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Transverse angular momentum: New results

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Summary. — We describe briefly the two methods of deriving angular-momentum sum rules, the quick way using the rotational properties of states and the cumbersome way utilizing the energy momentum tensor. Though cumbersome, the latter approach allows us to derive a new relation between the expectation value of the transverse component of the Belinfante version of the angular momentum $\langle J_T^{bel} \rangle$ of a quark in a transversely polarized nucleon in terms of the Generalized Parton Distributions H and E, namely $\langle J_T^{bel}(\text{quark}) \rangle = \frac{1}{2M} [P_0 \int_{-1}^{1} \mathrm{d}x x E_q(x,0,0) + M \int_{-1}^{1} \mathrm{d}x x H_q(x,0,0)]$, where P_0 is the energy of the nucleon and where "quark" implies the sum of quark and antiquark of a given flavor. A similar relation holds for gluons. The result is remarkably similar to Ji's relation for the case of longitudinal polarization.

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1. – Derivation of a sum rule

There are two steps:

1) Derive an expression for

$$\langle \text{Nucleon}; P, S \,|\, \boldsymbol{J} \,|\text{Nucleon}; P, S \rangle \tag{1}$$

in terms of P and S

2) Express $|\text{Nucleon}; P, S \rangle$ as a Fock expansion in terms of the constituents of the nucleon.

1˙1. *The super-quick approach.* – We know what a *rotation* does to a state, so we know matrix elements of R. But, *e.g.*

$$R_z(\beta) = e^{-i\beta \boldsymbol{J}_z}, \tag{2}$$

so that we get the matrix element of J_z using

$$\boldsymbol{J}_z = i\frac{\mathrm{d}}{\mathrm{d}\beta} R_z(\beta)\big|_{\beta=0}. \tag{3}$$

1˙2. *The approach via the energy momentum tensor.* – Typically the angular momentum density involves the Belinfante form of the energy-momentum tensor density $t^{\mu\nu}_{bel}(x)$ in the form, *e.g.*,

$$\boldsymbol{J}_z = \boldsymbol{J}^3 = \int \mathrm{d}V \left[x t^{02}_{bel}(x) - y t^{01}_{bel}(x) \right]. \tag{4}$$

The factors x, y cause trouble. One ends up with things like $\int \mathrm{d}V x \langle P, S | t^{02}_{bel}(0) | P, S \rangle$

The matrix element is independent of x so we are faced with $\int \mathrm{d}V x = \infty$? or $= 0$? Totally ambiguous!

The problem is an old one: In ordinary QM plane wave states give infinities. The solution is an old one: Build a wave packet, a superposition of physical plane-wave states...... but.... it is a long, complicated calculation. Both approaches give the same result for the nucleon's angular momentum

$$\langle \text{Nucleon}; P, S \,|\, \boldsymbol{J} \,|\text{Nucleon}; P, S \,\rangle = \frac{1}{2}\boldsymbol{s} + \text{delta function}, \tag{5}$$

where $\boldsymbol{s}$ is the *rest frame spin vector.*

A KEY POINT: *This result is independent of whether* $\boldsymbol{s}$ *is longitudinal or transverse.*

Although painful, the traditional approach is fruitful, because it connects matrix elements of $\boldsymbol{J}$ with matrix elements of the energy momentum tensor $t^{\mu\nu}$.

The most general form of the matrix elements of $t^{\mu\nu}_{bel}$, say for quarks, is (similar for gluons)

$$\begin{aligned} \langle\, P', S' \,|\, t^{\mu\nu}_{bel}(\text{quark}; 0) \,|\, P, S \,\rangle &= [\bar{u}'\gamma^\mu u \,\bar{P}^\nu + (\mu \leftrightarrow \nu)]\mathbb{D}_q(\Delta^2)/2 \\ &+ \left[\frac{i\Delta_\rho}{4M}\,\bar{u}'\sigma^{\mu\rho}u\,\bar{P}^\nu + (\mu \leftrightarrow \nu)\right][2\mathbb{S}_q(\Delta^2) - \mathbb{D}_q(\Delta^2)] \\ &+ \frac{\bar{u}'u}{2M}\left[\frac{1}{2}[\mathbb{G}_q(\Delta^2) - \mathbb{H}_q(\Delta^2)](\Delta^\mu\Delta^\nu - \Delta^2 g^{\mu\nu}) + M^2\mathbb{R}_q(\Delta^2)g^{\mu\nu}\right], \end{aligned} \tag{6}$$

where $\mathbb{D}$, $\mathbb{S}$, $\mathbb{G}$, $\mathbb{H}$ are scalar form factors, and

$$u \equiv u(P, S) \qquad u' \equiv u(P', S') \qquad \Delta = P' - P. \tag{7}$$

2. – Connection with generalized parton distributions

Comparing with the definition of GPDs [1] one finds

$$\int_{-1}^{1} \mathrm{d}x x H_q(x,0,0) = \mathbb{D}_q = \text{momentum fraction carried by quarks.} \tag{8}$$

Further one sees that

$$\int_{-1}^{1} \mathrm{d}x x E_q(x,0,0) = (2\,\mathbb{S}_q - \mathbb{D}_q). \tag{9}$$

From eqs. (8), (9) one has that

$$\int_{-1}^{1} \mathrm{d}x x H_q(x,0,0) + \int_{-1}^{1} \mathrm{d}x x E_q(x,0,0) = 2\,\mathbb{S}_q. \tag{10}$$

3. – Connection with angular momentum: old and new results

We now show how the expectation values of $\boldsymbol{J}$ are related to the GPDs.

3·1. *Longitudinally polarized nucleon.* – For the case of a *longitudinally* polarized nucleon moving in the z-direction Bakker, Leader and Trueman (BLT) [2] proved that $\mathbb{S}$ measures the expectation value of the z-component of $\boldsymbol{J}$. Hence eq. (10) can be written

$$\frac{1}{2}\int_{-1}^{1} \mathrm{d}x x [H_q(x,0,0) + E_q(x,0,0)] = \langle\, J_z^{bel}(\text{quark})\,\rangle, \tag{11}$$

which is the relation first derived by Ji [3].

Now as mentioned above $\int_{-1}^{1} \mathrm{d}x x H_q(x,0,0)$ measures the fraction of the nucleon's momentum carried by quarks and antiquarks of a given flavour, so that adding the gluon contribution([1])

$$\sum_{flavours} \int_{-1}^{1} \mathrm{d}x x H_q(x,0,0) + \int_{0}^{1} \mathrm{d}x x H_G(x,0,0) = 1. \tag{12}$$

Hence, summing eq. (11) over flavors and adding the analogous equation for gluons, one obtains

$$\begin{aligned} \frac{1}{2} + \sum_{flavors} \int_{-1}^{1} \mathrm{d}x x E_q(x,0,0) + \\ \int_{0}^{1} \mathrm{d}x x E_G(x,0,0) &= \sum_{flavours} \langle\, J_z^{bel}(\text{quark})\,\rangle + \langle\, J_z^{bel}(\text{gluon})\,\rangle \\ &= \frac{1}{2} \end{aligned} \tag{13}$$

([1]) For gluons the integrals run from 0 to 1.

so that

$$\sum_{flavors} \int_{-1}^{1} \mathrm{d}x x E_q(x,0,0) + \int_{0}^{1} \mathrm{d}x x E_G(x,0,0) = 0. \tag{14}$$

This fundamental sum rule has wide ramifications and can be shown to correspond to the vanishing of the nucleon's anomalous gravitomagnetic moment.

3˙2. *Transversely polarized nucleon.* – For the case of a *transversely* polarized nucleon, moving along the positive z-axis, it follows from BLT that

$$\langle\, J_T^{bel}(\mathrm{quark})\,\rangle = \frac{1}{2M}\,\left[P_0\,(2\,\mathbb{S}_q - \mathbb{D}_q) + M\,\mathbb{D}_q\right] \tag{15}$$

Substituting eqs. (8), (9) we obtain the new result

$$\langle\, J_T^{bel}(\mathrm{quark})\,\rangle = \frac{1}{2M}\left[P_0 \int_{-1}^{1} \mathrm{d}x x E_q(x,0,0) + M \int_{-1}^{1} \mathrm{d}x x H_q(x,0,0)\right], \tag{16}$$

where P_0 is the energy of the nucleon.

The factor P_0 may seem unintuitive. However if we go the rest frame eq. (16) reduces to the Ji result eq. (11), as it should, since in the rest frame there is no distinction between x and z directions. Moreover, if one calculates the orbital angular momentum in a simple classical model of a quark rotating about the center of the nucleon at rest, and then boosts the system one finds that the transverse angular momentum grows like P_0. Finally, if one sums eq. (16) over flavors and adds the analogous gluon equation, one finds that the term proportional to P_0 disappears as a consequence of eq. (14), and using eq. (12), one obtains the correct result

$$\sum_{flavors} \langle\, J_T^{bel}(\mathrm{quark})\,\rangle + \langle\, J_T^{bel}(\mathrm{gluon})\,\rangle = \frac{1}{2}\,. \tag{17}$$

The relation eq. (16) can be used to test model results and, possibly, lattice calculations.

Now BLT derived a sum rule for the angular momentum of a transversely polarized nucleon, namely

$$\frac{1}{2} = \frac{1}{2} \sum_{\text{flavours}} \int \mathrm{d}x\,[\Delta_T q(x) + \Delta_T \bar{q}(x)] + \sum_{q,\,\bar{q},\,G} \langle L_T \rangle, \tag{18}$$

where $\Delta_T q(x) \equiv h_1(x)$ is the quark transversity distribution. Note that here there is a *sum* over quark and antiquark transversities, so this quantity is *not* the expectation value of a *local* operator and is *not* related to the nucleon's tensor charge.

In this context it is important to realize that the quark part of eq. (18), *i.e.*

$$\frac{1}{2} \sum_{\text{flavours}} \int \mathrm{d}x\,[\Delta_T q(x) + \Delta_T \bar{q}(x)] + \sum_{q,\,\bar{q}} \langle L_T \rangle, \tag{19}$$

cannot be identified with $\langle J_T^{bel}(\text{quark}) \rangle$ in eq. (16). The reason is the following. While for the *total* angular momentum there is no difference between Belinfante and canonical angular momentum, *i.e.*

$$\langle J_T^{bel}(\text{total}) \rangle = \langle J_T^{can}(\text{total}) \rangle, \tag{20}$$

this is not true for the separate quark and gluon pieces, *i.e.*

$$\langle J_T^{bel}(\text{quark}) \rangle \neq \langle J_T^{can}(\text{quark}) \rangle \tag{21}$$

and in deriving eq. (18) BLT used the property that $\boldsymbol{J}$ is the generator of rotations. As explained in detail in [4] it is the canonical versions of the operators, $\boldsymbol{J}_{can}$, which are the generators of rotations. Thus the expression in eq. (19) corresponds to $\langle J_T^{can}(\text{quark}) \rangle$ and should not be confused with $\langle J_T^{bel}(\text{quark}) \rangle$.

In summary I have derived a rigorous relation between the expectation value of the Belinfante version of the transverse angular momentum carried by quarks in a transversely polarized nucleon and the generalized parton distributions H and E, which is closely analogous to Ji's relation for the longitudinal component of the quark angular momentum in a longitudinally polarized nucleon. Neither relation is a genuine sum rule, but both offer interesting possibilities for testing models and lattice calculations.

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