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Contemporary gravitational waves from primordial black holes

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Summary. — Stochastic background of gravitational waves (GW) generated by the interactions between primordial black holes (PBH) in the early universe and by PBH evaporation is considered. If PBHs dominated in the cosmological energy density prior to their evaporation, GWs from the earlier stages (*e.g.*, inflation) would be noticeably diluted. On the other hand, at the PBH dominance period they could form dense clusters where PBH binary formation might be significant. These binaries would be efficient sources of the gravitational waves.

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The registration of the gravitational waves generated in the early universe could bring an important information about inflation, possible (first order) cosmological phase transitions, topological defects, such as cosmic strings, etc. Very sensitive GW detectors such as LIGO and LISA may make the discovery opening a new era of gravitational wave astronomy. In particular, an observation of stochastic cosmological background of low frequency GWs could be a final proof of inflation. However, an absence of a such background would not mean that the universe was not in inflationary stage. First, the present day density of GWs depends upon the model of inflation and, second, there could be a mechanism which suppresses the density of GWs at post-inflationary stage. Such a mechanism is described in my talk. Though the inflationary background of GWs could be noticeably suppressed, a new higher frequency GWs would be generated by the suggested mechanism. The talk is based on two papers [1, 2]. In the second one a detailed reference list is presented which is reduced here due to lack of space.

We consider GWs produced by the interactions between primordial black holes (PBH), as well as by their evaporation. PBHs are supposed to be very light, so they evaporated before the big bang nucleosynthesis (BBN) leaving no trace in the present day universe, except for GWs. The lifetime of an evaporating black hole with mass M is equal to [3]

$$\tau_{BH} = \frac{10240\,\pi}{N_{eff}}\,\frac{M^3}{m_{Pl}^4}\,, \tag{1}$$

where $m_{Pl} = 1.22 \cdot 10^{19}\,\mathrm{GeV} = 2.176 \cdot 10^{-5}\,\mathrm{g}$ is the Planck mass and N_{eff} is the number of particle species with masses smaller than the black hole temperature:

$$T_{BH} = \frac{m_{Pl}^2}{8\pi M}\,. \tag{2}$$

The corrections due to the propagation and back-capture of the evaporated particles [4], the so called grey factor, change this result by a factor of order unity and are not included here.

According to ref. [5], to avoid a conflict with BBN the lifetime of PBHs should be shorter than $t \approx 10^{-2}\,\mathrm{s}$ and thus the black holes should be lighter than

$$M < 1.75 \cdot 10^8 \left(\frac{N_{eff}}{100}\right)^{1/3} \mathrm{g}. \tag{3}$$

The temperature of such PBHs exceeds $3 \cdot 10^4\,\mathrm{GeV}$ and correspondingly $N_{eff} \geq 10^2$.

Formation of PBHs from primordial density perturbations in the early Universe was considered in pioneering papers [6,7]. PBHs were formed when the density contrast, $\delta\rho/\rho$, at horizon was of the order unity or, in other words, when the Schwarzschild radius of the perturbation was of the order of the horizon scale. If PBHs were created at the radiation dominated stage, when the cosmological energy density was $\rho(t) = 3m_{Pl}^2/(32\pi t^2)$, and the horizon was $l_h = 2t$, the mass of such PBHs would be

$$M(t) = m_{Pl}^2 t \simeq 4 \cdot 10^{38} \left(\frac{t}{\mathrm{s}}\right) \mathrm{g}\,, \tag{4}$$

where t is the cosmological time.

The fraction, Ω_p, of the cosmological energy density of PBH produced by this mechanism depends upon the spectrum of the primordial density perturbations. If the usual flat Harrison-Zeldovich spectrum is assumed, then Ω_p would be quite small. We have not calculated Ω_p but have taken it as a free parameter of the model. One reason for that is that the spectrum of the density perturbations at small wavelengths is unknown. Moreover, there could be other mechanisms of PBH formation. In particular, in refs. [8,9] a model of PBH formation has been proposed which might lead to considerably larger probability of PBH formation. The mass spectrum of PBHs produced by the latter mechanism has the log-normal form

$$\frac{\mathrm{d}N}{\mathrm{d}M} = C \exp\left[\frac{(M - M_0)^2}{M_1^2}\right], \tag{5}$$

where C, M_0, and M_1 are some model-dependent parameters. Quite naturally the central value of PBH mass distribution may be in the desired range $M_0 < 10^9\,\mathrm{g}$. In this model the value of Ω_p may be much larger than in the conventional model based on the flat spectrum of the primordial fluctuations. We will not further speculate on the value of Ω_p and on the form of the mass spectrum of PBH. In what follows we assume for an order of magnitude estimate that the spectrum is well localized near some fixed mass M and that Ω_p is an arbitrary parameter. Different mechanisms of PBH production are reviewed, *e.g.*, in ref. [10].

The energy density of nonrelativistic PBHs drops down as $1/a^3$, while the energy density of the initially dominant relativistic matter drops as $1/a^4$, where $a = a(t)$ is the cosmological scale factor. So the relative contribution of PBH into the total energy density rises as

$$\Omega_{BH}(t) = \Omega_p a(t)/a_p = \Omega_p (t/t_p)^{1/2}, \tag{6}$$

here $t_p = M/m_{Pl}^2$ is the PBH production time, related to their mass by eq. (4). So initially $\Omega_{BH}(t)$ rises as $t^{1/2}$ and at some stage it reaches unity and after that Ω_{BH} remains constant till the PBH evaporation. PBHs would begin to dominate in the cosmological energy density at $t = t_{eq} = M(m_{Pl}^2 \Omega_p^2)$, if $t_{eq} > \tau_{BH}$, eq. (1). This can be transformed into the lower limit on the PBH mass:

$$M > 6 \cdot 10^{-2} \left(\frac{N_{eff}}{100}\right)^{1/2} \frac{m_{Pl}}{\Omega_p} \simeq 10^{-7}\,\mathrm{g}\;\Omega_p^{-1}\,. \tag{7}$$

If condition (7) was fulfilled, the universe expansion regime was initially relativistic, radiation dominated (RD), then after $t = t_{eq}$ it became non-relativistic, matter dominated (MD). Later after PBH evaporation, $t > \tau_{BH}$ the universe returned to RD stage again, and only after very long time, $t = t_{LSS} \sim 10^5$ years, the expansion became matter dominated. After that time the large scale structures (galaxies, their clusters, etc.) began to form. As is known, cosmological structure formation took place at MD stage, when initially small primordial density perturbations started to rise due to gravitational instability. In our case the density perturbations started to rise at $t > t_{eq}$. According to the theory, at MD stage $\Delta \equiv \delta\rho/\rho \sim a(t)$, till the perturbations remain small, $\Delta \ll 1$. When Δ reaches unity, the perturbations quickly rise and as a result they become quite large, $\Delta \gg 1$. In the present day universe $\Delta \sim 10^5$ at the galactic scale.

In our scenario we expect formation of high density clusters of PBHs with density contrast which rose as $\Delta(t) = \Delta_{in}(t/t_{in})^{2/3}$, where $t_{in} \geq t_{eq}$ is the moment when the perturbation comes inside the cosmological horizon. The density contrast would reach unity at $t_1(t_{in})$ such that

$$\Delta[t_1(t_{in})] = \Delta_{in}[t_1(t_{in})/t_{in}]^{2/3} = 1 \quad \text{or} \quad t_1(t_{in}) = t_{in}\Delta_{in}^{-3/2}\,. \tag{8}$$

To this end the PBH lifetime should be longer than t_1.

After the density contrast has reached unity, the cluster would decouple from the general cosmological expansion. In other words, the cluster stopped expanding together with the universe and, on the opposite, it would begin to shrink when gravity took over the free streaming of PBHs. So the cluster size would drop down and both n_{BH} and ρ_b would rise. The density contrast would quickly rise from unity to $\Delta_b = \rho_b/\rho_c \gg 1$, where ρ_c and ρ_b are, respectively, the average cosmological energy density and the density of PBHs in the cluster (bunch). It looks reasonable that the density contrast of the evolved cluster could rise up to $\Delta_b = 10^5$–10^6, as in the contemporary galaxies. After the size of the cluster stabilized, the number density of PBH, n_{BH}, as well as their mass density, ρ_{BH}, would be constant too. But the density contrast, Δ_b would continue to rise as $(t/t_1)^2$ because ρ_c drops down as $1/t^2$. From time $t = t_1$ to $t = \tau_{BH}$ the density contrast would additionally rise by the factor

$$\Delta(\tau_{BH}) = \Delta_b \left(\frac{\tau_{BH}}{t_1}\right)^2 . \tag{9}$$

This rise is associated with the drop of the average cosmological energy density, $\rho \sim 1/t^2$, but not with the absolute rise of $\delta\rho$. This effect is absent in the present day universe because the time when Δ reached unity was close to the present universe age.

GWs could be generated in the processes of PBH scattering in the high density clusters and, in particular, the GW emission could proceed from the PBH binaries. Both processes are strongly enhanced in the clusters. The probability of scattering and binary formation rate are proportional to the square of the number density of PBHs, n_{BH}. However, the net effect on the cosmological energy density of the emitted GWs is linear in n_{BH} because it is normalized to the total cosmological energy density.

The cross-section of the graviton bremsstrahlung was calculated in ref. [11] for the case of two spineless particles (here black holes) with masses m and M under assumption that $m \ll M$. In non-relativistic approximation, the differential cross section is

$$\mathrm{d}\sigma = \frac{64 M^2 m^2}{15 m_{pl}^6}\frac{\mathrm{d}\xi}{\xi}\left[5\sqrt{1-\xi} + \frac{3}{2}(2-\xi)\ln\frac{1+\sqrt{1-\xi}}{1-\sqrt{1-\xi}}\right], \tag{10}$$

where ξ is the ratio of the emitted graviton frequency, $\omega = 2\pi f$, to the kinetic energy of the incident black hole, *i.e.* $\xi = 2m\omega/\mathbf{p}^2$. In what follows we will use this expression for a simple estimate assuming that it is approximately valid for $m \sim M$.

The energy density of gravitational waves emitted at the time interval t and $t + \mathrm{d}t$ in the frequency range ω and $\omega + \mathrm{d}\omega$ is given by

$$\frac{\mathrm{d}\rho_{GW}}{\mathrm{d}\omega} = v_{rel} n_{BH}^2 \omega \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\omega}\right) \mathrm{d}t , \tag{11}$$

where n_{BH} is the number density of PBH and v_{rel} is their relative velocity. The latter is close to the virial velocity of PBHs in the cluster and can be about 0.1. As noted by the authors of ref. [11], Weizsäcker-Williams approximation is not valid. This means that there could be some difference between classical and quantum graviton emission.

The graviton bremsstrahlung proceeded till the PBH evaporation. Hence to find the total energy of the produced gravitons we need to integrate their energy spectrum over frequencies and redshift from τ_{BH} down to the moment of the cluster formation. Thus we obtain for the cosmological energy fraction of GWs:

$$\Omega_{GW}^{(brems)}(\omega_{max}, \tau_{BH}) \approx 16 Q \left(\frac{v_{rel}}{0.1}\right)\left(\frac{\Delta}{10^5}\right)\left(\frac{N_{eff}}{100}\right)\left(\frac{\omega_{max}}{M}\right). \tag{12}$$

Here coefficient $Q > 1$ reflects the uncertainty in the cross-section due to the unaccounted for Sommerfeld enhancement [12]. Note that Δ may be considerably larger than 10^5. For the details see ref. [2].

The frequency f_* of GW produced at time t_* during PBH evaporation, is redshifted down to the present day value, f, as

$$f = f_* \left[\frac{a(t_*)}{a_0}\right] = 0.34\, f_* \frac{T_0}{T_*}\left[\frac{100}{g_S(T_*)}\right]^{1/3}, \tag{13}$$

where $T_0 = 2.725\,\mathrm{K}$ is the temperature of the cosmic microwave background radiation at the present time, $T_* \equiv T(t_*)$ is the plasma temperature at the moment of radiation of the gravitational waves, and $g_S(T_*)$ is the number of species contributing to the entropy of the primeval plasma at temperature T_*. It is convenient to express T_0 in frequency units, $T_0 = 2.7\,\mathrm{K} = 5.4 \cdot 10^{10}\,\mathrm{Hz}$.

The density parameter of the gravitational waves at the present time is related to cosmological time t_* as

$$\Omega_{GW}(t_0) = \Omega_{GW}(t_*) \left(\frac{a(t_*)}{a(t_0)}\right)^4 \left(\frac{H_*}{H_0}\right)^2, \tag{14}$$

where $H_0 = 100 h_0\,\mathrm{km/s/Mpc}$ is the Hubble parameter and $h_0 = 0.74 \pm 0.04$ [13].

Using expression for redshift (13) and taking the emission time $t_* = \tau_{BH}$ we obtain

$$\Omega_{GW}(t_0) = 1.67 \times 10^{-5} h_0^{-2} \left(\frac{100}{g_S(T_{BH})}\right)^{1/3} \Omega_{GW}(\tau_{BH})\,. \tag{15}$$

Now we find that the total density parameter of gravitational waves integrated up to the maximum frequency is

$$h_0^2 \Omega_{GW}(t_0) \approx 0.6 \cdot 10^{-21}\,\mathrm{K} \left(\frac{10^5\,\mathrm{g}}{M}\right)^2, \tag{16}$$

where K is a numerical coefficient:

$$K = \left(\frac{v_{rel}}{0.1}\right) \left(\frac{\Delta}{10^5}\right) \left(\frac{N_{eff}}{100}\right) \left(\frac{Q}{100}\right) \left(\frac{100}{g_S(T_{BH})}\right)^{1/3}. \tag{17}$$

Presumably K is of order unity but since Δ may be much larger than 10^5, see eq. (9), K may also be large.

Classical emission of GW at the scattering of non-relativistic bodies is well described in quadrupole approximation. If the minimal distance between the bodies is larger than their gravitational radii, the energy of gravitational waves emitted in a single scattering process is equal to

$$\delta E_{GW} = \frac{37\pi}{15} \frac{M^2 m^2 v}{b^3 m_{Pl}^6}, \qquad v \ll 1, \tag{18}$$

where b is the impact parameter.

The differential cross-section of the gravitational scattering of two PBHs in non-relativistic regime, $q^2 \ll 2M^2$, is

$$\mathrm{d}\sigma = \frac{M^2}{m_{Pl}^2} \frac{\mathrm{d}q^2}{q^4} = \frac{2M^2}{m_{Pl}^2} b\,\mathrm{d}b \tag{19}$$

and the rate of the energy emission by the GWs is given by

$$\mathrm{d}\rho_{GW} = \frac{74\pi v_{rel}}{15} \rho_{BH}^2 \frac{M^4}{m_{Pl}^8} \frac{\mathrm{d}\omega}{2\pi} \mathrm{d}t\,. \tag{20}$$

The energy density parameter of GW at the moment of BH evaporation can be obtained integrating this expression over time and frequency. Thus we obtain

$$\Omega_{GW}(\tau_{BH}) = 2\cdot 10^{-10}\left(\frac{v_{rel}}{0.1}\right)^2\left(\frac{\Delta_b}{10^5}\right)\left(\frac{N_{eff}}{100}\right)\left(\frac{10^5\,\mathrm{g}}{M}\right). \tag{21}$$

If we allow for $b \sim r_g$, the energy density of GWs at the moment of PBHs evaporation might be comparable to unity.

Now we can calculate the relative energy density of GWs per logarithmic frequency interval at the present time:

$$\Omega_{GW}(f;t_0) \equiv \frac{1}{\rho_c}\frac{\mathrm{d}\rho_{GW}}{\mathrm{d}\ln f} \approx 2.4\cdot 10^{-12}\alpha'\left(\frac{f}{\mathrm{GHz}}\right)\left(\frac{10^5\,\mathrm{g}}{M}\right)^{1/2}, \tag{22}$$

where α' is the coefficient at least of order of unity:

$$\alpha' = \left(\frac{v_{rel}}{0.1}\right)\left(\frac{\Delta_b}{10^5}\right)\left(\frac{N_{eff}}{100}\right)^{3/2}\left(\frac{100}{g_S(T_{BH})}\right)^{1/4}. \tag{23}$$

It may be much larger, if $\Delta_b \gg 10^5$.

More efficient mechanism of GW emission may be radiation from the PBH binaries, if their number in the high density clusters is sufficiently high. To form the binary bound state PBHs should sufficiently cool down losing their kinetic energy. The cooling could be achieved by the energy loss to the gravitational wave radiation discussed above and by the dynamical friction [14]. A particle moving in the cloud of other particles would transfer its energy to these particles due to their gravitational interaction. However, one should keep in mind that the case of dynamical friction is essentially different from the energy loss due to gravitational radiation. In the latter case the energy leaks out of the system cooling it down, while dynamical friction does not change the total energy of the cluster. Nevertheless a particular pair of black holes moving toward each other with acceleration may transmit their energy to the rest of the system and became gravitationally captured forming a binary.

The dynamical friction time was estimated in ref. [15]. In both cases $v > \sigma$ and $v < \sigma$, where σ is the velocity dispersion, the characteristic time was of the order of

$$\tau_{DF} \approx \left(\frac{\sigma}{0.1}\right)^3\left[\frac{25}{\ln(10^{-6}/\Omega_p)}\right]\left(\frac{100}{N_{eff}}\right)\left(\frac{M}{1\,\mathrm{g}}\right)\left(\frac{10^6}{\Delta}\right)\tau_{BH}\,. \tag{24}$$

For PBH masses below a few grams dynamical friction would be an efficient mechanism of PBH cooling leading to frequent binary formation. Moreover, dynamical friction could result in the collapse of small PBHs into much heavier black hole. Even the whole high density cluster of PBHs could form a single black hole. These processes of heavier black hole formation would be accompanied by a strong burst of gravitational radiation.

The emission of GWs from a binary results in the energy loss which is compensated by a decrease of the radius of the binary and of the rotation period. As a result the system goes into the so-called inspiral regime. Ultimately the two rotating bodies coalesce and produce a burst of the gravitational waves. To reach this stage the characteristic time of the coalescence should be shorter than the lifetime of the system. In our case it is the lifetime of PBH with respect to the evaporation. The coalescence time of the binary made of two BH with masses M_1 and M_2 can be easily calculated, see, *e.g.*, book [16]:

$$\tau_{co} = \frac{5R_0^4\, m_{Pl}^6}{256 M_1 M_2 (M_1+M_2)}\,, \tag{25}$$

where R_0 is the initial radius of the binary. This result is true for a circular orbit of the binary. In the case of elliptic orbit the eccentricity drops down due to GW emission and the system approaches to the circular one. We may use eq. (25) for an order of estimate of the lifetime of the binary.

There are two interesting limiting cases, when $\tau_{co} \gg \tau_{BH}$ and vice versa. In the first case the stationary orbit approximation is valid and each binary emits GWs with fixed frequency equal to twice the orbital frequency and the frequency spectrum is determined by the distribution of the binaries on their radius. As is shown in ref. [2], if the stationary regime was realized, the spectral density parameter today would be:

$$\Omega_{GW}^{(stat)}(f;t_0) \approx 10^{-8}\epsilon \left[\frac{N_{eff}}{100}\right]^{2/3} \left[\frac{100}{g_S(T(\tau_{BH}))}\right]^{1/18} \left[\frac{M}{10^5\,\mathrm{g}}\right]^{1/3} \left[\frac{f}{\mathrm{GHz}}\right]^{10/3}, \tag{26}$$

where ϵ is the fraction of binaries with respect to the total number of PBHs in the cluster and $g_S(T(\tau_{BH}))$ is the number of the entropy degrees of freedom at the moment of PBH evaporation when the plasma temperature was equal to $T(\tau_{BH})$. Here the possibly weak redshift dilution of GWs by the factor $(\tau_{co}/\tau_{BH})^{2/9}$ is neglected.

If the system goes to the inspiral phase, then we would expect today a continuous spectrum in the range from $f_{min} \sim 10^7$ Hz to $f_{max} \sim 3\cdot 10^{14}$ Hz. However if we take into account the redshift of the early formed binaries from the moment of their formation to the PBH decay, the lower value of the frequency may move to about 1 Hz.

PBHs could also directly produce gravitons by evaporation. The total energy emitted by BH per unit time and frequency ω (energy) of the emitted particles, is approximately given by the equation (see, *e.g.*, book [17]):

$$\left(\frac{\mathrm{d}E}{\mathrm{d}t\mathrm{d}\omega}\right) = \frac{2N_{eff}}{\pi}\,\frac{M^2}{m_{Pl}^4}\,\frac{\omega^3}{e^{\omega/T_{BH}}-1}\,, \tag{27}$$

where T is the BH temperature (2). Due to the impact of the gravitational field of BH on the propagation of the evaporated particles, their spectrum is distorted [4] by the so-called grey factor $g(\omega)$, but we disregard it in what follows.

The frequency spectrum of the evaporated gravitons is not thermal because of the different redshifts in the course of the evaporation. According to the calculations of ref. [2] the spectral density parameter of GWs at $t = \tau_{BH}$ is equal to

$$\Omega_{GW}(\omega_*;\tau_{BH}) \approx \frac{2.9\cdot 10^3 M^4 \omega_*^4}{\pi\, m_{Pl}^8}\, I\left(\frac{\omega_*}{T_{BH}}\right), \tag{28}$$

where

$$I\left(\frac{\omega_*}{T_{BH}}\right) = \int_0^{z_{max}} \frac{\mathrm{d}z\,(1+z)^{1/2}}{\exp\left[(z+1)\omega_*/T_{BH}\right]-1}\,, \tag{29}$$

and

$$1 + z_{max} = \left(\frac{\tau_{BH}}{t_{eq}}\right)^{2/3}\left(\frac{t_{eq}}{t_p}\right)^{1/2} = \left(\frac{32170}{N_{eff}}\right)^{2/3}\left(\frac{M}{m_{Pl}}\right)^{4/3}\Omega_p^{1/3}. \tag{30}$$

With respect to the thermal spectrum, spectrum (28) has more power at small frequencies due to redshift of higher frequencies into lower band and less power at high ω_*.

The spectral density of the evaporated gravitons today would be

$$\Omega_{GW}(f;t_0) = 2.7\cdot 10^{-27}\left(\frac{N_{eff}}{100}\right)^2\left(\frac{10^5\,\mathrm{g}}{M}\right)^2\left(\frac{f}{10^{10}\,\mathrm{Hz}}\right)^4\cdot I\left(\frac{2\pi\cdot f}{T_0}\right), \tag{31}$$

where T_0 is the BH temperature redshifted to the present time:

$$T_0 = 4.5\cdot 10^{15}\,\mathrm{Hz}\left(\frac{100}{g_S(T_{BH})}\right)^{1/12}\left(\frac{100}{N_{eff}}\right)^{1/2}\left(\frac{M}{10^5\,\mathrm{g}}\right)^{1/2}. \tag{32}$$

The mechanisms of GWs generation considered here could create quite high cosmological fraction of the energy density of the relic gravitational waves at very high frequencies. Unfortunately at the lower part of the spectrum Ω_{GW} significantly drops down making such GWs outside the reach of LISA or LIGO. Still the planned interferometers DECIGO/BBO and detectors based on the resonance graviton-photon transformation could be sensitive to the predicted high-frequency GWs.

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