IL NUOVO CIMENTO Vol. 33 C, N. 1 Gennaio-Febbraio 2010
DOI 10.1393/ncc/i2010-10567-3

Colloquia: ICTT2009

Interaction of a glow discharge with an ion beam

B. Lucquin-Desreux
Lab. J.-L. Lions, CNRS UMR 7598, Université Pierre et Marie Curie
4 Place Jussieu, 75252 Paris Cedex 05, France

(ricevuto il 23 Ottobre 2009; approvato il 15 Gennaio 2010; pubblicato online il 5 Marzo 2010)

Summary. — The aim is to derive a fluid model, of hydrodynamic/diffusion type, describing the interaction of a glow discharge with an ion beam at atmospheric pressure.

PACS 52.40.Mj – Particle beam interactions in plasmas.
PACS 41.75.Ak – Positive-ion beams.
PACS 52.20.Hv – Atomic, molecular, ion, and heavy-particle collisions.
PACS 52.80.-s – Electric discharges.

1. – Introduction

We intend to continue the study of macroscopic models for partially ionized plasmas at atmospherique pressure, that we started in [1] and completed in [2,3], covering a wide range of discharges, such as thermal arc discharges, glow discharges, etc. We refer to the above references for a precise description of the physical context. The aim here is to modelize the interaction of an ion beam with a glow discharge.

2. – The scaled kinetic system

2˙1. *The kinetic collisional model.* – The plasma we consider is a mixture made of electrons (e), ions (i) and neutrals (n), which interact all together through elastic and inelastic collisions. Denoting by f_α ($\alpha = e, i, n$) the distribution function of the α species, the kinetic collisional system modelling this mixture is given by [1]

$$\partial_t f_\alpha + v_\alpha \cdot \nabla_x f_\alpha + \frac{F_\alpha}{m_\alpha} \cdot \nabla_{v_\alpha} f_\alpha = (\partial_t f_\alpha)_c, \tag{1}$$

where m_α is the mass of the α species and F_α the force term; moreover, we have

$$(\partial_t f_\alpha)_c = Q_{\alpha\alpha}(f_\alpha, f_\alpha) + Q_{\alpha\beta}(f_\alpha, f_\beta) + Q_{\alpha\gamma}(f_\alpha, f_\gamma) + Q_{\alpha,ir}(f_\alpha, f_\beta, f_\gamma),$$

where $\alpha, \beta, \gamma = e, i, n$ with $\alpha \neq \beta \neq \gamma \neq \alpha$. The three first terms represent the elastic collisions of a particle of the α species with the particles of the three species; the operators are classical Landau-Fokker-Planck operators if the two species involved in the collisional process are both charged particles and is of Boltzmann type otherwise [4]. The ionization-recombination operators are of the following form (see [1] for details):

$$(2a) \quad Q_{e,ir}(f_e, f_i, f_n)(v_e) = \int_{I\!R^{12}} \sigma^r \delta_v \delta_{\mathcal{E}} \left(f_{e'} f_{e^\star} f_i - \mathcal{F}_0 f_e f_n \right) \mathrm{d}v_e{}' \mathrm{d}v_e^\star \mathrm{d}v_i \mathrm{d}v_n$$
$$+2 \int_{I\!R^{12}} \sigma^{r\prime} \delta_{v'} \delta_{\mathcal{E}'} \left(\mathcal{F}_0 f_{e'} f_n - f_e f_{e^\star} f_i \right) \mathrm{d}v_e{}' \mathrm{d}v_e^\star \mathrm{d}v_i \mathrm{d}v_n,$$

$$(2b) \quad Q_{i,ir}(f_e, f_i, f_n)(v_i) = \int_{I\!R^{12}} \sigma^r \, \delta_v \, \delta_{\mathcal{E}} \left(\mathcal{F}_0 \, f_e \, f_n - f_{e'} \, f_{e^\star} \, f_i \right) \mathrm{d}v_e \, \mathrm{d}v_e{}' \, \mathrm{d}v_e^\star \, \mathrm{d}v_n,$$

$$(2c) \quad Q_{n,ir}(f_e, f_i, f_n)(v_n) = \int_{I\!R^{12}} \sigma^r \, \delta_v \, \delta_{\mathcal{E}} \left(f_{e'} \, f_{e^\star} \, f_i - \mathcal{F}_0 \, f_e \, f_n \right) \mathrm{d}v_e \, \mathrm{d}v_e{}' \, \mathrm{d}v_e^\star \mathrm{d}v_i.$$

In particular, $\mathcal{F}_0$ is a positive constant, which represents the efficiency of the dissociation with respect to the recombination and the notations $\delta_{\mathcal{E}}$ and δ_v hold for the energy and momentum conservation during this inelastic collisional process, *i.e.*

$$\delta_{\mathcal{E}} = \delta\Big(m_e |v_e|^2 + m_n |v_n|^2 - [m_e(|v_e{}'|^2 + |v_e^\star|^2) + m_i |v_i|^2 + 2\Delta] \Big),$$
$$\delta_v = \delta\Big(m_e v_e + m_n v_n - [m_e(v_e{}' + v_e^\star) + m_i v_i] \Big),$$

where δ denotes the Dirac measure, and Δ the ionization energy.

2˙2. *The diffusion scaling.* – Let us first describe the physical scalings. We first have a small parameter ε representing the mass ratio defined by

$$\varepsilon = \sqrt{\frac{m_e}{m_n}} = \sqrt{\frac{m_e}{m_i + m_e}} \ll 1.$$

We suppose that the temperature of the three species are of the same order of magnitude T_0, and define the characteristic velocities $(v_\alpha)_0$ as the respective thermal velocities, *i.e.*

$$(v_\alpha)_0 = \sqrt{\frac{k_B T_0}{m_\alpha}}, \qquad \text{for} \quad \alpha = e, i, n,$$

k_B being the Boltzmann constant. Consequently, these velocities only depend on the masses, and more precisely we have: $(v_n)_0 = \sqrt{1 - \varepsilon^2}\,(v_i)_0 = \varepsilon\,(v_e)_0$. We denote by δ_e and δ_i the two small paremeters defined by

$$\delta_e = \frac{(\rho_e)_0}{(\rho_n)_0}, \quad \delta_i = \frac{(\rho_i)_0}{(\rho_n)_0}.$$

For a glow discharge, we typically have $\delta_e = \delta_i = \varepsilon^2$. Here, in order to modelize the interaction of an ion beam with a glow discharge, we set: $\delta_e = \varepsilon^2$ and $\delta_i = \varepsilon$. We choose $x_0 = t_0\,(v_e)_0$ as reference length, the reference time t_0 being the smallest time scale defined by: $t_0 = \tau_{en} = \delta_i \tau_{ee} = \delta_i \tau_{ir}$ ($\tau_{\alpha\beta}$ denoting the characteristic collision time of a particle of the α species againts the β one and τ_{ir} the characteristic time of inelastic collisions). Last, we choose a force field unit F_0 such that $F_0 x_0 = k_B T_0$. At the electronic diffusion scale ($t \to \varepsilon^2 t, x \to \varepsilon x$), the dimensionless kinetic system then writes

$$\partial_t f_e + \frac{1}{\varepsilon}\Big(v_e \cdot \nabla_x + F_e \cdot \nabla_{v_e}\Big) f_e = \frac{1}{\varepsilon^2}\left[Q^{\varepsilon}_{en}(f_e, f_n) + Q^{\varepsilon}_{ei}(f_e, f_i)\right] + \frac{1}{\varepsilon}\left[Q^{\varepsilon}_{e,ir}(f_e, f_i, f_n) + Q_{ee}(f_e, f_e)\right], \tag{3a}$$

$$\partial_t f_i + \frac{1}{\sqrt{1-\varepsilon^2}}\Big(v_i \cdot \nabla_x + F_i \cdot \nabla_{v_i}\Big) f_i = \frac{1}{\varepsilon\sqrt{1-\varepsilon^2}}\left[Q^{\varepsilon}_{in}(f_i, f_n) + Q_{ii}(f_i, f_i)\right] + Q^{\varepsilon}_{i,ir}(f_e, f_i, f_n) + Q^{\varepsilon}_{ie}(f_i, f_e), \tag{3b}$$

$$\Big(\partial_t + v_n \cdot \nabla_x + F_n \cdot \nabla_{v_n}\Big) f_n = \frac{1}{\varepsilon} Q_{nn}(f_n, f_n) + \frac{1}{\sqrt{1-\varepsilon^2}} Q^{\varepsilon}_{ni}(f_n, f_i) + \varepsilon\left[Q^{\varepsilon}_{n,ir}(f_e, f_i, f_n) + Q^{\varepsilon}_{ne}(f_n, f_e)\right], \tag{3c}$$

where all the collision operators, except the intra species one, depend on ε.

3. – The coupled fluid model for the mixture

We use a classical Hilbert method by expanding each distribution function f^{α} in terms of ε ($f_\alpha = f^0_\alpha + \varepsilon f^1_\alpha + \ldots$), but also each collision operator (see Lemma A.1) and identifying terms of equal power in each side of the equations in the above kinetic system.

3$\cdot$1. *The equilibrium states.* – We start with the identification of the lowest-order terms, in order to derive the three equilibrium states. We find in fact

$$Q^0_{en}(f^0_e, f^0_n) + Q^0_{ei}(f^0_e, f^0_i) = 0, \tag{4a}$$

$$Q^0_{in}(f^0_i, f^0_n) + Q_{ii}(f^0_i, f^0_i) = 0, \tag{4b}$$

$$Q_{nn}(f^0_n, f^0_n) = 0. \tag{4c}$$

We get from (4c) that $f^0_n = \rho_n M_{u,T}$ is a classical Maxwellian of density ρ_n, while we easily deduce from (4b) that $f^0_i = \rho_i M_{u,T}$ is also a Maxwellian that shares with f^0_n the same mean velocity u and mean temperature T. Last, we get from (4a) and the properties of the operator L_e defined by $L_e = Q^0_{en}(\cdot, f^0_n) + Q^0_{ei}(\cdot, f^0_i)$ (see Lemma A.1) that f^0_e is an isotropic function (that will be precised in the next section).

3$\cdot$2. *The electronic order-one correction.* – We go on with the identification of the next terms, in order to compute the order-one corrections. Concerning the electrons, we get

$$L_e f^1_e = S^1_e, \quad \text{with} \tag{5a}$$

$$S^1_e = (v_e \cdot \nabla_x + F_e \cdot \nabla_{v_e}) f^0_e - Q^1_{en}(f^0_e, f^0_n) - Q^1_{ei}(f^0_e, f^0_i) - Q_{ee}(f^0_e, f^0_e) - Q^0_{e,ir}(f^0_e, f^0_i, f^0_n) \tag{5b}$$

The solvability condition for this equation writes (see Lemma A.1): for all $W > 0$ (W represents the energy $|v|^2/2$), the integral of S^1_e on the sphere of radius W is zero, *i.e.* $\int_{S_W} S_e^1 \, \mathrm{d}N(v) = 0$. As the first four terms in S_e^1 are odd [4], this reduces to

$$\text{for all } W > 0, \quad \int_{S_W} [Q_{ee}(f_e^0, f_e^0) + Q^0_{e,ir}(f_e^0, f_i^0, f_n^0)] \, \mathrm{d}N(v) = 0.$$

Integrating against the isotropic function $H_e = \log(\rho_i f_e^0/(\rho_n \mathcal{F}_0))$, we get

$$\int_{\mathbb{R}^3} Q_{ee}(f_e^0, f_e^0) \, H_e \, \mathrm{d}v_e + \int_{\mathbb{R}^3} Q^0_{e,ir}(f_e^0, f_i^0, f_n^0) \, H_e \, \mathrm{d}v_e = 0.$$

But as each one of these two integrals is non-positive (this is classical for the first one [5]; we refer to Lemma A.2 for the second one), this means that they are both zero. We deduce that f_e^0 is a centered Maxwellian $f_e^0 = \rho_e M_{0,T_e}$, with ρ_e given by the following Saha law:

$$\rho_e = \frac{\mathcal{F}_0 \, \rho_n}{\rho_i} \, (2\pi T_e)^{3/2} \, \exp\Big[-\frac{\Delta}{T_e}\Big]. \tag{6}$$

Moreover, it is possible to compute explicitly the order-one correction f_e^1; this is due to the fact that the operator L_e is of Lorentz type, like in [6]. We have the following expression, which is a generalization of Lemma 7.1 of [2]: $f_e^1 = f_e^{1,o} + f_e^{1,e}$, where $f_e^{1,e}$ is an arbitrary isotropic function while $f_e^{1,o} = f_e^0 v_e \cdot \phi_e^1$, with ϕ_e^1 given by (α_n and α_i are both isotropic functions entirely given in terms of the Boltzman and Fokker-Planck kernels in Q^0_{en} and Q^0_{in})

$$\phi_e^1(v_e) = \frac{u}{T_e} - \frac{1}{2[\alpha_n(|v_e|)\rho_n + \alpha_i(|v_e|)\rho_i]} \left[\nabla_x\Big(\frac{\mu_e}{T_e}\Big) - \frac{F_e}{T_e} + \frac{|v_e|^2}{2} \nabla_x\Big(-\frac{1}{T_e}\Big)\right]. \tag{7}$$

3·3. *The fluid system.* – Let us go on with the Hilbert method, looking for the order-one correction f_n^1. This one satisfies the equation: $2Q_{nn}(f_n^1, f_n^0) = (\partial_t + v_n \cdot \nabla_x + F_n \cdot \nabla_{v_n}) f_n^0$. The solvability condition for this equation gives a classical Euler system for (ρ_n, u, T) which does not depend on the other species (see system (14) below). Concerning the ions, we find that f_i^1 satisfies the equation: $L_i(f_i^1) = S_i^1$, where L_i is the linear operator defined by $L_i = Q^0_{in}(\cdot, f_n^0) + 2Q_{ii}(\cdot, f_i^0)$ and S_i^1 is given by

$$S_i^1 = (\partial_t + v_i \cdot \nabla_x + F_i \cdot \nabla_{v_i}) f_i^0 - Q^0_{in}(f_i^0, f_n^1).$$

We have in fact $Q_{ie}(\cdot, f_e^0) = 0$ because f_e^0 is isotropic and $Q^0_{i,ir}(f_e^0, f_i^0, f_n^0) = 0$, on account of the Saha law. It is easy to check that the kernel of the operator L_i is the straight line generated by f_i^0, so that the solvability conditions reduce to one equation on the ionic density ρ_i, with no source term, because the operator Q^0_{in} conserves mass (see eq. (15) below). We finally go on, looking for the order-two correction f_e^2. We find the following equation: $L_e f_e^2 = S_e^2$, with

$$\begin{aligned} S_e^2 = \partial_t f_e^0 + [v_e \cdot \nabla_x + F_e \cdot \nabla_{v_e}] f_e^1 - Q^0_{en}(f_e^1, f_n^1) - Q^1_{en}(f_e^1, f_n^0) - Q^1_{en}(f_e^0, f_n^1) - Q^2_{en}(f_e^0, f_n^0) \\ - Q^0_{ei}(f_e^1, f_i^1) - Q^1_{ei}(f_e^1, f_i^0) - Q^1_{ei}(f_e^0, f_i^1) - Q^2_{ei}(f_e^0, f_i^0) - 2Q_{ee}(f_e^1, f_e^0) - S^1_{e,ir}, \end{aligned}$$

where $S^1_{e,ir}$ is the order-one term in the asymptotic expansion of $Q_{i,ir}(f_e^\varepsilon, f_i^\varepsilon, f_n^\varepsilon)$ in terms of ε. The solvability condition for this equation writes: $\forall W > 0$, $\int_{S_W} S_e^2(v) \, \mathrm{d}N(v) = 0$.

As $Q^1_{en}(f_e^{1,e}, f_n^0)$, $Q^1_{ei}(f_e^{1,e}, f_i^0)$ and $Q_{ee}(f_e^{1,o}, f_0^e)$ are odd functions of v (and $Q^0_{en}(f_e^{1,e}, \cdot) = Q^0_{en}(f_e^{1,e}, \cdot) = 0$), it remains:

$$\text{(8)}\quad \forall W > 0, \quad \int_{S_W} \tilde{S}_e^2(v)\, \mathrm{d}N(v) = 0, \quad \text{with}$$

$$\tilde{S}_e^2 = \partial_t f_e^0 + (v_e \cdot \nabla_x - E \cdot \nabla_{v_e}) f_e^{1,o} - Q^0_{en}(f_e^{1,o}, f_n^1) - Q^1_{en}(f_e^{1,o}, f_n^0) - Q^1_{en}(f_e^0, f_n^1)$$
$$-Q^2_{en}(f_e^0, f_n^0) - Q^0_{ei}(f_e^{1,o}, f_i^1) - Q^1_{ei}(f_e^{1,o}, f_i^0) - Q^1_{ei}(f_e^0, f_i^1) - Q^2_{ei}(f_e^0, f_i^0)$$
$$-2Q_{ee}(f_e^{1,e}, f_e^0) - S^1_{e,ir}.$$

Remarking that $f_e^{1,e}$ only appears in $2Q_{ee}(f_e^{1,e}, f_e^0) = \frac{\mathrm{d}Q_{ee}}{\mathrm{d}f_e}(f_e^0)\,(f_e^{1,e})$, we deduce that condition (8) entirely determines $f_e^{1,e}(W)$, for any W. Now, multiplying (8) by 1, W, and integrating with respect to W, we get in particular

$$\text{(9)}\qquad \int_{I\!R^3} \tilde{S}_2^e(v_e)\, \mathrm{d}v_e = 0, \quad \int_{I\!R^3} \tilde{S}_2^e(v_e)\, \frac{|v_e|^2}{2}\, \mathrm{d}v_e = 0.$$

From the definition of Q^0_{en} (see Lemma A.1) and the properties of $q_e^{\mathcal{B}}$ (self-adjointness, kernel made of isotropic functions, see [4]), we have, for any f and g (and setting $C_g = (\int_{I\!R^3} g(v) \mathrm{d}v)$):

$$\int_{I\!R^3} Q^0_{en}(f, g) \begin{pmatrix} 1 \\ \frac{1}{2}|v_e|^2 \end{pmatrix} \mathrm{d}v_e = C_g \int_{I\!R^3} f(v_e) \begin{pmatrix} q_e^{\mathcal{B}}(1) \\ q_e^{\mathcal{B}}(\frac{1}{2}|v_e|^2) \end{pmatrix} \mathrm{d}v_e = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

In the same way, as $Q^1_{en}(f_e^0, f_n^1) = q_e^{\mathcal{B}}(\nabla f_e^0) \int_{I\!R^3} v_n f_n^1 \mathrm{d}v_n$, we also have

$$\int_{I\!R^3} Q^1_{en}(f_e^0, f_n^1) \begin{pmatrix} 1 \\ \frac{1}{2}|v_e|^2 \end{pmatrix} \mathrm{d}v_e = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

These two properties are also valid for Q^0_{ei} and $Q^1_{ei}(f_e^0, f_i^1)$. From [7], we get

$$\int_{I\!R^3} [Q^1_{en}(f_e^{1,o}, f_n^0) + Q^2_{en}(f_e^0, f_n^0) + Q^1_{ei}(f_e^{1,o}, f_i^0) + Q^2_{ei}(f_e^0, f_i^0)] \begin{pmatrix} 1 \\ \frac{1}{2}|v_e|^2 \end{pmatrix} \mathrm{d}v_e = \begin{pmatrix} 0 \\ U_e \end{pmatrix},$$

where we have set (λ_n and λ_i are two coefficients which can be explicitly computed in terms of the Boltzmann and Fokker-Planck kernels in Q^0_{en} and Q^0_{ei}, see [3, 4, 7]):

$$\text{(10)}\qquad U_e = 2u \cdot [\nabla_x(\rho_e T_e) - \rho_e F_e] + 3\rho_e(\lambda_n \rho_n + \lambda_i \rho_i) \frac{T - T_e}{T_e}.$$

As the linearized Fokker-Planck operator classicaly conserves mass and energy, we have

$$\int_{I\!R^3} Q_{ee}(f_e^{1,e}, f_e^0) \begin{pmatrix} 1 \\ \frac{1}{2}|v_e|^2 \end{pmatrix} \mathrm{d}v_e = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Now, setting $R_e = \int_{I\!R^3} S^1_{e,ir}\, \mathrm{d}v_e$, we get: $\int_{I\!R^3} S^1_{e,ir} |v_e|^2/2\, \mathrm{d}v_e = -\Delta R_e$ (see Lemma A.2). Last, concerning the transport terms, we refer to the computations already done in [7, 8, 1, 6, 2, 3]. We then derive from (9) the following energy-transport model:

$$\partial_t \rho_e + \mathrm{div}\big(\rho_e(u+u_J)\big) = R_e, \tag{11a}$$

$$\partial_t\big(\tfrac{3}{2}\rho_e T_e\big) + \mathrm{div}\Big[\tfrac{5}{2}\rho_e u T_e + \rho_e v_J\Big] - \rho_e(u+u_J)\cdot F_e = S_e \tag{11b}$$

with $S_e = U_e - \Delta R_e$ and

$$\begin{pmatrix} u_J \\ v_J \end{pmatrix} = -D \begin{pmatrix} \nabla_x\Big(\frac{\mu_e}{T_e}\Big) - \frac{F_e}{T_e} \\ \nabla_x\Big(-\frac{1}{T_e}\Big) \end{pmatrix}, \tag{12}$$

the entries D_{kl} of the diffusion matrix D being given by

$$D_{kl} = D_{lk} = \frac{1}{6 \times 2^{k+l-2}} \int_{\mathbb{R}^3} \frac{|v|^{2(k+l-1)}}{\rho_n \alpha_n(|v|) + \rho_i \alpha_i(|v|)} M_{0,T_e}(v)\, \mathrm{d}v. \tag{13}$$

Note that R_e in fact depends on the isotropic part $f_e^{1,e}$ that appears in $S^1_{e,ir}$. But, thanks to the expression of S_e, it is possible to eliminate R_e between the two equations in (11); from a physical point of view, this corresponds to the fact that the energy which is conserved by the ionization is $|v|^2/2 + \Delta$. We then get the following fluid model for the mixture:

Theorem: *The equilibrium states for the heavy species $f_n^0 = \rho_n M_{u,T}$ and $f_i^0 = \rho_i M_{u,T}$ are characterized by the same mean velocity u and temperature T; the electronic distribution function $f_e^0 = \rho_e M_{0,T_e}$ is a centered Maxwellian, whose density ρ_e is given in terms of the two other densities ρ_n and ρ_i, and the electronic temperature T_e, by the Saha law* (6). *The neutral particles satisfy the standard Euler system (setting $W_n = \frac{1}{2}\rho_n|u|^2 + \frac{3}{2}\rho_n T$):*

$$\partial_t \rho_n + \mathrm{div}\big(\rho_n u\big) = 0, \tag{14a}$$

$$\partial_t\big(\rho_n u\big) + \mathrm{div}\left[\rho_n\,(u \otimes u)\right] + \nabla_x\,(\rho_n T) - \rho_n F_n = 0, \tag{14b}$$

$$\partial_t W_n + \mathrm{div}\Big[u\,\big(W_n + \rho_n T\big)\Big] - \rho_n u \cdot F_n = 0, \tag{14c}$$

which is totally independent of the other species. The fluid model for ions reduces to a mass conservation equation:

$$\partial_t \rho_i + \mathrm{div}(\rho_i u) = 0. \tag{15}$$

The electronic macroscopic quantities (ρ_e, T_e) satisfy the following diffusion equation:

$$\partial_t\big(\frac{3}{2}\rho_e T_e + \Delta\rho_e\big) + \mathrm{div}(j_{T_e} + \Delta j_{\rho_e}) - j_{\rho_e}\cdot F_e = U_e, \tag{16}$$

where we have set, for simplicity: $j_{\rho_e} = \rho_e(u+u_J)$, $j_{T_e} = \frac{5}{2}\rho_e u T_e + \rho_e v_J$, with u_J and v_J defined by (12), (13) *and U_e by* (10).

4. – Conclusion and comments

Equation (15) corresponds to the charge conservation, while eq. (16) represents the balance sheet of the electronic energy. The coupling between the three species appears through the Saha law and the transport coefficients in eq. (16).

The model we have here derived looks like the fluid model obtained in [3]. We first mention however that the derivation from the kinetic system is far different here, essentially on account of the orderings of the inelastic collisions. Secondly, the model we have obtained differs by the expression of the transport coefficients which are here simpler and can be explicitly computed; this is due to the fact that the operator L_e is here a Lorentz-type operator, like in [6]. In [3], the diffusion coefficients are more complex, because they depend on the ionization, which is stronger than in the case under consideration here.

Moreover, let us note that the model we have here obtained directly from the kinetic system can also be recovered from a glow plasma model, of SHE type, in which the Saha law appears as a source term in the equation on the ionic density (see Proposition 4 of [2]), by multiplying this relaxation term by a factor $1/\varepsilon$ and looking for the limit $\varepsilon \to 0$.

Let us finally mention that the derivation we have here obtained is purely formal. From a mathematical point of view, one main difficulty comes from the strong coupling between the neutrals and the other species and the fact that the Euler system for the neutrals can develop solutions with shocks.

Appendix A.

In the two Lemmas below, we only recall, for simplicity, the first-order terms of the asymptotic expansion of the collisional operators in terms of ε and their main properties. For the other terms, we refer to previous references.

Lemma A.1 [4,6,9]: i) *Let* α, $\beta = i$, n, $\alpha \neq \beta$; *then* $Q^\varepsilon_{\alpha\beta}(f_\alpha, f_\beta) = Q^0_{\alpha\beta}(f_\alpha, f_\beta) + \mathcal{O}(\varepsilon^2)$, *with*

$$Q^0_{\alpha\beta}(f_\alpha, f_\beta)(v_\alpha) = \int_{I\!R^3\times S^2} B^{\mathcal{B}}_\star \left(v_\alpha - v^\star_\beta, \Omega\right) \left(f_{\alpha'} f_{\beta'_\star} - f_\alpha f_{\beta_\star}\right) \mathrm{d}v^\star_\beta \, \mathrm{d}\Omega.$$

ii) *Let* α, $\beta = e$, n, $\alpha \neq \beta$; *then* $Q^\varepsilon_{\alpha\beta}(f_\alpha, f_\beta) = Q^0_{\alpha\beta}(f_\alpha, f_\beta) + \mathcal{O}(\varepsilon)$, *with*

$$Q^0_{en}(f_e, f_n)(v_e) = q^{\mathcal{B}}_e(f_e)(v_e) \int_{I\!R^3} f_n(v_n)\mathrm{d}v_n,$$

$$Q^0_{ne}(f_n, f_e)(v_n) = -2\,\nabla_{v_n} f_n(v_n) \cdot \int_{I\!R^3\times S^2} B^{\mathcal{B}}(v_e, \Omega)\, \frac{(v_e\cdot\Omega)^2}{|v_e|^2}\, v_e\, f_e(v_e)\, \mathrm{d}v_e\, \mathrm{d}\Omega.$$

Moreover, $q^{\mathcal{B}}_e(f_e)(v_e) = \int_{S^2} B^{\mathcal{B}}(v_e, \Omega)[f_e(v_e - 2(v_e, \Omega)\Omega) - f_e(v_e)]\mathrm{d}\Omega$.

iii) $Q^\varepsilon_{ei}(f_e, f_i) = Q^0_{ei}(f_e, f_i) + \mathcal{O}(\varepsilon)$, $Q^\varepsilon_{ie}(f_i, f_e) = Q^0_{ie}(f_i, f_e) + \mathcal{O}(\varepsilon)$, *with*

$$Q^0_{ei}(f_e, f_i)(v_e) = q^{\mathcal{F}}_e(f_e)(v_e) \int_{I\!R^3} f_i(v_i)\mathrm{d}v_i, \quad q^{\mathcal{F}}_e(f_e) = \nabla_{v_e}\cdot [B^{\mathcal{F}}\, S\, \nabla_{v_e} f_e],$$

$$Q^0_{ie}(f_i, f_e)(v_i) = -2\,\nabla_{v_i} f_i(v_i) \cdot \int_{I\!R^3} \frac{B^{\mathcal{F}}(v_e)}{|v_e|^2}\, v_e\, f_e(v_e)\, \mathrm{d}v_e.$$

iv) *For any* f_n, f_i, *the kernel of the operators* $L_e = Q^0_{en}(\cdot, f_n) + Q^0_{in}(\cdot, f_i)$ *is made of isotropic functions. Introducing the energy variable* $W(v) = |v|^2/2$, *the sphere* $S_W = \{v \in I\!R^3, W(v) = W\}$, *and* $\mathrm{d}N(v) = \frac{\mathrm{d}S_W(v)}{\sqrt{2W}}$ *(*$\mathrm{d}S_W$ *is the Euclidean surface element on* S_W*), we get from the co-area formula that the equation* $L_e f = g$ *has a solution if and only if the right-hand side satisfies the following orthogonality relation:*

$$\forall W > 0, \quad \int_{S_W} g(v)\,\mathrm{d}N(v) = 0.$$

Lemma A.2 [1-3]: i) *We set:* $\rho_\alpha = \int_{I\!R^3} f_\alpha(v_\alpha)\mathrm{d}v_\alpha$, $\alpha \in \{e, i, n\}$; *then* $Q_{\alpha,ir} = Q^0_{\alpha,ir} + R^1_{\alpha,ir} + O(\varepsilon^2)$, *where* $Q^0_{\alpha,ir}$ *is given by* (2)*, but with* δ_v, $\delta_{\mathcal{E}}$ *replaced by their limit when* ε *goes to zero, i.e. by* $\delta^0_v = \delta(v_i - v_n)$, $\delta^0_{\mathcal{E}} = \delta(|v_e|^2 - [|v_e{}'|^2 + |v_e^\star|^2 + 2\Delta])$. *Moreover*

$$Q^0_{i,ir}\Big(f_e, f_i, f_n\Big)(v) = -Q^0_{n,ir}\Big(f_e, f_i, f_n\Big)(v) = A_1(f_e)\, f_n(v) - A(f_e)\, f_i(v), \textit{with}$$
$$A_1(f_e) = \textstyle\int_{I\!R^9} \sigma^r \delta^0_{\mathcal{E}} \mathcal{F}_0\, f_e(v_e)\,\mathrm{d}v_e \mathrm{d}v_e{}'\mathrm{d}v_e^\star, \quad A(f_e) = \int_{I\!R^9} \sigma^r \delta^0_{\mathcal{E}} f_e(v_e{}') f_e(v_e^\star)\,\mathrm{d}v_e\mathrm{d}v_e{}'\mathrm{d}v_e^\star.$$

ii) *The operator* $Q^0_{e,ir}(f_e, f_i, f_n)$ *satisfies*

$$\int_{I\!R^3} Q^0_{e,ir}\Big(f_e, f_i, f_n\Big)(v_e)\begin{pmatrix} 1 \\ \frac{1}{2}|v_e|^2 \end{pmatrix}\mathrm{d}v_e = [\rho_n A_1(f_e) - \rho_i A(f_e)]\begin{pmatrix} 1 \\ -\Delta \end{pmatrix}.$$

iii) *Moreover, let* $H(f_e) = \log(\mathcal{F}_0^{-1}\, \rho_n^{-1}\, \rho_i\, f_e)$, *and let* σ^r *be positive, then*

$$\int_{I\!R^3} Q^0_{e,ir}\Big(f_e, f_i, f_n\Big)(v_e) H\Big(f_e\Big)(v_e)\mathrm{d}v_e = -\rho_i \int_{I\!R^9} \sigma^r \delta^0_{\mathcal{E}}\Big[\mathcal{F}_0 \frac{\rho_n}{\rho_i} f_e(v_e) - f_e(v_e{}') f_e(v_e^\star)\Big]$$
$$\times\Big[\log\Big(\mathcal{F}_0 \frac{\rho_n}{\rho_i} f_e(v_e)\Big) - \log\Big(f_e(v_e{}')\, f_e(v_e^\star)\Big)\Big]\,\mathrm{d}v_e\mathrm{d}v_e{}'\mathrm{d}v_e^\star \leq 0.$$

If f_e *is isotropic and* $Q^0_{e,ir}(f_e, f_i, f_n) = 0$, *then:* $f_e = \rho_e M_{0,T_e}$, *with* $\rho_e = \frac{\mathcal{F}_0\,\rho_n}{\rho_i}(2\pi T_e)^{3/2}\exp[-\frac{\Delta}{T_e}]$; *moreover,* $\int_{I\!R^3} S^1_{e,ir}(\Delta + |v|^2/2)\,\mathrm{d}v_e = 0$.

REFERENCES

[1] Choquet I. and Lucquin-Desreux B., *J. Stat. Phys.*, **119** (2005) 197.

[2] Choquet I., Degond P. and Lucquin-Desreux B., *Discret. Contin. Dyn. Syst.-Ser. B*, **8** (2007) 735.

[3] Choquet I., Degond P. and Lucquin-Desreux B., *Math. Comput. Model.*, **49** (2009) 88.

[4] Degond P. and Lucquin-Desreux B., *Math. Mod. Meth. Appl. Sci.*, **6** (1996) 405.

[5] Cercignani C., *The Boltzmann Equation and its Applications* (Springer Verlag, New York) 1988.

[6] Lucquin-Desreux B., *Math. Mod. Meth. Appl. Sci.*, **10** (2000) 409.

[7] Degond P. and Lucquin-Desreux B., *Transp. Theory Stat. Phys.*, **26** (1996) 595.

[8] Lucquin-Desreux B., *Transp. Theory Stat. Phys.*, **27** (1998) 99.

[9] Ben Abdallah N. and Degond P., *J. Math. Phys.*, **37** (1996) 3306.