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Coulomb law and energy levels in a superstrong magnetic field

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Summary. — The analytical expression for the Coulomb potential in the presence of a superstrong magnetic field is derived. The structure of hydrogen levels originating from LLL is analyzed.

PACS 11.10.Kk – Field theories in dimensions other than four.
PACS 30.31.J- – Relativistic and quantum electrodynamics (QED) effects in atoms, molecules, and ions.

1. – Introduction

The long awaited discovery of Higgs boson is planned during the next two years at LHC. For the first time what is called now the Higgs phenomenon was used in the Ginzburg-Landau phenomenological theory of superconductivity to expel the magnetic field from a superconductor.

Quite unexpectedly in the superstrong magnetic field a photon also gets a (quasi) mass. In this talk we have discussed this phenomenon and how it affects the atomic energy levels. The talk is based on papers [1].

In what follows the strong magnetic field is $B > m_e^2 e^3$; the superstrong magnetic field is $B > m_e^2/e^3$; the critical magnetic field is $B_{cr} = m_e^2/e$ and we use Gauss units: $e^2 = \alpha = 1/137$.

The Landau radius of an electron orbit in the magnetic field B is $a_H = 1/\sqrt{eB}$ and it is much smaller than the Bohr atomic radius for $B \gg e^3 m_e^2$. For such strong B electrons on Landau levels feel a weak Coulomb potential moving along the magnetic field. In [2] a numerical solution of the Schrödinger equation for a hydrogen atom in strong B was performed. According to this solution the ground level goes to $-\infty$ when B goes to $+\infty$. However, the photon mass leads to the Coulomb potential screening and the ground level remains finite at $B \to \infty$ [3]. Since the electron at the ground Landau level moves freely along the magnetic field, the problem resembles $D = 2$ QED and we will start our discussion from this theory.

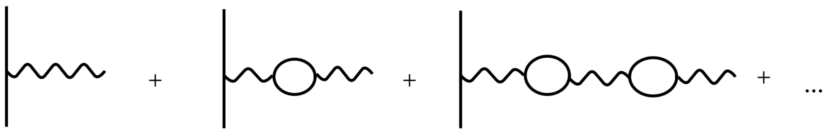

Fig. 1. – Modification of the Coulomb potential due to the dressing of the photon propagator.

2. – $D = 2$ **QED: screening of** Φ

The following equation for an electric potential of the point-like charge holds; see fig. 1:

$$\Phi(\bar{k}) \equiv A_0(\bar{k}) = \frac{4\pi g}{\bar{k}^2}\,; \qquad \boldsymbol{\Phi} \equiv \boldsymbol{A}_0 = D_{00} + D_{00}\Pi_{00}D_{00} + \dots\,. \tag{1}$$

Summing the series we get

$$\boldsymbol{\Phi}(k) = -\frac{4\pi g}{k^2 + \Pi(k^2)}\,, \quad \Pi_{\mu\nu} \equiv \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\Pi(k^2)\,, \tag{2}$$

$$\Pi(k^2) = 4g^2\left[\frac{1}{\sqrt{t(1+t)}}\ln(\sqrt{1+t} + \sqrt{t}) - 1\right] \equiv -4g^2 P(t)\,, \tag{3}$$

where $t \equiv -k^2/4m^2$, $[g] = \text{mass}$.

Taking $k = (0, k_\parallel)$, $k^2 = -k_\parallel^2$ for the Coulomb potential in the coordinate representation, we get

$$\boldsymbol{\Phi}(z) = 4\pi g \int_{-\infty}^{\infty} \frac{e^{ik_\parallel z}\mathrm{d}k_\parallel/2\pi}{k_\parallel^2 + 4g^2 P(k_\parallel^2/4m^2)}\,, \tag{4}$$

and the potential energy for the charges $+g$ and $-g$ is finally $V(z) = -g\boldsymbol{\Phi}(z)$.

The asymptotics of $P(t)$ are

$$P(t) = \begin{cases} \frac{2}{3}t, & t \ll 1\,, \\ 1, & t \gg 1\,. \end{cases} \tag{5}$$

Let us take as an interpolating formula for $P(t)$ the following expression:

$$\overline{P}(t) = \frac{2t}{3 + 2t}\,. \tag{6}$$

The accuracy of this approximation is not worse than 10% for the whole interval of t

variation, $0 < t < \infty$. Substituting an interpolating formula in (4) we get

$$\begin{aligned}(7)\qquad \mathbf{\Phi} &= 4\pi g \int_{-\infty}^{\infty} \frac{e^{ik_\parallel z}\mathrm{d}k_\parallel/2\pi}{k_\parallel^2 + 4g^2(k_\parallel^2/2m^2)/(3+k_\parallel^2/2m^2)} \\ &= \frac{4\pi g}{1+2g^2/3m^2}\int_{-\infty}^{\infty}\left[\frac{1}{k_\parallel^2} + \frac{2g^2/3m^2}{k_\parallel^2+6m^2+4g^2}\right] e^{ik_\parallel z}\frac{\mathrm{d}k_\parallel}{2\pi} \\ &= \frac{4\pi g}{1+2g^2/3m^2}\left[-\frac{1}{2}|z| + \frac{g^2/3m^2}{\sqrt{6m^2+4g^2}}\exp\left(-\sqrt{6m^2+4g^2}|z|\right)\right].\end{aligned}$$

In the case of heavy fermions ($m \gg g$) the potential is given by the tree level expression; the corrections are suppressed as g^2/m^2.

In the case of light fermions ($m \ll g$)

$$(8)\qquad \mathbf{\Phi}(z)|_{m\ll g} = \begin{cases} \pi e^{-2g|z|}\,, & z \ll \frac{1}{g}\ln(\frac{g}{m})\,, \\ -2\pi g\left(\frac{3m^2}{2g^2}\right)|z|\,, & z \gg \frac{1}{g}\ln(\frac{g}{m})\,, \end{cases}$$

$m = 0$ corresponds to the Schwinger model; the photon gets a mass.

Light fermions make the transition from $m > g$ to $m = 0$ continuous.

3. – $D = 4$ QED

In order to find the potential of a point-like charge we need an expression for P in strong B. One starts from the electron propagator G in strong B. The solutions of the Dirac equation in the homogeneous constant in time B are known, so one can write the spectral representation of the electron Green function. The denominators contain $k^2 - m^2 - 2neB$, and for $B \gg m^2/e$ and $k_\parallel^2 \ll eB$ in sum over levels the lowest Landau level (LLL, $n = 0$) dominates. In the coordinate representation a transverse part of LLL wave function is $\Psi \sim \exp((-x^2 - y^2)eB)$ which in the momentum representation gives $\Psi \sim \exp((-k_x^2 - k_y^2)/eB)$ (we suppose that B is directed along the z-axis).

Substituting the electron Green functions we get the expression for the polarization operator in superstrong B.

For $B \gg B_{cr}$, $k_\parallel^2 \ll eB$ the following expression is valid [4]:

$$\begin{aligned}(9)\qquad \Pi_{\mu\nu} &\sim e^2 eB \int \frac{\mathrm{d}q_x \mathrm{d}q_y}{eB}\exp\left(-\frac{q_x^2+q_y^2}{eB}\right) \\ &\quad * \exp\left(-\frac{(q+k)_x^2 + (q+k)_y^2}{eB}\right)\mathrm{d}q_0\mathrm{d}q_z\gamma_\mu \frac{1}{\hat{q}_{0,z} - m}(1 - i\gamma_1\gamma_2)\gamma_\nu \\ &\quad * \frac{1}{\hat{q}_{0,z} + \hat{k}_{0,z} - m}(1 - i\gamma_1\gamma_2) = e^3 B * \exp\left(-\frac{k_\perp^2}{2eB}\right) * \Pi_{\mu\nu}^{(2)}(k_\parallel \equiv k_z)\,.\end{aligned}$$

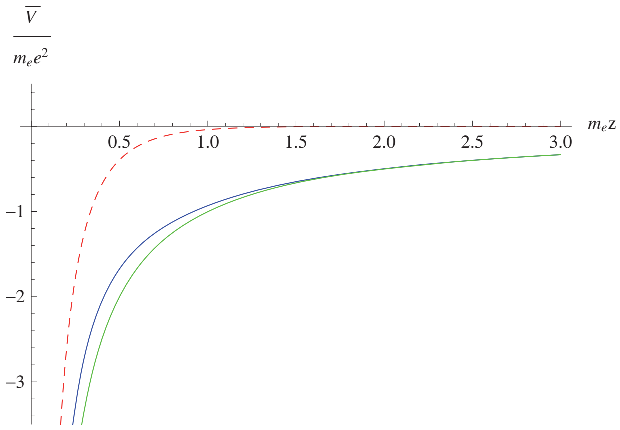

Fig. 2. – (Colour on-line) A modified Coulomb potential at $B = 10^{17}$G (blue, dark solid) and its long distance (green, pale solid) and short distance (red, dashed) asymptotics.

With the help of it, the following result was obtained in [1]:

$$\mathbf{\Phi}(k) = \frac{4\pi e}{k_\parallel^2 + k_\perp^2 + \frac{2e^3 B}{\pi}\exp\left(-\frac{k_\perp^2}{2eB}\right) P\left(\frac{k_\parallel^2}{4m^2}\right)}, \tag{10}$$

$$\mathbf{\Phi}(z) = 4\pi e \int \frac{e^{ik_\parallel z} \mathrm{d}k_\parallel \mathrm{d}^2 k_\perp/(2\pi)^3}{k_\parallel^2 + k_\perp^2 + \frac{2e^3 B}{\pi}\exp(-k_\perp^2/(2eB))(k_\parallel^2/2m_e^2)/(3 + k_\parallel^2/2m_e^2)} \tag{11}$$
$$= \frac{e}{|z|}\left[1 - e^{-\sqrt{6m_e^2}|z|} + e^{-\sqrt{(2/\pi)e^3 B + 6m_e^2}|z|}\right].$$

For the magnetic fields $B \ll 3\pi m^2/e^3$ the potential is Coulomb up to small power suppressed terms:

$$\mathbf{\Phi}(z)\left|_{e^3 B \ll m_e^2} = \frac{e}{|z|}\left[1 + O\left(\frac{e^3 B}{m_e^2}\right)\right]\right., \tag{12}$$

in full accordance with the $D = 2$ case, $e^3 B \to g^2$.

In the opposite case of the superstrong magnetic fields $B \gg 3\pi m_e^2/e^3$ we get

$$\mathbf{\Phi}(z) = \begin{cases} \frac{e}{|z|} e^{(-\sqrt{(2/\pi)e^3 B}|z|)}, & \frac{1}{\sqrt{(2/\pi)e^3 B}} \ln\left(\sqrt{\frac{e^3 B}{3\pi m_e^2}}\right) > |z| > \frac{1}{\sqrt{eB}}, \\ \frac{e}{|z|}(1 - e^{(-\sqrt{6m_e^2}|z|)}), & \frac{1}{m} > |z| > \frac{1}{\sqrt{(2/\pi)e^3 B}} \ln\left(\sqrt{\frac{e^3 B}{3\pi m_e^2}}\right), \\ \frac{e}{|z|}, & |z| > \frac{1}{m}, \end{cases} \tag{13}$$

$$\bar{V}(z) = -e\mathbf{\Phi}(z). \tag{14}$$

In fig. 2 the plot of a Coulomb potentiala modified by the superstrong B as well as its short- and long-distance asymptotics are presented.

4. – Electron in the magnetic field

The spectrum of the Dirac equation in the homogeneous magnetic field constant in time is given by [5]

$$\varepsilon_n^2 = m_e^2 + p_z^2 + (2n + 1 + \sigma_z)eB\,, \tag{15}$$

$n = 0, 1, 2, 3, \ldots;\quad \sigma_z = \pm 1$.

For $B > B_{cr} \equiv m_e^2/e$ the electrons are relativistic with only one exception: the electrons from the lowest Landau level (LLL, $n = 0,\quad \sigma_z = -1$) can be nonrelativistic. In what follows we will find the spectrum of electrons from LLL in the screened Coulomb field of the proton.

The spectrum of the Schrödinger equation in cylindrical coordinates (ρ, z) is [6]

$$E_{p_z n_\rho m \sigma_z} = \left(n_\rho + \frac{|m| + m + 1 + \sigma_z}{2}\right)\frac{eB}{m_e} + \frac{p_z^2}{2m_e}\,, \tag{16}$$

LLL: $n_\rho = 0, \sigma_z = -1, m = 0, -1, -2, \ldots$,

$$R_{0m}(\rho) = \left[\pi(2a_H^2)^{1+|m|}(|m|!)\right]^{-1/2} \rho^{|m|} e^{(im\varphi - \rho^2/(4a_H^2))}\,. \tag{17}$$

Now we should take into account the electric potential of the atomic nuclei situated at $\rho = z = 0$. For $a_H \ll a_B$ the adiabatic approximation is applicable and the wave function in the following form should be looked for:

$$\Psi_{n0m-1} = R_{0m}(\rho)\chi_n(z)\,, \tag{18}$$

where $\chi_n(z)$ is the solution of the Schrödinger equation for electron motion along the magnetic field

$$\left[-\frac{1}{2m}\frac{\mathrm{d}^2}{\mathrm{d}z^2} + U_{eff}(z)\right]\chi_n(z) = E_n\chi_n(z)\,. \tag{19}$$

Without screening the effective potential is given by the following formula:

$$U_{eff}(z) = -e^2\int\frac{|R_{0m}(\rho)|^2}{\sqrt{\rho^2 + z^2}}\mathrm{d}^2\rho\,, \tag{20}$$

For $|z| \gg a_H$ the effective potential equals the Coulomb one

$$U_{eff}(z)\,|_{z\gg a_H} = -\frac{e^2}{|z|} \tag{21}$$

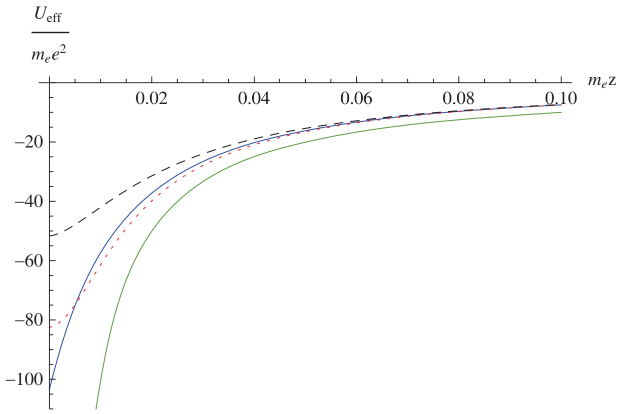

Fig. 3. – (Colour on-line) Effective potential with screening for $m = 0$ (dark solid (blue) curve) and $m = -1$ (long-dashed curve), (24); simplified potential (short-dashed (red) curve) (25). The curves correspond to $B = 3 \times 10^{17}$G. The Coulomb potential (pale solid (green)) is also shown.

and it is regular at $z = 0$

$$U_{eff}(0) \sim -\frac{e^2}{|a_H|} . \tag{22}$$

Since $U_{eff}(z) = U_{eff}(-z)$, the wave functions are odd or even under the reflection $z \to -z$; the ground states (for $m = 0, -1, -2, \ldots$) are described by the even wave functions. The energies of the odd states are

$$E_{\text{odd}} = -\frac{m_e e^4}{2n^2} + O\left(\frac{m_e^2 e^3}{B}\right) , \quad n = 1, 2, \ldots . \tag{23}$$

So, for the superstrong magnetic fields $B > m_e^2/e^3$ they coincide with the Balmer series.

5. – Energies of even states: screening

When screening is taken into account the expression for the effective potential transforms into [1]

$$\tilde{U}_{eff}(z) = -e^2 \int \frac{|R_{0m}(\vec{\rho})|^2}{\sqrt{\rho^2 + z^2}} \mathrm{d}^2\rho \left[1 - e^{-\sqrt{6m_e^2}\, z} + e^{-\sqrt{(2/\pi)e^3 B + 6m_e^2}\, z}\right] . \tag{24}$$

For $m = 0$ the following simplified formula can be used:

$$U_{simpl}(z) = -e^2 \frac{1}{\sqrt{a_H^2 + z^2}} \left[1 - e^{-\sqrt{6m_e^2}\, z} + e^{-\sqrt{(2/\pi)e^3 B + 6m_e^2}\, z}\right] . \tag{25}$$

In fig. 3 the plots of the effective potentials for $m = 0$ and $m = -1$ are presented.

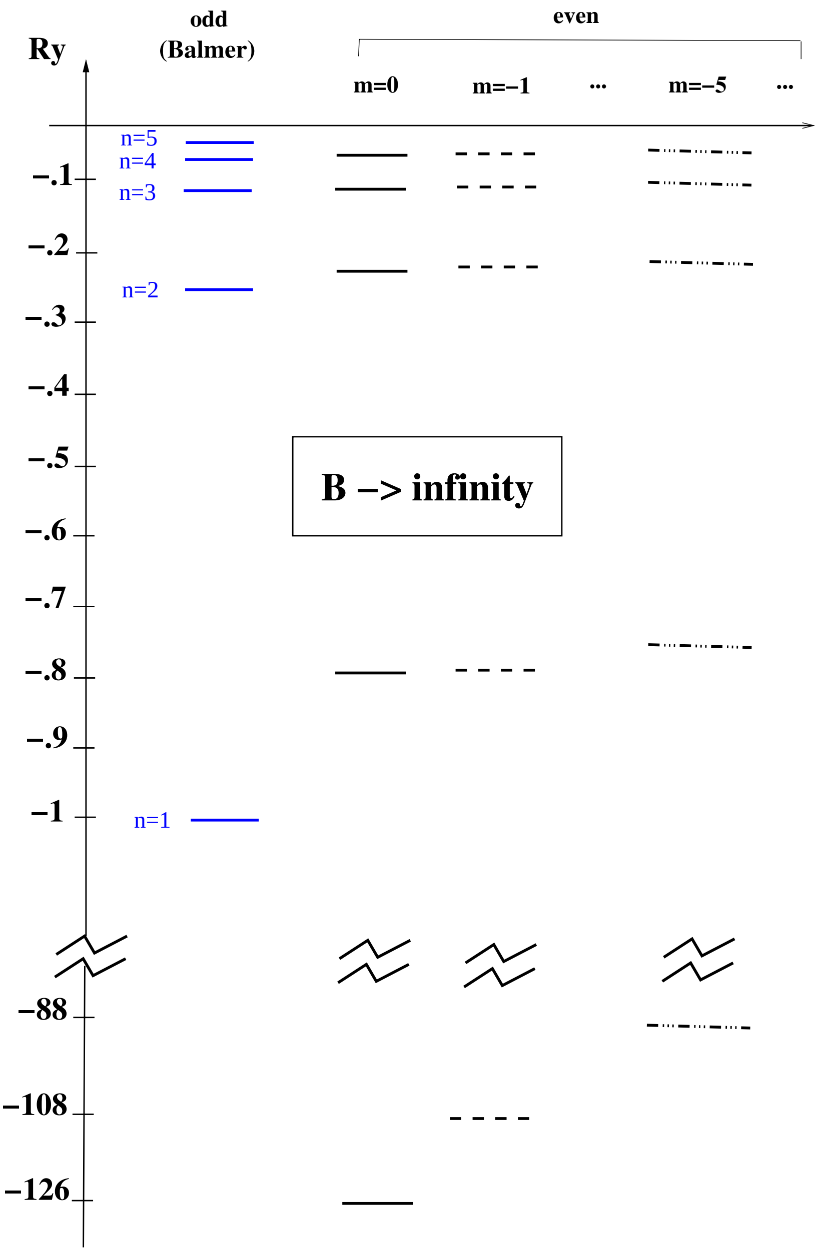

Fig. 4. – Spectrum of the hydrogen levels in the limit of the infinite magnetic field. Energies are given in Rydberg units, $Ry \equiv 13.6\,\mathrm{eV}$.

6. – Karnakov-Popov equation

It provides a several percent accuracy for the energies of even states for $H > 10^3$ ($H \equiv B/(m_e^2 e^3)$), see [7].

The main idea is to integrate the Shrödinger equation with the effective potential from $x = 0$ till $x = z$, where $a_H \ll z \ll a_B$ and to equate the obtained expression for $\chi'(z)$ to the logarithmic derivative of Whittaker function—the solution of Shrödinger equation with Coulomb potential, which exponentially decreases at $z \gg a_B$

$$\begin{aligned} 2\ln\left(\frac{z}{a_H}\right) + \ln 2 - \psi(1+|m|) + O(a_H/z) = \\ 2\ln\left(\frac{z}{a_B}\right) + \lambda + 2\ln\lambda + 2\psi\left(1 - \frac{1}{\lambda}\right) + 4\gamma + 2\ln 2 + O(z/a_B)\,, \end{aligned} \tag{26}$$

where $\psi(x)$ is the logarithmic derivative of the gamma-function and

$$E = -(m_e e^4/2)\lambda^2\,. \tag{27}$$

The modified KP equation, which takes screening into account, looks like [1]

$$\ln\left(\frac{H}{1+\frac{e^6}{3\pi}H}\right) = \lambda + 2\ln\lambda + 2\psi\left(1-\frac{1}{\lambda}\right) + \ln 2 + 4\gamma + \psi(1+|m|)\,. \tag{28}$$

The spectrum of the hydrogen atom in the limit $H \to \infty$ is shown in fig. 4.

7. – Conclusions

- Atomic energies at superstrong B is the only known (for me) case when the radiative "correction" determines the energy of states.
- The analytical expression for the charged particle electric potential in $d = 1$ is given; for $m < g$ screening takes place at all distances.
- The analytical expression for the charged particle electric potential at superstrong B in $d = 3$ is found; screening takes place at the distances $|z| < 1/m_e$.
- An algebraic formula for the energy levels of a hydrogen atom originating from the lowest Landau level in superstrong B has been obtained.

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