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On global boundedness of higher velocity moments for solutions to the linear Boltzmann equation with hard sphere collisions

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Summary. — This paper considers the time- and space-dependent linear Boltzmann equation for elastic or inelastic (granular) collisions. First, in the angular cut-off case or with hard sphere collisions, mild L^1-solutions are constructed as limits of iterate functions. Then, in the case of hard sphere collisions together with, *e.g.*, specular boundary conditions, global boundedness in time of higher velocity moments is proved, using our old collision velocity estimates together with a Jensen inequality. This generalizes our earlier results for hard inverse collision forces, and also results given by other authors from the space-homogeneous case to our space-dependent one.

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1. – Introduction

The linear Boltzmann equation is frequently used for mathematical modelling in physics (*e.g.*, for describing the neutron distribution in reactor physics, cf. [1-3]). In our papers [4-9] we have studied the linear Boltzmann equation, both in the elastic and the inelastic (granular) case for a function $f(\mathbf{x}, \mathbf{v}, t)$ representing the distribution of particles with mass m colliding binary with other particles of mass m_*, which have a given (known) distribution function $Y(\mathbf{x}, \mathbf{v}_*)$. Thereby we have also got results on boundedness (in time) of higher velocity moments for hard inverse collision forces. The purpose of this paper is to use another method to get similar results for hard sphere collisions in the time and space dependent case; cf. ref. [10] and [11] for the space-homogeneous case.

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So we will study collisions between particles with mass m and particles with mass m_*, such that momentum is conserved,

$$m\mathbf{v} + m_*\mathbf{v}_* = m\mathbf{v}' + m_*\mathbf{v}'_*, \tag{1}$$

where $\mathbf{v}$, $\mathbf{v}_*$ are velocities before and $\mathbf{v}'$, $\mathbf{v}'_*$ are velocities after a collision.

In the elastic case, where also kinetic energy is conserved, one finds that the velocities after a binary collision terminate on two concentric spheres, so all velocities $\mathbf{v}'$ lie on a sphere around the center of mass, $\bar{\mathbf{v}} = (m\mathbf{v}+m_*\mathbf{v}_*)/(m+m_*)$, with radius $\frac{m_* w}{m+m_*}$, where $w = |\mathbf{v}-\mathbf{v}_*|$, and all velocities $\mathbf{v}'_*$ lie on a sphere with the same center $\bar{\mathbf{v}}$ and with radius $\frac{mw}{m+m_*}$, cf. fig. 1 in [4].

In the granular, inelastic case we assume the following relation between the relative velocity components normal to the plane of contact of the two particles:

$$\mathbf{w}' \cdot \mathbf{u} = -a(\mathbf{w} \cdot \mathbf{u}), \tag{2}$$

where a is a constant, $0 < a \leq 1$, and $\mathbf{w} = \mathbf{v} - \mathbf{v}_*, \mathbf{w}' = \mathbf{v}' - \mathbf{v}'_*$ are the relative velocities before and after the collision, and $\mathbf{u}$ is a unit vector in the direction of impact, $\mathbf{u} = (\mathbf{v} - \mathbf{v}')/|\mathbf{v} - \mathbf{v}'|$. Then we find that $\mathbf{v}' = \mathbf{v}'_a$ lies on the line between $\mathbf{v}$ and $\mathbf{v}'_1$, where $\mathbf{v}'_1$ is the postvelocity in the case of elastic collision, *i.e.* with $a = 1$, and $\mathbf{v}'_{*a}$ lies on the (parallel) line between $\mathbf{v}_*$ and $\mathbf{v}'_{*1}$.

Now the following relations hold for the velocities in a granular, inelastic collision:

$$\mathbf{v}' = \mathbf{v} - (a+1)\frac{m_*}{m+m_*}(\mathbf{w}\cdot\mathbf{u})\mathbf{u}, \qquad \mathbf{v}'_* = \mathbf{v}_* + (a+1)\frac{m}{m+m_*}(\mathbf{w}\cdot\mathbf{u})\mathbf{u}, \tag{3}$$

where $\mathbf{w}\cdot\mathbf{u} = w\cos\theta$, $w = |\mathbf{v}-\mathbf{v}_*|$, if the unit vector $\mathbf{u}$ is given in spherical coordinates,

$$\mathbf{u} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \quad 0 \leq \theta \leq \pi/2,\ 0 \leq \phi < 2\pi. \tag{4}$$

Moreover, if we change notations, and let $'\mathbf{v}, '\mathbf{v}_*$ be the velocities before, and $\mathbf{v}, \mathbf{v}_*$ the velocities after a binary inelastic collision, then by (2) and (3), cf. [7-11],

$$'\mathbf{v} = \mathbf{v} - \frac{(a+1)m_*}{a(m+m_*)}(\mathbf{w}\cdot\mathbf{u})\mathbf{u}, \qquad '\mathbf{v}_* = \mathbf{v}_* + \frac{(a+1)m}{a(m+m_*)}(\mathbf{w}\cdot\mathbf{u})\mathbf{u}. \tag{5}$$

2. – Preliminaries

We consider the time-dependent transport equation for a distribution function $f(\mathbf{x},\mathbf{v},t)$, depending on a space variable $\mathbf{x} = (x_1, x_2, x_3)$ in a bounded convex body D with (piecewise) C^1-boundary $\Gamma = \partial D$, and depending on a velocity variable $\mathbf{v} = (v_1, v_2, v_3) \in V = \mathbb{R}^3$ and a time variable $t \in \mathbb{R}_+$. Then the linear Boltzmann equation is in the strong form

$$\frac{\partial f}{\partial t}(\mathbf{x},\mathbf{v},t) + \mathbf{v}\cdot\text{grad}_{\mathbf{x}} f(\mathbf{x},\mathbf{v},t) = (Qf)(\mathbf{x},\mathbf{v},t), \quad \mathbf{x}\in D,\ \mathbf{v}\in V=\mathbb{R}^3,\ t\in\mathbb{R}_+, \tag{6}$$

supplemented by initial data

$$f(\mathbf{x},\mathbf{v},0)=f_0(\mathbf{x},\mathbf{v}),\quad \mathbf{x}\in D,\ \mathbf{v}\in V. \tag{7}$$

The collision term can, in the case of inelastic (granular) collision, be written, cf. [7-11],

$$(Qf)(\mathbf{x},\mathbf{v},t)= \int_V\int_\Omega \big[J_a(\theta,w)Y(\mathbf{x},{}'\mathbf{v}_*)f(\mathbf{x},{}'\mathbf{v},t)-Y(\mathbf{x},\mathbf{v}_*)f(\mathbf{x},\mathbf{v},t)\big]B(\theta,w)\mathrm{d}\mathbf{v}_*\,\mathrm{d}\theta\,\mathrm{d}\phi \tag{8}$$

with $w=|\mathbf{v}-\mathbf{v}_*|$, where $Y\geq 0$ is a known distribution, $B\geq 0$ is given by the collision process, and finally J_a is a factor depending on the granular process (and giving mass conservation, if the gain and the loss integrals converge separately). For elastic collisions $J_a=1$, and in the case of hard sphere collisions $J_a=a^{-2}$, cf. [10] and [11]. Furthermore, ${}'\mathbf{v}$, ${}'\mathbf{v}_*$ in (8) are the velocities before and $\mathbf{v}$, $\mathbf{v}_*$ the velocities after the binary collision, cf. (5), and $\Omega=\{(\theta,\phi):0\leq\theta<\hat\theta,\ 0\leq\phi<2\pi\}$ represents the impact plane, where $\hat\theta<\frac{\pi}{2}$ in the angular cut-off case, and $\hat\theta=\frac{\pi}{2}$ in the infinite range case. For hard sphere collisions one can also take $\hat\theta=\frac{\pi}{2}$, cf. (10) below. The collision function $B(\theta,w)$ is in the physically interesting case with inverse k:th power collision forces given by

$$B(\theta,w)=b(\theta)w^\gamma,\qquad \gamma=\frac{k-5}{k-1},\qquad w=|\mathbf{v}-\mathbf{v}_*|, \tag{9}$$

with hard forces for $k>5$, Maxwellian for $k=5$, and soft forces for $3<k<5$, where $b(\theta)$ has a non-integrable singularity for $\theta=\frac{\pi}{2}$. But in the case of hard sphere collisions, then (for $\gamma=1$) the collision function is given by

$$B(\theta,w)=\mathrm{const}\cdot w\sin\theta\cos\theta. \tag{10}$$

So in the angular cut-off case and also in the hard sphere case, the gain and the loss terms in (8) can be separated

$$(Qf)(\mathbf{x},\mathbf{v},t)=(Q^+f)(\mathbf{x},\mathbf{v},t)-(Q^-f)(\mathbf{x},\mathbf{v},t), \tag{11}$$

where the gain term can be written (with a kernel K_a)

$$(Q^+f)(\mathbf{x},\mathbf{v},t)=\int_V K_a(\mathbf{x},{}'\mathbf{v}\to\mathbf{v})f(\mathbf{x},{}'\mathbf{v},t)\mathrm{d}\,{}'\mathbf{v}, \tag{12}$$

and the loss term is written with the collision frequency $L(\mathbf{x},\mathbf{v})$ as

$$(Q^-f)(\mathbf{x},\mathbf{v},t)=L(\mathbf{x},\mathbf{v})f(\mathbf{x},\mathbf{v},t). \tag{13}$$

In the case of non-absorbing body we have that

$$L(\mathbf{x},\mathbf{v})=\int_V K_a(\mathbf{x},\mathbf{v}\to\mathbf{v}')\mathrm{d}\mathbf{v}'. \tag{14}$$

Furthermore, eqs. (6)-(8) are in the space-dependent case supplemented by (general) boundary conditions

$$(15)\qquad f_-(\mathbf{x},\mathbf{v},t) = \int \frac{|\mathbf{n}\tilde{\mathbf{v}}|}{|\mathbf{n}\mathbf{v}|} R(\mathbf{x},\tilde{\mathbf{v}}\to\mathbf{v}) f_+(\mathbf{x},\tilde{\mathbf{v}},t)\mathrm{d}\tilde{\mathbf{v}},$$
$$\mathbf{n}\mathbf{v} < 0,\ \mathbf{n}\tilde{\mathbf{v}} > 0,\ \mathbf{x}\in\Gamma = \partial D,\ t\in\mathbb{R}_+,$$

where $\mathbf{n} = \mathbf{n}(\mathbf{x})$ is the unit outward normal at $\mathbf{x}\in\Gamma=\partial D$. The function $R \geq 0$ satisfies (in the non-absorbing boundary case)

$$(16)\qquad \int_V R(\mathbf{x},\tilde{\mathbf{v}}\to\mathbf{v})\mathrm{d}\mathbf{v} \equiv 1,$$

and f_- and f_+ represent the ingoing and outgoing trace functions corresponding to f. In the specular reflection case the function R is represented by a Dirac measure $R(\mathbf{x},\tilde{\mathbf{v}}\to\mathbf{v}) = \delta(\mathbf{v}-\tilde{\mathbf{v}}+2(\mathbf{n}\tilde{\mathbf{v}})\mathbf{n})$, and in the diffuse reflection case $R(\mathbf{x},\tilde{\mathbf{v}}\to\mathbf{v}) = |\mathbf{n}\mathbf{v}|W(\mathbf{x},\mathbf{v})$ with some given function $W\geq 0$, (*e.g.*, Maxwellian function).

Let $t_b \equiv t_b(\mathbf{x},\mathbf{v}) = \inf_{\tau\in\mathbb{R}_+}\{\tau : \mathbf{x}-\tau\mathbf{v}\notin D\}$, and $\mathbf{x}_b \equiv \mathbf{x}_b(\mathbf{x},\mathbf{v}) = \mathbf{x}-t_b\mathbf{v}$, where t_b represents the time for a particle going with velocity $\mathbf{v}$ from the boundary point $\mathbf{x}_b$ to the point $\mathbf{x}$.

Then, using differentiation along the characteristics, eq. (6) can formally be transformed to a *mild* equation, and also to an *exponential* form of equation in the angular cut-off case or in the hard sphere case, cf. [4-9].

3. – Construction of solutions

We construct L^1-solutions to our problems as limits of iterate functions f^n, when $n\to\infty$. Let first $f^{-1}(\mathbf{x},\mathbf{v},t)\equiv 0$. Then define for given f^{n-1} the next iterate f^n, first at the ingoing boundary (using the appropriate boundary condition), and then inside D and at the outgoing boundary (using the exponential form of the equation),

$$(17)\qquad f^n_-(\mathbf{x},\mathbf{v},t) = \int_V \frac{|\mathbf{n}\tilde{\mathbf{v}}|}{|\mathbf{n}\mathbf{v}|} R(\mathbf{x},\tilde{\mathbf{v}}\to\mathbf{v}) f^{n-1}_+(\mathbf{x},\tilde{\mathbf{v}},t)\mathrm{d}\tilde{\mathbf{v}},$$

$$(18)\qquad f^n(\mathbf{x},\mathbf{v},t) = \bar{f}^n(\mathbf{x},\mathbf{v},t)\exp\left[-\int_0^t L(\mathbf{x}-s\mathbf{v},\mathbf{v})\mathrm{d}s\right] +$$
$$+\int_0^t \exp\left[-\int_0^\tau L(\mathbf{x}-s\mathbf{v},\mathbf{v})\mathrm{d}s\right]\int_V K_a(\mathbf{x}-\tau\mathbf{v},{}'\mathbf{v}\to\mathbf{v}) f^{n-1}(\mathbf{x}-\tau\mathbf{v},{}'\mathbf{v},t-\tau)\mathrm{d}'\mathbf{v}\,\mathrm{d}\tau,$$

where

$$(19)\qquad \bar{f}^n(\mathbf{x},\mathbf{v},t) = \begin{cases} f_0(\mathbf{x}-t\mathbf{v},\mathbf{v}), & 0\leq t\leq t_b,\\ f^n_-(\mathbf{x}_b,\mathbf{v},t-t_b), & t>t_b.\end{cases}$$

Let also $f^n(\mathbf{x},\mathbf{v},t)\equiv 0$ for $\mathbf{x}\in\mathbb{R}^3\setminus D$. Now we get a monotonicity lemma, $f^n(\mathbf{x},\mathbf{v},t) \geq f^{n-1}(\mathbf{x},\mathbf{v},t)$, which is essential and can be proved by induction.

Then, by differentiation along the characteristics and integration (with Green's formula), we find (using the equations above, cf. [7]), that

$$\int_D \int_V f^n(\mathbf{x}, \mathbf{v}, t) \mathrm{d}\mathbf{x} \, \mathrm{d}\mathbf{v} \leq \int_D \int_V f_0(\mathbf{x}, \mathbf{v}) \mathrm{d}\mathbf{x} \, \mathrm{d}\mathbf{v}, \tag{20}$$

so Levi's theorem (on monotone convergence) gives existence of (mild) L^1-solutions

$$f(\mathbf{x}, \mathbf{v}, t) = \lim_{n \to \infty} f^n(\mathbf{x}, \mathbf{v}, t). \tag{21}$$

Proposition (Existence). Assume (for inelastic or elastic collisions), that the function B is given by (10), or (9) with angular cut-off, and that K_a, L and R are non-negative, measurable functions, such that (14) and (16) hold, and $L(\mathbf{x}, \mathbf{v}) \in L^1_{\mathrm{loc}}(D \times V)$.

Then for every $f_0 \in L^1(D \times V)$ there exists a mild L^1-solution $f(\mathbf{x}, \mathbf{v}, t)$ to the problem (6)-(8) with (15), satisfying the corresponding inequality in (20). Furthermore, if $L(\mathbf{x}, \mathbf{v}) f(\mathbf{x}, \mathbf{v}, t) \in L^1(D \times V)$, then equality in (20) for the limit function f holds, giving mass conservation together with uniqueness in the relevant function space (cf. [4-6] and also Proposition 3.3, Chapt. 11 in [3]).

Remark 1. The assumption $Lf \in L^1(D \times V)$ is, for instance, satisfied for the solution f in the case of inverse power collision forces, cf. (9), together with, *e.g.*, specular boundary reflections. This follows from a statement on global boundedness (in time) of higher velocity moments (cf. Theorem 4.1 and Corollary 4.1 in [7]). Compare also the results in the next section.

Remark 2. There holds also both in the elastic and inelastic cases an *H-theorem* for a general relative entropy functional

$$H_F^{\Phi}(f)(t) = \int_D \int_V \Phi\left(\frac{f(\mathbf{x}, \mathbf{v}, t)}{F(\mathbf{x}, \mathbf{v})} \right) F(\mathbf{x}, \mathbf{v}) \mathrm{d}\mathbf{x} \, \mathrm{d}\mathbf{v}, \tag{22}$$

giving that this H-functional is nonincreasing in time, if $\Phi = \Phi(z), \mathbb{R}_+ \to \mathbb{R}$, is a convex C^1-function, and if there exists a corresponding stationary solution $F(\mathbf{x}, \mathbf{v})$ with the same total mass as the initial data $f_0(\mathbf{x}, \mathbf{v})$ for the time-dependent solution $f(\mathbf{x}, \mathbf{v}, t)$; cf. Theorem 5.1 in [7]. By using this H-functional one can prove that every time-dependent solution $f(\mathbf{x}, \mathbf{v}, t)$ converges to the corresponding stationary solution $F(\mathbf{x}, \mathbf{v})$ as time goes to infinity, cf. Remark 5.1 in [7] and further references.

4. – Higher velocity moment estimates

In this section we will generalize a result on global boundedness of higher velocity moments to the case of hard sphere collisions. Then we start with some (old) velocity estimates for a binary collision, and also give the corresponding moment estimates, cf. Propositions 1.1 and 1.2 in [4]. Compare ref. [12] for analogous results.

Proposition A. If $\mathbf{v}$ and $\mathbf{v}'_a(\theta, \phi)$ are the velocities before and after a (granular) binary collision, then, with $w = |\mathbf{v} - \mathbf{v}_*|$,

$$|\mathbf{v}'_a(\theta, \phi)|^2 - |\mathbf{v}|^2 \leq 2\,(a+1) \frac{m_*}{m + m_*} w \cos\theta \left[3|\mathbf{v}_*| - \frac{m_*}{m + m_*} |\mathbf{v}| \cos\theta \right].$$

Proposition B. If $\sigma > 0$, there exist constants $c_1 > 0$, $c_2 > 0$ (depending on σ, m, m_* and a) such that

$$\big(1+|\mathbf{v}_a'(\theta,\phi)|^2\big)^{\sigma/2} - \big(1+|\mathbf{v}|^2\big)^{\sigma/2} \le c_1 w\cos\theta\big(1+|\mathbf{v}_*|\big)^{\max(1,\sigma-1)}\big(1+|\mathbf{v}|^2\big)^{\frac{\sigma-2}{2}} \\ -c_2 w\cos^2\theta\big(1+|\mathbf{v}|^2\big)^{\frac{\sigma-1}{2}}.$$

By using these propositions we have earlier got results on boundedness of higher velocity moments for hard inverse collision forces, $0 \le \gamma < 1$, but now we will use a Jensen inequality to get the analogous results for hard sphere collisions, $\gamma = 1$, in the space-dependent case with, *e.g.*, specular reflection boundary.

We start with an elementary lemma (used in the theorem below) for the velocities in a binary collision, where $v = |\mathbf{v}|$, $v_* = |\mathbf{v}_*|$ and $w = |\mathbf{w}|$, cf. [4].

Lemma. For $\gamma \ge 0$ it holds that

$$-w^{\gamma+1} \le \big(1+v_*\big)^{\gamma+1} - 2^{-\gamma}\big(1+v^2\big)^{\frac{\gamma+1}{2}} \quad \text{where } \mathbf{w} = \mathbf{v}-\mathbf{v}_* \text{ is the relative velocity.}$$

Proof: We have that

$$\big(1+v^2\big)^{1/2} \le \big(1+v^2+2v\big)^{1/2} = 1+v, \quad \text{where } v = |\mathbf{v}_*+\mathbf{w}| \le v_*+w, \\ \text{so } \big(1+v^2\big)^{1/2} \le 1+v_*+w.$$

The convexity (for $\gamma > 0$) gives that

$$\big(1+v^2\big)^{\frac{\gamma+1}{2}} = \Big(\big(1+v^2\big)^{1/2}\Big)^{\gamma+1} \le \big(1+v_*+w\big)^{\gamma+1} = \left(\frac{1+v_*+w}{2}\right)^{\gamma+1} 2^{\gamma+1} \\ \le \frac{(1+v_*)^{\gamma+1}+w^{\gamma+1}}{2}2^{\gamma+1} = \Big[\big(1+v_*\big)^{\gamma+1}+w^{\gamma+1}\Big]2^{\gamma},$$

and the lemma follows.

Now we can formulate our main result on global boundedness (in time) for hard sphere collisions, *i.e.* with $\gamma = 1$, in the following theorem. Compare Theorem 4.1 in [7] for the case of hard inverse forces.

Theorem. Assume for hard sphere collisions with $\gamma = 1$ that the function $B(\theta, w)$ is given by eq. (10), and suppose that the function $Y(\mathbf{x}, \mathbf{v}_*)$ satisfies the following conditions:

$$(23) \quad \int_V \big(1+v_*\big)^{\gamma+\max(2,\sigma)} \sup_{\mathbf{x}\in D} Y(\mathbf{x},\mathbf{v}_*)\mathrm{d}\mathbf{v}_* < \infty \qquad \text{and} \qquad \int_V \inf_{\mathbf{x}\in D} Y(\mathbf{x},\mathbf{v}_*)\mathrm{d}\mathbf{v}_* > 0.$$

Let the boundary conditions (15) be given for a "non-heating" boundary (*e.g.*, specular reflections) by

$$(24) \qquad R(\mathbf{x}, \tilde{\mathbf{v}} \to \mathbf{v}) = 0 \qquad \text{for} \qquad |\mathbf{v}| > |\tilde{\mathbf{v}}|.$$

Then the higher velocity moments belonging to the mild solution $f(\mathbf{x},\mathbf{v},t)$ given by (21) are all bounded (globally in time),

$$\int_D \int_V \left(1+v^2\right)^{\sigma/2} f(\mathbf{x},\mathbf{v},t)\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{v} \le C_\sigma < \infty, \quad \sigma > 0,\ t>0,\ 0<a\le 1,$$
$$\text{if } \left(1+v^2\right)^{\sigma/2} f_0(\mathbf{x},\mathbf{v}) \in L^1(D\times V).$$

Proof: Start from the definition of the iterate function $f^n(\mathbf{x},\mathbf{v},t)$ in eqs. (17)-(19), and differentiate along the characteristics, using the corresponding mild form of the equation, and then multiply by $(1+v^2)^{\sigma/2}$, where $v=|\mathbf{v}|$ and $\sigma>0$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[(1+v^2)^{\sigma/2} f^n(\mathbf{x}+t\mathbf{v},\mathbf{v},t)\big] = \int_V K_a(\mathbf{x}+t\mathbf{v},{}'\mathbf{v}\to\mathbf{v})(1+v^2)^{\sigma/2} f^{n-1}(\mathbf{x}+t\mathbf{v},{}'\mathbf{v},t)\mathrm{d}\,'\mathbf{v} -$$
$$-L(\mathbf{x}+t\mathbf{v},\mathbf{v})(1+v^2)^{\sigma/2} f^n(\mathbf{x}+t\mathbf{v},\mathbf{v},t).$$

Integrating $\iiint \ldots \mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{v}\,\mathrm{d}\tau$ (with Green's formula) gives

$$\begin{aligned}(25)\qquad &\int_D\int_V (1+v^2)^{\sigma/2} f^n(\mathbf{x},\mathbf{v},t)\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{v} = \\ &= \int_D\int_V \left(1+v^2\right)^{\sigma/2} f_0(\mathbf{x},\mathbf{v})\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{v} + \\ &\quad + \int_0^t\int_\Gamma \Big[\int_V \left(1+v^2\right)^{\sigma/2} f_-^n(\mathbf{x},\mathbf{v},\tau)|\mathbf{n}\mathbf{v}|\mathrm{d}\mathbf{v} - \\ &\quad - \int_V \left(1+v^2\right)^{\sigma/2} f_+^n(\mathbf{x},\mathbf{v},\tau)|\mathbf{n}\mathbf{v}|\mathrm{d}\mathbf{v}\Big]\mathrm{d}\Gamma\,\mathrm{d}\tau + \\ &\quad + \int_0^t\int_D\int_V\int_V K_a(\mathbf{x},{}'\mathbf{v}\to\mathbf{v})\left(1+v^2\right)^{\sigma/2} f^{n-1}(\mathbf{x},\mathbf{v},\tau)\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{v}\,\mathrm{d}'\mathbf{v}\,\mathrm{d}\tau - \\ &\quad - \int_0^t\int_D\int_V L(\mathbf{x},\mathbf{v})\left(1+v^2\right)^{\sigma/2} f^n(\mathbf{x},\mathbf{v},\tau)\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{v}\,\mathrm{d}\tau,\end{aligned}$$

where all integrals exist inductively.

Let the velocity moment be defined by

$$(26)\qquad M_\sigma(t) \equiv M_\sigma^n(t) = \int_D\int_V \left(1+v^2\right)^{\sigma/2} f^n(\mathbf{x},\mathbf{v},t)\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{v}$$

and differentiate eq. (25). For the boundary terms in (25) use the assumption (24), together with the monotonicity lemma, $f^{n-1}\le f^n$, to get that

$$\int_V \left(1+v^2\right)^{\sigma/2} f_-^n(\mathbf{x},\mathbf{v},t)|\mathbf{n}\mathbf{v}|\mathrm{d}\mathbf{v} \le \int_V \left(1+v^2\right)^{\sigma/2} f_+^n(\mathbf{x},\mathbf{v},t)|\mathbf{n}\mathbf{v}|\mathrm{d}\mathbf{v}.$$

And for an estimate of the gain and loss terms in (25), use Proposition B and the Lemma above together with the assumptions (23) to get (for general $\gamma\ge 0$), cf. [7], that

$$(27)\qquad M'_\sigma(t) \le K_1 M_{\sigma+\gamma-1}(t) + K_2 M_{\sigma-1}(t) - K_0 M_{\sigma+\gamma}(t),$$

with constants $K_1 > 0$, $K_2 > 0$, $K_0 > 0$. Then we get for $\gamma = 1$ that

$$M'_\sigma(t) \le K_3 M_\sigma(t) - K_0 M_{\sigma+1}(t), \tag{28}$$

with $K_3 = K_1 + K_2$ (where estimate (28) would lead to a creation of moments of arbitrary order, cf. [13, 14], and forthcoming papers).

Next we will use Jensen's inequality $\Phi(\int_\Omega g\,\mathrm{d}\mu) \le \int_\Omega \Phi(g)\mathrm{d}\mu$, where $\mu(\Omega) = 1$, $g \in L^1(\mu)$ and Φ is a convex function. Here we take $\mathrm{d}\mu = f^n \mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{v}/\|f^n\|$, where $\|f^n\| = \int_D \int_V f^n \mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{v} \le \int_D \int_V f_0 \mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{v}$, and $g(v) = (1+v^2)^{\sigma/2}$, together with the convex function $\Phi(z) = z^{\frac{\sigma+1}{\sigma}}$ for $\sigma > 0$. Then it follows that

$$\left[\int_D \int_V \left(1+v^2\right)^{\sigma/2} f^n \mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{v}/\|f^n\|\right]^{\frac{\sigma+1}{\sigma}} \le \int_D \int_V \left[\left(1+v^2\right)^{\sigma/2}\right]^{\frac{\sigma+1}{\sigma}} f^n \mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{v}/\|f^n\|,$$

$$\text{so } \left(\int_D \int_V \left(1+v^2\right)^{\sigma/2} f^n \mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{v}\right)^{\frac{\sigma+1}{\sigma}} \le \|f^n\|^{1/\sigma} \int_D \int_V \left(1+v^2\right)^{\frac{\sigma+1}{2}} f^n \mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{v},$$

and $-M_{\sigma+1}(t) \le -C(M_\sigma(t))^{\frac{\sigma+1}{\sigma}}$ with a positive constant $C > 0$.

Now we get a differential inequality (of Bernoulli type) for the velocity moment $M_\sigma(t)$,

$$M'_\sigma(t) \le A_1 M_\sigma(t) - A_0\left(M_\sigma(t)\right)^{\frac{\sigma+1}{\sigma}}, \tag{29}$$

which can be written as

$$\left(M_\sigma(t)\right)^{-\frac{\sigma+1}{\sigma}} M'_\sigma(t) - A_1\left(M_\sigma(t)\right)^{-1/\sigma} \le -A_0.$$

Let $y(t) = (M_\sigma(t))^{-1/\sigma}$, so $y'(t) = -\frac{1}{\sigma}(M_\sigma(t))^{-1-\frac{1}{\sigma}} M'_\sigma(t)$ and $y'(t) + a_1 y(t) \ge a_0$, where $a_1 = A_1/\sigma$ and $a_0 = A_0/\sigma$. Multiplication by $e^{a_1 t}$ and integration gives

$$y(t) \ge y(0)e^{-a_1 t} + \frac{a_0}{a_1}(1 - e^{-a_1 t}) \ge \min\left[y(0), \frac{a_0}{a_1}\right].$$

But $y(t) = (M_\sigma(t))^{-1/\sigma}$, so

$$(M_\sigma(t))^{1/\sigma} \le \left(\min\left[(M_\sigma(0))^{-1/\sigma}, \frac{a_0}{a_1}\right]\right)^{-1} = \max\left[(M_\sigma(0))^{1/\sigma}, \frac{a_1}{a_0}\right].$$

Then $M^n_\sigma(t) = M_\sigma(t) \le \max\left[M_\sigma(0), \left(\frac{A_1}{A_0}\right)^\sigma\right] < \infty$, and the theorem follows, because $f^n \uparrow f$ as $n \to \infty$.

REFERENCES

[1] Bellemo N., Palczewski A. and Toscani G., *Mathematical Topics in Nonlinear Kinetic Theory* (World Scientific) 1989.
[2] Cercignani C., *The Boltzmann Equation and Its Applications* (Springer-Verlag) 1988.
[3] Greenberg W., van der Mee C. and Protopopescu V., *Boundary Value Problems in Abstract Kinetic Theory* (Birkhäuser-Verlag) 1987.

[4] Pettersson R., *IMA J. Appl. Math.*, **38** (1987) 151.
[5] Pettersson R., *J. Stat. Phys.*, **59** (1990) 403.
[6] Pettersson R., *J. Stat. Phys.*, **72** (1993) 355.
[7] Pettersson R., *Transp. Theory Stat. Phys.*, **33** (2004) 527.
[8] Pettersson R., *AIP Conf. Proc. (RGD24, Bari)*, **762** (2005) 252.
[9] Pettersson R., *AIP Conf. Proc. (RGD26)*, **1084** (2008) 135.
[10] Lods B. and Toscani G., *J. Stat. Phys.*, **117** (2004) 635.
[11] Villani C., *J. Stat. Phys.*, **124** (2006) 781.
[12] Povzner A. Y., *Mat. Sbor. (N.S.)*, **58** (1962) 65.
[13] Desvillettes L., *Arch. Ration. Mech. Anal.*, **123** (1993) 387.
[14] Mischler S. and Wennberg B., *Ann. Inst. Henri Poincaré Anal. Non Linéaire*, **16** (1999) 467.