1$ is a consequence of the $*$-weak continuity of the group $\mathcal{G}(t)$ of operators (11) of the Heisenberg equation.

In order to construct an infinitesimal generator of the group $\{U^+(t)\}_{t\in\mathbb{R}}$ we firstly differentiate the n-th term of expansion (12) in the sense of the pointwise convergence of the space $\mathfrak{L}_\gamma$. If $g \in \mathcal{D}(\mathcal{N}) \subset \mathfrak{L}_\gamma(\mathcal{F}_\mathcal{H})$, for (1+$n$)-th–order cumulant (10), $n \geq 1$, we derive

$$\begin{aligned}&\lim_{t\to 0} \frac{1}{t} \mathfrak{A}_{1+n}\big(t, (Y\backslash X)_1, X\big) g_{s-n}(Y\backslash X)\psi_s = \sum_{\substack{Z\subset Y\backslash X,\\ Z\neq\emptyset}} \mathcal{N}_{\mathrm{int}}^{(|Z|+n)}(Z, X) g_{s-n}(Y\backslash X)\psi_s = \\ &= \sum_{k=1}^{s-n} \frac{1}{k!} \sum_{i_1\neq\ldots\neq i_k \in\{j_1,\ldots,j_{s-n}\}} \mathcal{N}_{\mathrm{int}}^{(k+n)}(i_1, \ldots, i_k, X) g_{s-n}(Y\backslash X)\psi_s.\end{aligned}$$

Then, according to this equality, for group (12) we obtain

$$\begin{aligned}&\lim_{t\to 0} \frac{1}{t}\Big((U^+(t)g)_s - g_s\Big)\psi_s = \lim_{t\to 0} \frac{1}{t}\big(\mathfrak{A}_1(t) g_s - g_s\big)\psi_s + \\ &+ \sum_{n=1}^{s} \frac{1}{(s-n)!} \sum_{j_1\neq\ldots\neq j_{s-n}=1}^{s} \lim_{t\to 0} \frac{1}{t} \mathfrak{A}_{1+n}\big(t, (Y\backslash X)_1, X\big) g_{s-n}(Y\backslash X)\psi_s = \\ &= \mathcal{N}_s g_s \psi_s + \sum_{n=1}^{s} \frac{1}{n!} \sum_{k=n+1}^{s} \frac{1}{(k-n)!} \sum_{j_1\neq\ldots\neq j_k=1}^{s} \mathcal{N}_{\mathrm{int}}^{(k)}(j_1, \ldots, j_k) g_{s-n}(Y\backslash\{j_1, \ldots, j_n\})\psi_s.\end{aligned}$$

Thus, if $g \in \mathcal{D}(\mathfrak{B}^+) \subset \mathfrak{L}_\gamma(\mathcal{F}_\mathcal{H})$ in the sense of the $*$-weak convergence of the space $\mathfrak{L}_\gamma(\mathcal{F}_\mathcal{H})$ we finally have: $\mathrm{w}^* - \lim_{t\to 0}(\frac{1}{t}(U^+(t)g - g) - \mathfrak{B}^+ g) = 0$, where the generator $\mathfrak{B}^+ = \bigoplus_{n=0}^{\infty} \mathfrak{B}_n^+$ of group of operators (12) is given by (13).

3$\dot{}$3. *The existence and uniqueness theorem.* – The following statement holds for abstract initial-value problem (2), (3) on the space $\mathfrak{L}_\gamma(\mathcal{F}_\mathcal{H})$.

A solution of the initial-value problem of the quantum dual BBGKY hierarchy (2), (3) is determined by expansion (9). If $G(0) \in \mathcal{D}(\mathfrak{B}^+) \subset \mathfrak{L}_\gamma(\mathcal{F}_\mathcal{H})$ it is a classical solution and for arbitrary initial data $G(0) \in \mathfrak{L}_\gamma(\mathcal{F}_\mathcal{H})$ it is a generalized solution.

Indeed, for the initial data $G(0) \in \mathcal{D}(\mathfrak{B}^+) \subset \mathfrak{L}_\gamma(\mathcal{F}_\mathcal{H})$, sequence (9) is a classical solution of initial-value problem (2), (3) in the sense of the $*$-weak convergence of the space $\mathfrak{L}_\gamma(\mathcal{F}_\mathcal{H})$.

Let us now show that in the general case $G(0) \in \mathfrak{L}_\gamma(\mathcal{F}_\mathcal{H})$ expansions (9) give a generalized solution of the initial-value problem to the quantum dual BBGKY hierarchy (2), (3). To this aim we consider the functional

$$(f, G(t)) := \sum_{s=0}^{\infty} \frac{1}{s!} \mathrm{Tr}_{1,\ldots,s}\, f_s\, G_s(t), \tag{15}$$

where $f \in \mathfrak{L}^1_{\alpha,0}$ is a finite sequence of the degenerate trace class operators with infinitely times differentiable kernels and with compact support. According to estimate (14) this functional exists provided that $\alpha = \gamma^{-1} > e$ (see sect. **4**).

Using (9), we can transform functional (15) as follows:

$$(f, G(t)) = (f, U^+(t)G(0)) = (U(t)f, G(0)). \tag{16}$$

In this equality the group $U^+(t)$ is defined by expression (12) and $U(t)$ is an adjoint mapping to the group $U^+(t)$

$$(U(t)f)_s(Y) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathrm{Tr}_{s+1,\ldots,s+n}\, \mathfrak{A}_{1+n}(-t, Y_1, X\backslash Y) f_{s+n}(X), \tag{17}$$

where $X \equiv \{1, \ldots, s+n\}$, *i.e.* $X\backslash Y \equiv \{s+1, \ldots, s+n\}$. In expansion (17) the evolution operator $\mathfrak{A}_{1+n}(-t, Y_1, X\backslash Y)$ is the $(1+n)$-th–order cumulant of group of operators (11)

$$\mathfrak{A}_{1+n}(-t, Y_1, X\backslash Y) = \sum_{\mathrm{P}:\, \{Y_1, X\backslash Y\} = \bigcup_i X_i} (-1)^{|\mathrm{P}|-1}(|\mathrm{P}|-1)! \prod_{X_i \subset \mathrm{P}} \mathcal{G}_{|X_i|}(-t, X_i),$$

where $\sum_\mathrm{P}$ is the sum over all possible partitions P of the set $\{Y_1, X \setminus Y\} = \{Y_1, s+1, \ldots, s+n\}$ into $|\mathrm{P}|$ nonempty mutually disjoint subsets $X_i \subset \{Y_1, X \setminus Y\}$. If $f \in \mathfrak{L}^1_{\alpha,0}$, series (17) converges, provided that $\alpha > e$ [10] and the functional $(U(t)f, G(0))$ exists.

The one-parameter family of operators $U(t)$ is differentiable with respect to t and for $f \in \mathfrak{L}^1_{\alpha,0}$ an infinitesimal generator of group (17) is defined by the following expression:

$$\begin{aligned}(\mathfrak{B}f)_s(Y) &= -\mathcal{N}_s(Y) f_s(Y) + \\ &+ \sum_{k=1}^{s} \frac{1}{k!} \sum_{i_1 \neq \ldots \neq i_k = 1}^{s} \sum_{n=1}^{\infty} \frac{1}{n!} \mathrm{Tr}_{s+1,\ldots,s+n} \Big(-\mathcal{N}_{\mathrm{int}}^{(k+n)} \Big)(i_1, \ldots, i_k, X\backslash Y) f_{s+n}(X).\end{aligned} \tag{18}$$

Since for bounded interaction potentials (1), if $f \in \mathfrak{L}^1_{\alpha,0}$, the operator $\mathfrak{B}U(t)f$ is a trace class operator, the operator $\mathfrak{B}U(t)fG(0)$ is also a trace class operator then the functional $(\mathfrak{B}U(t)f, G(0))$ exists. Moreover, there holds the equality: $(\mathfrak{B}U(t)f, G(0)) = (U(t)\mathfrak{B}f, G(0))$ and the following result:

$$\lim_{t\to 0}\left|\left(\frac{1}{t}(U(t)-I)f, G(0)\right) - (\mathfrak{B}f, G(0))\right| = \lim_{t\to 0}\left|\sum_{s=0}^{\infty}\frac{1}{s!}\,\mathrm{Tr}_{1,\ldots,s}\left(\frac{1}{t}(U(t)f - f)_s G_s(0) - \right.\right.$$
$$\left.\left.-(\mathfrak{B}f)_s G_s(0)\right)\right| \leq \|G(0)\|_{\mathfrak{L}_\gamma(\mathcal{F}_\mathcal{H})}\lim_{t\to 0}\left\|\frac{1}{t}(U(t)f - f) - \mathfrak{B}f\right\|_{\mathfrak{L}^1_{\gamma^{-1}}(\mathcal{F}_\mathcal{H})} = 0.$$

Hence equality (16) can be differentiated with respect to time and we get finally

$$\frac{\mathrm{d}}{\mathrm{d}t}(f, G(t)) = \big(U(t)\mathfrak{B}f, G(0)\big) = \big(\mathfrak{B}f, U^+(t)G(0)\big) = (\mathfrak{B}f, G(t)),$$

where the operator $\mathfrak{B}$ is defined by (18). These equalities mean that the sequence of operators (9) for arbitrary $G(0) \in \mathfrak{L}_\gamma(\mathcal{F}_\mathcal{H})$ is a generalized solution of the Cauchy problem to the quantum dual BBGKY hierarchy (2), (3).

3$\cdot$4. *The existence of the mean-value observable functional.* – As it was above mentioned, the functional of the mean value (5) defines a duality between marginal observables and marginal states. If $G(t) \in \mathfrak{L}_\gamma(\mathcal{F}_\mathcal{H})$ and $F(0) \in \mathfrak{L}^1_\alpha$, where $\mathfrak{L}^1_\alpha$ is the space of sequences of trace class operators defined in [10], then, according to estimate (14), the functional (5) exists, provided that $\alpha = \gamma^{-1} > e$, and the following estimate holds:

$$|\langle G(t)|F(0)\rangle| \leq e^2(1-\gamma e)^{-1}\|G(0)\|_{\mathfrak{L}_\gamma(\mathcal{F}_\mathcal{H})}\|F(0)\|_{\mathfrak{L}^1_{\gamma^{-1}}(\mathcal{F}_\mathcal{H})}.$$

Thus, marginal density operators from the space $\mathfrak{L}^1_\alpha$ describe finitely many quantum particles. Indeed, for such additive-type observable as the number of particles, *i.e.* one-component sequence $N(0) = (0, I, 0, \ldots)$, according to definition of cumulants, the expansion for solution (9) assumes the following form:

$$(N(t))_s(Y) = \mathfrak{A}_s(t, 1, \ldots, s)\sum_{j=1}^{s} I = I\delta_{s,1}, \quad s \geq 1,$$

and we have: $|\langle N(t)|F(0)\rangle| = |\,\mathrm{Tr}_1\ F_1(0,1)| \leq \|F(0)\|_{\mathfrak{L}^1_{\gamma^{-1}}(\mathcal{F}_\mathcal{H})} < \infty$.

Let us now remark that, if we try to extend our results to the description of infinitely many particles [1], the problem of the definition of functional (5) arises. As a matter of fact, marginal density operators have to belong to more general spaces than $\mathfrak{L}^1_\alpha(\mathcal{F}_\mathcal{H})$. For example, one could choose the space of sequences of bounded operators containing the equilibrium states, but in this case every term of expansions for the mean-value functional (5) contains the divergent traces [1,13,10] and the analysis of such a point for quantum systems remains an open problem.

4. – Conclusion

The concept of cumulants of groups of operators of the Heisenberg equations or cumulants of groups of operators of the von Neumann equations forms the basis of the groups of operators for quantum many-particle evolution equations as well as the quantum dual BBGKY hierarchy (2) for marginal observables and the BBGKY hierarchy for marginal density operators [10].

On the space $\mathfrak{L}_{\gamma}(\mathcal{F}_{\mathcal{H}})$ one-parameter mapping (12) is not a strong continuous group. The group $\{U^{+}(t)\}_{t\in\mathbb{R}}$ of operators (12) defined on the space $\mathfrak{L}_{\gamma}(\mathcal{F}_{\mathcal{H}})$ is dual to the strong continuous group $\{U(t)\}_{t\in\mathbb{R}}$ of operators (17) for the BBGKY hierarchy defined on the space of sequences of trace class operators $\mathfrak{L}^{1}_{\alpha}(\mathcal{F}_{\mathcal{H}})$ and the property that it is a C_0^*-group follows also from general theorems about dual semigroups [14].

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