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Study of a localized photon source in spaces of measures

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Summary. — In this paper we study a three-dimensional photon transport problem in an interstellar cloud, with a localized photon source inside. The problem is solved indirectly, by defining the adjoint of an operator acting on an appropriate space of continuous functions. By means of sun-adjoint semigroups theory of operators in a Banach space of regular Borel measures, we prove existence and uniqueness of the solution of the problem. A possible approach to identify the localization of the photon source is finally proposed.

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1. – Introduction

The interstellar space, far from being a vacuum, can be considered a "chemical laboratory" responsible of the birth of stars and galaxies. Developments in astrophysics during the last decades made this subject in the forefront of general scientific and popular interest.

The intergalactic medium is mainly composed of hydrogen (90%), about 10% of the atoms are helium, and a further 0.1% of atoms are carbon, nitrogen and oxygen. Other elements are even less abundant. Mixed with gas are "dust" grains of silicon and carbonates [1].

Matter is concentrated in big clouds (nebulae or intestellar clouds), whose dimensions are of the order of ten lights years, *i.e.* between 10^{-1} and 10 parsec (one parsec is about $3 \cdot 10^{13}$ kilometers). Note that the diameter of the solar system is of the order of 10^{-4} parsec. The numerical density of the particles inside an interstellar cloud ranges from 10^3 to 10^6 particles per cubic centimetre (Earth atmosphere density, at sea level, is approximately 10^{19} particles per cubic centimetre, whereas in the intergalactic vacuum one can find 10^0 particles/cm^3). This means that a nebula is extremely rarefied, but not so much as the intergalactic vacuum [2].

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Mathematical models in this context have been studied only in the last decade, starting from the one-dimensional case (see, for instance [3] and the references quoted therein). Also inverse procedures connected to this kind of problems represent a subject of interest because of their physical applications ([4] and therein). However, in some cases, the generalization of the model to the three-dimensional problem does not seem to be so easy, because of the difficulties related to the mathematical analysis. In particular, the study of the case of an emission nebula (like for example the Orion nebula), *i.e.* an interstellar cloud with a star inside (that can be modelled with a pointwise source, represented by a Dirac delta functional), has not been solved. From a mathematical point of view the difficulties connected to this kind of model are due to the delta-"function", which does not belong to a Banach space. The one-dimensional case has been solved, going to analyse the problem in a locally convex space, in particular in a space of distributions, and by using semigroups theory in locally convex spaces [5,6]. In this paper, we propose a method to solve the three-dimensional case, going to study the model in a space of measures and by using sun-adjoint semigroup theory. Applications of semigroup theory on measure spaces for the study of transport equation are present in the literature: see, for instance, [7,8] (Chapter XI, sect. 6) [9] and the references quoted therein.

We consider a photon transport in a homogeneous cloud, which occupies a closed and convex region of the space $V \subset \mathbb{R}^3$ (bounded by the closed regular suface ∂V and with V_i the interior of V, so that $V = V_i \cup \partial V$) with a source q (for example, a star), localized in a point $\mathbf{x} = \mathbf{x}_0$ inside the nebula. In particular, the pointwise photon source can be modelled by means of the Dirac delta-"function". The photon transport equation in the interstellar space reads as follows [3,10]:

$$(1) \qquad \frac{1}{c}\frac{\partial}{\partial t}U = -\mathbf{u}\cdot\nabla_{\mathbf{u}}U - \sigma U + \int_{S_2}\sigma_s U \mathrm{d}\mathbf{u}' + q,$$

where c is the light speed, $U = U(\mathbf{x}, \mathbf{u}, t)$ is the photon number density, so that $U(\mathbf{x}, \mathbf{u}, t)\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{u}$ represents the expected number of photons which, at time t, are in the volume element $\mathrm{d}\mathbf{x}$ centered at $\mathbf{x} \in V$ and having velocity within the solid angle $\mathrm{d}\mathbf{u}$ around the unit vector $\mathbf{u}$ belonging to the unit sphere S_2; $\sigma = \sigma(\mathbf{x}, \mathbf{u})$ and $\sigma_s = \sigma_s(\mathbf{x}, \mathbf{u})$ are the total and the scattering cross-sections, respectively, represented by two non-negative continuous functions on $V \times S_2$; $q = q(\mathbf{x}, t) = q_0(t)\delta(\mathbf{x} - \mathbf{x}_0)$ is the photon source localized in $\mathbf{x}_0 \in V_i$, with δ the three-dimensional Dirac delta functional and $q_0(t)$ a given continuous function of time.

In particular, we assume that U satisfies the so called no-reentry boundary conditions:

$$(2) \qquad U(\mathbf{x}, \mathbf{u}, t) = 0, \qquad \text{for } \mathbf{x} \in \partial V \text{ and ingoing } \mathbf{u},$$

i.e. no photons come in the nebula from outside.

In order to study the abstract version of problem (1), (2), we define the "standard" Boltzmann operator (we shall call it A). This has been studied usually as an operator acting on some L^p space. However, since the unknown function $U(\mathbf{x}, \mathbf{u}, t)$ is a distribution of photons and the source term is modelled with a Dirac delta-"function", it seems that the most natural space in which the Boltzmann operator should act is a space of measures on the set $V \times S_2 \subset \mathbb{R}^6$. In particular, we define this operator acting in a Banach space of regular Borel measures. We are able to define it indirectly as the adjoint operator of an operator B, acting on an appropriate space of continuous functions on $V \times S_2$. By means of the properties of the preadjoint operator, by using the adjoint of the semigroup generated by B, we prove existence and uniqueness of the transport equation in a suitable space of measures. Finally also an inverse procedure can be implemented, in order to localized the pointwise photon source inside the nebula.

2. – Study of the preadjoint operator

Let E be the following locally compact metric space:

$$E = \big\{(\mathbf{x}, \mathbf{u}) : \mathbf{x} \in V_i \text{ or } \mathbf{x} \in \partial V \text{ and } \mathbf{u} \neq 0, \mathbf{u} \text{ ingoing}\big\} \subset V \times S_2. \tag{3}$$

We consider two spaces of continuous functions defined as follows:

$$X = C(V \times S_2), \qquad Y = \big\{f \in X : f(\mathbf{x}, \mathbf{u}) = 0, \forall \mathbf{x} \in \partial V, \mathbf{u} \text{ outgoing}\big\} \subset X. \tag{4}$$

In particular, Y is a Banach space with the norm

$$\|f\| = \sup \big|f(\mathbf{x}, \mathbf{u})\big|, \quad \forall (\mathbf{x}, \mathbf{u}) \in E. \tag{5}$$

Remark 2.1. The space Y is the closure of the linear set of all continuous functions with compact support in the set E. Moreover, by applying the Riesz Representation Theorem, we have that the dual space Y^* of the space Y is isometrically isomorphic to the Banach space of all regular Borel measures $M(E)$ on E (which contains the Dirac delta functional). In particular, the correspondence between $M(E)$ and Y^* depends on the fact that for each linear functionals $F \in Y^*$, there exists a uniquely determined measure $\mu \in M(E)$ such that

$$F(f) = \int_E f(\mathbf{x}, \mathbf{u}) \mathrm{d}\mu(\mathbf{x}, \mathbf{u}), \, \forall f \in Y \qquad \text{and} \qquad \|f\| = \|\mu\|,$$

where

$$\|\mu\| = |\mu|(E) = \sup_{f \in Y, \|f\|=1} \int_E f(\mathbf{x}, \mathbf{u}) \mathrm{d}\mu(\mathbf{x}, \mathbf{u}),$$

with $|\mu|$ the total variation of μ [11]. ∎

Let us define the Boltzmann operator A such that

$$Af = -c\mathbf{u} \cdot \nabla_{\mathbf{u}} f - c\sigma f + c \int_{S_2} \sigma_s f \mathrm{d}\mathbf{u}' \tag{6}$$

with

$$D(A) = \big\{f \in X : Af \in X, f(\mathbf{x}, \mathbf{u}) = 0, \forall \mathbf{x} \in \partial V, \mathbf{u} \text{ ingoing}\big\}.$$

Note that $D(A)$ "contains" boundary conditions (2).

By using A, the abstract version of system (1)-(2) reads

$$\begin{cases} \dfrac{\mathrm{d}U(t)}{\mathrm{d}t} = AU(t) + q(t), \\ U(0) = U_0, \end{cases} \tag{7}$$

with U_0 a given function in X and where $U(t)$ must be considered as a function from $[0, +\infty)$ into X [3, 6].

To prove existence and uniqueness of the solution of system (7), we use a "preadjoint approach", *i.e.* we do not analyse properties of A directly, but we are going to study the properties of another operator, the preadjoint of the operator A. To this aim, let us define the operator B such that

$$Bf = c\mathbf{u} \cdot \nabla_{\mathbf{u}} f - c\sigma f + c \int_{S_2} \sigma_s f \mathrm{d}\mathbf{u}', \qquad D(B) = \{f \in Y : Bf \in Y\}. \tag{8}$$

Remark 2.2. The operator B is the preadjoint Boltzmann operator, *i.e.* it is the formal preadjoint of the operator A. In fact, its domain "contains" the adjoint boundary conditions (see (4)). ■

In particular, the operator B can be split into two parts $B = S + L$, such that

$$Sf = c\mathbf{u} \cdot \nabla_{\mathbf{u}} f - c\sigma f, \qquad D(S) = \{f \in Y : Sf \in Y\} \tag{9}$$

and

$$Lf = \int_{S_2} \sigma_s f \mathrm{d}\mathbf{u}', \qquad D(L) = Y, \qquad R(L) = X. \tag{10}$$

We want to study the properties of B ($D(S + L) = D(S)$), in order to prove that the preadjoint Boltzmann system

$$\begin{cases} \dfrac{\mathrm{d}f(t)}{\mathrm{d}t} = Bf(t), \\ f(0) = f_0, \end{cases} \tag{11}$$

has a unique solution in a suitable subset of Y, for a given f_0 in such a set.

If we extend the domain of the operator S, by defining the operator $\tilde{S}$, such that

$$\tilde{S}f = Sf, \qquad D(\tilde{S}) = \left\{f \in Y : \tilde{S}f \in X\right\}, \tag{12}$$

we can extend also the operator B, in such a way that $\tilde{B} = \tilde{S} + L$. The following lemma holds.

Lemma 2.1. If L is a bounded operator, there exists a linear subset $H \subset Y$, dense in Y, such that the operator $\tilde{B}_H$, resctriction of the operator $\tilde{B}$ on the set H, maps H into Y and it is the generator of a C_0-semigroup of operators $\{T(t); t \geq 0\}$ in Y.

Proof:
Since Y is a closed subspace of the Banach space X, S is the generator of a C_0-semigroup of operators acting on Y, $\tilde{S}$ is an operator such that its domain is dense in Y, $\tilde{S}f = Sf$, for any $f \in D(S)$ and, finally, $(\lambda I - \tilde{S})^{-1}$ exists as a bounded operator, for suitable λ (where I is the inclusion $Y \to X$ with $If = f$, for all $f \in Y$), hypotheses of Lemma 2.2 of [12] are satisfied. Hence, the lemma follows. ■

Let us now give a redefinition of B:

$$B = \tilde{B}_H, \qquad D(B) = H \subset Y. \tag{13}$$

Theorem 2.1. If L is a bounded operator, then system (11) *has a unique solution in Y, for any $f \in D(B)$.*

Proof:
The theorem follows from semigroup theory of operators [13]. ■

3. – Transport equation in the space of measures

In sect. **2**, we denoted by $M(E)$ the space of regular Borel measures on E (see (3) and Remark 2.2). The Banach space $M(E)$ is dual to the space Y. In this section, in order to use the properties proved in sect. **2**, we shall define the operator A (see (6)) in the space $M(E)$ as a restriction of the adjoint operator B^*. In fact, since the operator B, defined in (13), is unbounded, closed and densely defined, its adjoint operator B^* exists and it is closed. In particular, B^* needs not be densely defined, consequently it is not always generator of a C_0-semigroup. We use sun-adjoint semigroups theory of [14] and its notation.

Let $M^{\odot}$ be the subspace in $M(E)$, such that $M^{\odot} = \overline{D(B^*)}$. Then the operator

$$A : D(A) \subset M^{\odot} \to M^{\odot}, \qquad A\mu = B^*\mu, \tag{14}$$

with

$$D(A) = \left\{\mu \in D(B^*) : B^*\mu \subset M^{\odot}\right\} \subset M^{\odot}, \tag{15}$$

is a restriction of B^*, adjoint operator of the operator B, to the subspace $M^{\odot}$.

Lemma 3.1. If L is bounded operator, the operator A defined in (14)*, acting in the subspace $M^{\odot}$ of the space of measures on E, is the generator of a C_0-semigroup $\{Z(t); t \geq 0\}$ of positive operators.*

Proof:
Since the operator A is densely defined in $M^{\odot}$, the theorem follows from sun-adjoint semigroups theory in [14]. ■

Remark 3.1. The semigroup $\{Z(t); t \geq 0\}$ generated by A is connected to the semigroup $\{T(t); t \geq 0\}$ generated by B as follows:

$$Z(t)\mu = T^*(t)\mu, \qquad \forall \mu \in M^*, \tag{16}$$

where $\{T^*(t); t \geq 0\}$ is the adjoint semigroup of $Z(t)$. The subspace $M^{\odot}$ is exactly the set of measures $\mu \in M(S)$, for which the family is strongly continuous at $t = 0$. ■

Theorem 3.1. System (7)*, with A defined in* (14)*, is uniquely solvable, for any initial condition $U_0 \in D(A)$.*

Proof:
Since q is continuous in $t \geq 0$ and locally Lipschitz continuous in $\mathbf{x}$, uniformly on bounded intervals, there is a $T_{\max} \leq \infty$ such that system (7) admits a unique mild solution on $[0, T_{\max}[$, for any $U_0 \in D(A)$ (see Theorem 1.4, Chapter 6 of [15] on standard arguments of semigroups theory, and Theorem 4.3.11 of [14] on perturbation theory of sun-adjoint semigroups). ■

Hence, thanks to an indirect approach, which consists on defining A (acting in a suitable Banach space of regular Borel measures) as the adjoint operator of its preadjoint operator B (acting on an appropriate space of continuous functions on $V \times S_2$), we are able to prove existence and uniqueness of the solution of the initial system (1).

Remark 3.2. Assume that a suitable instrument (for instance, a radio-telescope) is located at $\hat{\mathbf{x}} \in \mathbb{R}^3 - V$ (where $\hat{\mathbf{x}} \in \mathbb{R}^3 - V$ is "far" from the cloud) and that it measures the photon flux arriving from the source localized at $\mathbf{x}_0 \in V$. This means that such an instrument detects photons passing through $\hat{\mathbf{x}}$ and having velocity $c\hat{\mathbf{u}}$, with $\hat{\mathbf{u}} = (\hat{\mathbf{x}} - \mathbf{x}_0)/|\hat{\mathbf{x}} - \mathbf{x}_0|$. It is possible to perform a procedure (inverse problem in trasport theory [16]), which permits to identify the position $\mathbf{x}_0$ of the photon source [17,18]. Also a numerical approach similar to that proposed in [4] can be given. ■

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REFERENCES

[1] Dyson J. E. and Williams D. A., *The Physics of the Interstellar Medium* (Institute of Physics Publishing, Bristol) 1997.
[2] Spitzer L., *Physical Processes in the Interstellar Medium* (John Wiley & and Sons, Toronto) 1978.
[3] Belleni Morante A., Lamb W. and McBride A. C., *Anal. Appl.*, **5** (2007) 77.
[4] Belleni Morante A., Monaco R., Riganti R. and Salvarani F., *Appl. Math. Comput.*, **154** (2004) 115.
[5] Choe Y. H., *J. Math. Anal. Appl.*, **16** (1985) 293.
[6] Lisi M. and Totaro S., *Math. Meth. Appl. Sci.*, **29** (2006) 1019.
[7] Beals R. and Protopopescu V., *J. Math. Anal. Appl.*, **121** (1987) 370.
[8] Greenberg W., van der Mee C. and Protopopescu V., *Boundary Value Problems in Abstract Kinetic Theory* (Birkhäuser, Basel) 1987.
[9] Nelson P. jr. and Victory H. D., *Ann. Mat. Pura Appl.*, **112** (1977) 69.
[10] Pomraning G. C., *The Equation of Radiation Hydrodynamics* (Pergamon Press, Oxford) 1973.
[11] Rudin W., *Functional Analysis* (McGraw Hill, New York) 1973.
[12] Suhadolc A. and Vidav I., *Math. Balk.*, **3** (1973) 514.
[13] Belleni Morante A., *Applied Semigroups and Evolution Equations* (Oxford University Press, Oxford) 1979.
[14] Van Neerven J. M. A. M., *The Adjoint of a Semigroup of Linear Operators* (Springer Verlag, Berlin Heidelberg) 1992.
[15] Pazy A., *Semigroups of Linear Operators and Applications to Partial Differential Equations* (Springer Verlag, New York) 1983.
[16] Zweifel P. F., *Transp. Theory Stat. Phys.*, **28** (1999) 171.
[17] Belleni Morante A., *Transp. Theory Stat. Phys.*, **32** (2003) 73.
[18] Lisi M. and Totaro S., *Transp. Theory Stat. Phys.*, **32** (2003) 327.