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Unitarity bound on the energy loss by a charged particle travelling near a periodic radiator

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Summary. — The linear energy loss, or braking force, $\langle \mathrm{d}W/\mathrm{d}z\rangle$ of a charged particle passing at constant distance b from a semi-infinite inhomogeneous or periodic medium is related to the reflection coefficient R of an evanescent wave. Assuming that $|R| < 1$ as for an ordinary wave, a bound $\langle \mathrm{d}W/\mathrm{d}z\rangle \leq Z^2/(2\pi \times 137\, b^2)$, in natural units, is obtained. Detailed bounds are also obtained for the frequency and angular spectrum of the Smith-Purcell and Cherenkov-at-distance radiations. Some examples in favor of the prolongation of $|R| < 1$ for evanescent waves, as well as some examples rising doubts about it, are presented.

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PACS 41.60.-m – Radiation by moving charges.
PACS 41.60.Bq – Cherenkov radiation.

1. – Introduction

We consider a particle of charge Ze in uniform linear motion in vacuum and at constant distance b from a semi-infinite medium. The particle trajectory is given by $(x, y, z) = (0, 0, vt)$. The medium, hereafter called "radiator", is periodic in z, uniform in y and is contained in the half-space $x \geq b$. Typical examples of such media are

a) Smith-Purcell radiator,

b) plate of transparent glass,

c) plate of resistive medium

(in cases b and c, the period in z has zero length). In the three cases, the particle looses energy, converted into: Smith-Purcell radiation (case a), "Cherenkov-at-distance effect"

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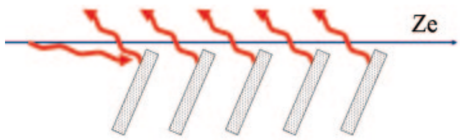

Fig. 1. – Smith-Purcell radiator made of inclined metallic foils.

(case b) or Joule effect of the currents induced by the moving Coulomb field of the particle (case c). This energy loss results in an average *braking force*, $\langle \mathrm{d}W/\mathrm{d}z \rangle$. We will neglect the resulting variation of the velocity.

An example of Smith-Purcell radiatior, made of inclined metallic foils, is schematized in fig. 1. The first foil intercepts quasi-real photons from the Coulomb field of the incoming electron and re-emits them as diffraction radiation. The following foils do the same, but with an incomplete Coulomb field, since each of these foils is in the *shadow* of the preceding one. Thus the radiation emitted by each following foil can be strongly reduced compared to the radiation from a single foil([1]). This fact has suggested [2] a universal bound on the braking force, of the form

$$\left\langle \frac{\mathrm{d}W}{\mathrm{d}z} \right\rangle \leq C \, \frac{Z^2 \hbar c}{137\, b^2}\,, \tag{1}$$

C being a constant of order unity, independent of the medium and its boundary, as well as of the particle energy.

In this paper, we express $\langle \mathrm{d}W/\mathrm{d}z \rangle$ in terms of the reflection coefficient of an evanescent wave on the radiator and show that (1) is obtained, with the precise value $C = 1/(2\pi)$, if the unitarity relation $|R| \leq 1$ can be prolongated to the case of evanescent waves. The bound applies to any of the devices a, b, c listed above.The main ingredients are

- decomposition of the total field in the half-space $x \leq b$ as the sum of the Coulomb field and the field *reflected* by the radiator. The braking force is a retro-action of the reflected field on the particle.
- Fourier expansions in the region $x \in [0, b]$ of both Coulomb and reflected fields, taking into account their common space-time periodicity. Some components have an *oscillating* behaviour in x, some have a real exponential (or *evanescent*) x-behaviour. The braking force is due to an evanescent reflected component.
- Conjectured prolongation of the unitarity relation $|R| \leq 1$ to the case of evanescent waves. In Appendix A, we present simple examples in favor of this conjecture, but also possible exceptions.

2. – Coulomb and reflected fields

Throughout this paper we use relativistic quantum units where $\hbar = 1$, $c = 1$ and rational definitions of charges and fields: $\nabla \cdot \boldsymbol{E} = \rho$, $e^2/(4\pi) = \alpha \simeq 1/137$. The Lorentz-

([1]) The shadow effect has been theoretically evaluated and experimentally observed in [1] for the case of diffraction radiation by two successive foils.

transformed Coulomb electric field is

$$\boldsymbol{E}_{\mathrm{C}}(t,x,y,z) \simeq \frac{Ze}{4\pi}\,\gamma\,\left[x^2+y^2+\gamma^2(z-vt)^2\right]^{-3/2}\begin{pmatrix} x \\ y \\ z-vt \end{pmatrix}, \tag{2}$$

with $\gamma=(1-v^2)^{-1/2}$. The partial Fourier transformation in t and y is

$$\boldsymbol{E}_{\mathrm{C}}(\omega,x,k_y,z) = \frac{-iZe}{2v}\,e^{-\mu|x|}\,e^{i\omega z/v}\begin{pmatrix} i\,\mathrm{sign}(x) \\ k_y/\mu \\ \omega/(\gamma^2 v\mu) \end{pmatrix}, \tag{3}$$

with

$$\mu = \sqrt{k_y^2+\omega^2/(\gamma v)^2}. \tag{4}$$

The total field in the region $x \le b$ is

$$\boldsymbol{E}(t,x,y,z) = \boldsymbol{E}_{\mathrm{C}}(t,x,y,z) + \boldsymbol{E}_{\mathrm{R}}(t,x,y,z). \tag{5}$$

$\boldsymbol{E}_{\mathrm{R}}$ is the field reflected by the radiator. It is the field which brakes the electron. The average braking force is

$$\langle \mathrm{d}W/\mathrm{d}z\rangle = -Ze\,\langle E_{z,\mathrm{R}}(t,0,0,vt)\rangle. \tag{6}$$

Magnetic fields, related to electric fields, will not be considered explicitly.

3. – Space-time periodicity and Fourier expansion

L being the spatial period of the radiator, the electron encounters periodically the same environment with time period $T=L/v$, whence the periodicity condition

$$\boldsymbol{E}(t,x,y,z) = \boldsymbol{E}(t+L/v,x,y,z+L). \tag{7}$$

This equation can be applied to the Coulomb and reflected fields separately. Edges effects in y and z for a finite radiator are neglected. We can make the following Fourier expansion, restricted to the free-space region $x\in[0,b]$ between the trajectory and the radiator:

$$\begin{aligned}\boldsymbol{E}(t,x,y,z) = \int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega}{2\pi}e^{-i\omega t}\sum_{n=-\infty}^{+\infty} e^{ik_z z}\int_{-\infty}^{+\infty}\frac{\mathrm{d}k_y}{2\pi}e^{ik_y y}\sum_{\xi=\pm1} e^{ik_x x}\\ \times\Big(F(\omega,k_x,k_y,n)\,\mathbf{e}^{(\mathrm{TM})}(\boldsymbol{k}) + G(\omega,k_x,k_y,n)\,\mathbf{e}^{(\mathrm{TE})(\boldsymbol{k})}\Big),\end{aligned} \tag{8}$$

with the following constraints:

$$k_z=\omega/v+nQ,\quad n \text{ integer},\quad Q=2\pi/L\quad\text{(periodicity)}, \tag{9}$$

$$\omega^2=k_x^2+k_y^2+k_z^2\quad\text{(massless Klein-Gordon equation)}. \tag{10}$$

The first summation in (8) is over the discrete values of k_z of (9). The second summation is over the sign ξ of k_x for positive k_x^2 (oscillating modes), or k_x/i for negative k_x^2 (evanescent modes). For definiteness, we write

$$k_x = i\xi \, \{k_y^2 + k_z^2 - (\omega + i0)^2\}^{1/2}, \quad \xi = \pm 1. \tag{11}$$

Thus $\xi = +1$ corresponds to a right-moving wave ($k_x/\omega > 0$) for positive k_x^2, or a right-evanescent wave ($k_x/i > 0$) for negative k_x^2.

The vectors

$$\mathbf{e}^{(\mathrm{TM})}(\boldsymbol{k}) = \begin{pmatrix} -k_x k_z \\ -k_y k_z \\ k_x^2 + k_y^2 \end{pmatrix}, \qquad \mathbf{e}^{(\mathrm{TE})}(\boldsymbol{k}) = \begin{pmatrix} -k_y \, \omega \\ +k_x \, \omega \\ 0 \end{pmatrix}, \tag{12}$$

are basic electric field vectors for the transverse magnetic ($B_z = 0$) and transverse electric ($\mathcal{E}_z = 0$) polarisations (we did not normalize them to unity). The reality of $\boldsymbol{E}(t, x, y, z)$ implies

$$F^*(\omega, k_x, k_y, n) = F(-\omega, -k_x^*, -k_y, -n) \quad \text{(same for } G). \tag{13}$$

In the region $0 \leq x \leq b$, the Coulomb field has only a TM, right-evanescent component, with $n = 0$. Equations (3), (4) give $k_x = +i\mu$ and

$$F_C(\omega, +i\mu, k_y, n) = iZe \, \delta_{n,0}/(2\omega\mu), \quad F_C(\omega, -i\mu, k_y, n) = 0, \quad G_C = 0. \tag{14}$$

Only the ($n = 0$, TM) reflected wave contributes to the average braking force (6). It is characterized by $k_z = \omega/v$, $k_x = -i\mu$ and $\mathrm{e}_z^{(\mathrm{TM})} = k_x^2 + k_y^2 = -\omega^2/(\gamma^2 v^2)$. Thus

$$\frac{\mathrm{d}W}{\mathrm{d}z} = -Ze \int_{-\infty}^{+\infty} \frac{\mathrm{d}\omega}{2\pi} \left(\frac{-\omega^2}{\gamma^2 v^2} \right) \int_{-\infty}^{+\infty} \frac{\mathrm{d}k_y}{2\pi} \, F_{\mathrm{R}}(\omega, -i\mu, k_y, 0). \tag{15}$$

To sum up, in the region $x \in [0, b]$ we have three kinds of waves, represented in fig. 2:

- the Coulomb field, with $n = 0$, $k_z = \omega/v$ and $k_x = i\mu$ (right-evanescent).
- oscillating, left-moving reflected waves, making the Smith-Purcell radiation.
- left-evanescent reflected waves. Among them, the ($n = 0$, TM) reflected wave brakes the particle. It has $k_x = -i\mu$.

4. – The reflection matrix

The reflection on the radiator conserves ω and k_y, but can change n, that is to say k_z (due to the diffraction) and the polarisation mode TM or TE. It can thus be described by an infinite-dimensional *reflection matrix*

$$\langle n', s' | \mathbf{R}(\omega, k_y) | n, s \rangle \tag{16}$$

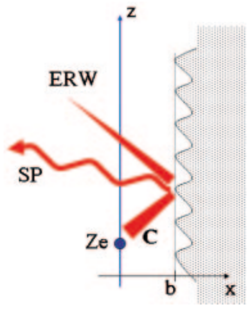

Fig. 2. – Coulomb field (C), Smith-Purcell radiation (SP) and evanescent reflected wave (ERW).

with s and $s' =$ TM or TE. The variables k_x, k_z, k'_x, k'_z of the incident and reflected modes are fixed by (9)-(11). The Fourier amplitude of the braking mode is given by

$$F_R(\omega, -i\mu, k_y, 0) = R(\omega, k_y)\ F_C(\omega, +i\mu, k_y, 0) \tag{17}$$

where $R(\omega, k_y)$ stands for the diagonal element $\langle 0, \mathrm{TM}|\mathbf{R}(\omega, k_y)|0, \mathrm{TM}\rangle$. Taking into account (13)-(15) and (17) the braking force is

$$\left\langle \frac{\mathrm{d}W}{\mathrm{d}z} \right\rangle = \frac{-Z^2}{137\pi(\gamma v)^2} \int_0^{+\infty} \mathrm{d}\omega \int_{-\infty}^{+\infty} \mathrm{d}k_y\, \frac{\omega}{\mu}\, \mathrm{Im}\{R(\omega, k_y)\}, \tag{18}$$

Expression (18) includes the energy spent in Smith-Purcell radiation, Cherenkov-at-distance radiation and the energy deposit in the medium.

5. – Unitarity bound on the reflection coefficient

A reflection coefficient $R = (\textit{reflected amplitude})/(\textit{incident amplitude})$ depends on the reference plane where the wave amplitudes are measured. In (17), this plane was chosen to be at $x = 0$, containing the electron trajectory. Alternatively we may choose the plane $x = b$, thus defining a new coefficient $\hat{R}$ which differs from R by a propagation phase:

$$R = \exp[2ik_x b]\, \hat{R}, \tag{19}$$

where k_x is the incident momentum. In our case (see (17), (18)), $k_x = i\mu$, therefore

$$R(\omega, k_y) = \exp[-2\mu b]\, \hat{R}(\omega, k_y). \tag{20}$$

In the *oscillating* case, a reflected wave cannot have more intensity than the incident wave, wherefrom the unitarity condition $|R| = |\hat{R}| \leq 1$. In our case (evanescent wave), the incident wave has no flux in the x direction and the preceding argument cannot be used. All we can say for the moment is

$$\mathrm{Im}\{R(\omega, k_y)\} \leq 0, \tag{21}$$

which simply garantees that the energy loss is positive. Nevertheless, some examples shown in Appendix A suggest the analytic prolongation of $|R| \leq 1$ to the case of an evanescent wave. Then

$$|\hat{R}(\omega, k_y)| \leq 1 \Rightarrow |R(\omega, k_y)| \leq e^{-2\mu b}. \tag{22}$$

Some possible exceptions are also pointed out. In the next section, assuming that its validity, we draw the consequences of (22) on the average braking force.

6. – Consequences of the unitarity bound for evanescent waves

Putting (22) in (18) we obtain

$$\left\langle \frac{\mathrm{d}W}{\mathrm{d}z} \right\rangle \leq \frac{Z^2}{137\pi(\gamma v)^2} \int_0^{+\infty} \mathrm{d}\omega \int_{-\infty}^{+\infty} \mathrm{d}k_y \, \frac{\omega}{\mu} \, e^{-2\mu b}, \tag{23}$$

with μ given by (4). The double integration([2]) yields

$$\left\langle \frac{\mathrm{d}W}{\mathrm{d}z} \right\rangle \leq \frac{Z^2}{2\pi \times 137\, b^2}\,. \tag{24}$$

We may also apply (23) in the half-integrated form

$$\left\langle \frac{\mathrm{d}^2W}{\mathrm{d}z\,\mathrm{d}\omega} \right\rangle \leq \frac{Z^2}{137\pi\,(\gamma v)^2} \int_{-\infty}^{+\infty} \mathrm{d}k_y \, \frac{\omega}{\mu} \, e^{-2\mu b} = \frac{2Z^2\omega}{137\pi\,(\gamma v)^2} \, K_0\left(\frac{2\omega b}{\gamma v}\right), \tag{25}$$

or in the non-integrated form

$$\left\langle \frac{\mathrm{d}^3W}{\mathrm{d}z\,\mathrm{d}\omega\,\mathrm{d}k_y} \right\rangle \leq \frac{Z^2}{137\pi\,(\gamma v)^2} \, \frac{\omega}{\mu} \, e^{-2\mu b}, \tag{26}$$

Equation (24) is of the form (1) with $C = 1/(2\pi)$. To give an idea, for $Z = \pm 1$ and b equal to the Bohr radius 0.529 Å, the bound is 0.82 GeV/cm. For $b = 1$ mm, it is 0.23 eV/km. An electron beam of 1 A at distance 1 mm from a radiator delivers a maximum power of 0.23 milliwatt/metre([3]).

The bound (25) bears on the energy radiated (by Smith-Purcell and distant Cherenkov effects) in the infinitesimal frequency interval $[\omega, \omega + \mathrm{d}\omega]$. It also includes the Joule effect of the induced currents of this frequency. For high ω the Joule effect is replaced by the excitations or ionisation of atoms of the medium, $\hbar\omega$ being the excitation energy.

The bound (26) applies to the energy radiated at frequency ω and at given k_y. Since k_z is discrete according to (9), fixing k_y selects one value of $|k_x|$ for each n, that is to say one Smith-Purcell direction for each n and one Cherenkov direction for $n = 0$. The bound applies to the sum of the intensities in these directions.

([2]) For the integration, one may use the variables (r, θ) given by $r\cos\theta = 2b\omega/(\gamma v)$, $r\sin\theta = 2bk_y$.

([3]) Here we neglect a possible coherence between the radiations by different electrons. This is not valid if the electron beam is bunched on a scale smaller than a typical emitted wavelength.

7. – Conclusion

We have not yet been able to prove or disprove the existence of a universal bound on $\langle \mathrm{d}W/\mathrm{d}z \rangle$ of the form (1). Nevertheless, we have shown that $\langle \mathrm{d}W/\mathrm{d}z \rangle$ is related to the imaginary part of the reflection coefficient of an evanescent wave. The problem is then reduced to find a possible bound on the latter quantity. We think that such a bound may be obtained by analytic prolongation of the ordinary unitarity relation $|R| \leq 1$. Causality, from which analyticity is deduced, would therefore be the deep reason for the bound.

A tentative value $C = 1/(2\pi)$ is obtained if we simply assume $|R| \leq 1$, or at least $|\,\mathrm{Im}\, R| \leq 1$ in the evanescent domain. Such assumption also gives the detailed bounds (25), (26) for the ω and k_y spectrum.

If the assumption $|\,\mathrm{Im}\, R| \leq 1$ proves to be wrong, then either the bound exists with a coefficient $C > 1/(2\pi)$, or there is no bound at all, *i.e.*, $C = \infty$. At fixed impact parameter, the Coulomb field of the particle reaches a finite limit when $v \to 1$ ($\gamma \to \infty$), therefore $\langle \mathrm{d}W/\mathrm{d}z \rangle$ is expected to reach a saturation limit. The form (1) is the only one having a finite limit at $v \to 1$ and allowed by dimensional arguments. The case $C = \infty$ is unlikely.

Bounds similar to (1) should occur when a charged particle passes between *two* radiators, one at $x \geq b$, the other at $x \leq -b'$. The formalism developed in this paper could be generalised, but taking into account an infinite series of reflections of the wave between the two radiators.

A bound of the form $\langle \mathrm{d}W/\mathrm{d}z \rangle \leq C'\, Z^2 \hbar c/(137\, r^2)$, with $C' > C$, should be looked for a particle moving along the axis of a cylindrical hole in a medium. It may be applied to the energy loss of an electron in travelling wave tubes and in a series of accelerating cavities.

Appendix A.

Bounds on the reflection coefficient of an evanescent wave

The results of sect. **6** are based on the inequality (22), which comes from a generalisation of $|R| \leq 1$ to the case of evanescent wave. In this appendix we present arguments in favor of this hypothesis, but also possible exceptions.

Let us consider first the case of a *scalar* wave ψ incident on a semi-infinite, homogeneous and transparent medium filling the region $x \geq 0$. In the vacuum region we have

$$\psi = e^{-i\omega t} \left(e^{ipx+iqy} + R e^{-ipx+iqy} \right),$$

and in the medium, $\psi = T\, e^{-i\omega t} e^{ip'x+iqy}$. The flux conservation implies that $J = \mathrm{Im}\{\psi^* \partial_x \psi\}$ is positive. For an oscillating wave (real p) it leads to $|R| \leq 1$. For an evanescent wave ($p = +i|p|$) it leads to $\mathrm{Im}\, R > 0$. Both inequalities are satisfied by the analytical result

$$R = (p - p')/(p + p'), \tag{A.1}$$

in the following three physical cases:

– p' real positive (oscillating wave in the medium),

– $p' = +i|p'|$ (evanescent wave in the medium),

– $\mathrm{Re}\, p' > 0$ and $\mathrm{Im}\, p' > 0$ (absorbed wave in the medium).

In addition, we see that $|R| \leq 1$ is also satisfied for an incident evanescent wave ($p = +i|p|$). More precisely,

$$1 - |R|^2 = 2\,\mathrm{Im}(R)\,\mathrm{Im}(p')/\,\mathrm{Re}(p'). \tag{A.2}$$

Considering a wave in the medium coming from $x = +\infty$ and suffering total reflection, with reflection coefficient $R' = -R$, we can derive (A.2) from the fact that no flux passes through the $x = +0$ plane.

A proof of the prolongation of $|R| \leq 1$ may be obtained using causality, which implies that R is analytic in ω in the half-plane $\mathrm{Im}\,\omega > 0$.

Let us now consider an *electromagnetic* wave. We have two reflection coefficients,

$$R_\perp = (p - p'/\mu)/(p + p'/\mu), \qquad R_\parallel = (p - p'/\varepsilon)/(p + p'/\varepsilon), \tag{A.3}$$

corresponding to the polarisations respectively parallel and perpendicular to the plane of incidence (the convention for the polarisation basis vectors is that given by, *e.g.*, Figure 7.10 of ref. [3]). ε and μ are the permitivity and permeability of the medium. For a transparent medium, with μ and ε real and positive, we get the same conclusions that for the scalar case, just replacing $p' = \sqrt{\varepsilon\mu\omega^2 - q^2}$ by p'/μ or p'/ε.

If the medium is absorbing, for instance due to $\mathrm{Im}\,\varepsilon > 0$, it may happen that $\mathrm{Im}(p'/\varepsilon)$ is negative in spite of the fact that $\mathrm{Im}\, p'$ is positive. In this case $|R| > 1$. Due to this possible counter-example, we cannot consider the results of sect. **6** as absolutely certain. All we can hope is that (22) is valid in most of the integration domain of (18).

The TM/TE polarisation vectors (12) used in (16) are combinations of the $\parallel$ and $\perp$ polarisation vectors, so that $\mathbf{R}(\omega, k_y)$ is not diagonal in the TM/TE basis. The TM $\to$ TM coefficient is given by

$$\langle 0, \mathrm{TM}|\mathbf{R}(\omega, k_y)|0, \mathrm{TM}\rangle \equiv R(\omega, k_y) = \frac{-\xi\zeta\, R_\parallel + \eta\,(\xi + \eta + \zeta)\, R_\perp}{(\xi + \eta)\,(\eta + \zeta)}\,, \tag{A.4}$$

with $\xi = k_x^2$, $\eta = k_y^2$ and $\zeta = k_z^2$, $\xi + \eta + \zeta = \omega^2$. In our case, $\xi < 0$, $\xi + \eta = -\omega^2/(\gamma v)^2$ and $\zeta = \omega^2/v^2$.

If $R_\parallel = -R_\perp$, then $R(\omega, k_y) = R_\perp$. If $R_\parallel = +R_\perp$, then $R(\omega, k_y)/R_\perp < -1$. In this case, the inequality $|R_\parallel| = |R_\perp| \leq 1$ does not necessarily leads to $|R(\omega, k_y)| \leq 1$. This is a second possibility of the violation of inequality (22).

To sum up, for a homogeneous radiator medium, there are indications that, at least for the scalar wave model, the unitarity relation $|R| \leq 1$ can be prolongated in the domain of evanescent waves. For electromagnetic waves, we found possible exceptions.

The case of a medium which is periodic in z (*e.g.*, a Smith-Purcell radiator) is more complicated since R becomes an infinite-dimensional matrix. All we can say for the moment from flux conservation is that, for an incident oscillating wave, the diagonal element (which corresponds to the specular reflection) has its modulus less than unity.

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