IL NUOVO CIMENTO Vol. 36 C, N. 3 Maggio-Giugno 2013
DOI 10.1393/ncc/i2013-11520-8

Colloquia: MSQS 2012

Singular potentials: Confinement and Riemann hypothesis

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ricevuto l'8 Febbraio 2013

Summary. — We analyze the role of singular potentials in quantum mechanics and field theory. In particular, we focus on the conformal invariant potential $V(x) = 1/x^2$ which governs interesting physical phenomena like Efimov effect in atomic and nuclear compounds, quark confinement in QCD, scaling dimensions in the AdS/CFT correspondence, impurity effects in graphene and even models of the Pólya-Hilbert approach to the Riemann hypothesis.

PACS 12.38.Aw – General properties of QCD (dynamics, confinement, etc.).
PACS 73.22.Pr – Electronic structure of graphene.
PACS 02.10.De – Algebraic structures and number theory.
PACS 11.25.Tq – Gauge/string duality.

1. – Introduction

One of the main technical differences between field theory and standard quantum mechanics is the appearance of ultraviolet divergences and the need of a renormalization mechanism of the coupling constants. However, there are some quantum-mechanical systems with singular interactions which exhibit similar ultraviolet behaviors to those of quantum field theories and involve renormalization prescriptions. Among these systems there is a very special one which to some extent provides a paradigm: conformal quantum mechanics. The system describes the motion of a free particle in a $1/x^2$ potential. The systems is classically conformally invariant but its quantization involves a renormalization mechanism which in some case breaks conformal symmetry.

This very simple quantum system exhibits some phenomena which describe interesting physical effects: the Efimov effect in nuclear physics, the Gribov confinement mechanism in gauge theories, the behavior of charged impurites in graphene, field theory in anti-deSitter space times and the Pólya-Hilbert conjecture on the Riemann hypothesis.

The Efimov effect is the striking appearance of an infinity of bound states with energies in a geometric sequence $E_{n+1} = aE_n$ in some nuclear and molecular compounds of three bodies. The confinement of quarks in QCD has been numerically revealed but there is not an analytic understanding of the phenomenon. The Gribov approach to confinement from first principles is based on the existence of instabilities in the effective potential induced by the presence of heavy quarks. The relativistic behavior of electrons in a graphene plate with charged impurites can also be described in terms of the $1/x^2$ conformal potential. Due to the conformal anomaly the system becomes unstable for strong values of the impurity charges. Another interesting system with conformal symmetry is the Calogero-Moser system which is an integrable system that attracted a lot of attention in the last years.

The list of physical systems where the potential $1/x^2$ play a fundamental role is expanding very fast and currently is already very impresive. In this note we will only review a selected list of applications of conformal quantum mechanics.

2. – Conformal quantum mechanics

The simplest quantum system with singular potential is [1, 2]

$$H = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}\frac{g}{x^2}\,. \tag{1}$$

The $1/x^2$ potential appears as centrifugal term in two body problems with central potential. It also appears in 3-body problems in nuclear physics [3] and in N body problems with conformal couplings like the Calogero-Moser systems [4, 5].

The pathological behavior show up in the fact H is a symmetric operator on functions vanishing at the origin but it is not self-adjoint.

The Hamiltonian H can be made self-adjoint with different choices of boundary conditions at the singularity. There are three different regimes depending on the strength of the coupling constant g [6-8]. If $g > \frac{3}{4}$ the Hamiltonian is essentially self-adjoint and has a unique self-adjoint extension with a continuum positive spectrum. All the states with finite energy vanish at the origin. If $-\frac{1}{4} < g < \frac{3}{4}$ there are many self-adjoint extensions which correspond to different boundary conditions

$$\lim_{x\to 0} 2x\psi'(x) = \lim_{x\to 0}\left(1 + 2\nu \coth[\nu \log(\Lambda\, x)]\right)\psi(x), \tag{2}$$

where $\nu = \sqrt{1/4 + g}$. The arbitrary parameter Λ introduced by the boundary condition (2) corresponds to a renormalization scale which breaks conformal invariance. In this case apart from the continuum spectrum with positive energies arises a bound state

$$\psi_0(x) = K_\nu(\sqrt{2|E_0|}x)(\sqrt{2|E_0|}x)^{\frac{1}{2}} \tag{3}$$

with negative energy

$$E_0 = -2\Lambda^2\left(\frac{\Gamma(1+\nu)}{\Gamma(1-\nu)}\right)^{\frac{1}{\nu}}\,. \tag{4}$$

Finally, if $g < -\frac{1}{4}$ there is a family of self-adjoint extensions

$$\lim_{x\to 0}\left(1 + 2i\nu \cot[i\nu \log(\Lambda\, x)]\right)\psi_n(x) = \lim_{x\to 0} 2r\psi_n'(x) \tag{5}$$

with an infinity of bound states

$$\psi_n(x) = K_\nu(\sqrt{2|E_n|}x)(\sqrt{2|E_n|}x)^{-\frac{1}{2}} \tag{6}$$

with negative energies

$$E_n = -2\Lambda^2 \exp\left(\frac{2\pi n}{\nu} i + \frac{1}{\nu}\log\frac{\Gamma(1+\nu)}{\Gamma(1-\nu)}\right). \tag{7}$$

In this case, conformal invariance is again broken by the parameter Λ introduced by the boundary condition (5) but not completely since a discrete conformal symmetry is preserved. The boundary condition and the spectrum are invariant under the discrete rescaling of $\Lambda \to \Lambda\ e^{2\pi i/\nu}$ [8].

3. – Gribov's picture of confinement

The standard picture of confinement is provided by the dual superconductor scenario where the QCD vacuum behaves like a dual superconductor generated by the condensation of chromo-magnetic monopoles. The chromo-electric flux is expelled in that vacuum by the dual Meissner effect [9, 10]. A heavy quark-antiquark pair in such a magnetic superconducting vacuum generates a concentration of the chromo-electric flux lines around the segment connecting the two particles. This implies that the effective quark-antiquark potential grows linearly with the distance. At large distances the quark-antiquark flux tube behaves like a string which leads to quark confinement. The dual superconductor picture has been numerically confirmed but an analytic proof from first principles is lacking.

An alternative picture for confinement was suggested by Gribov [11, 12], motivated by the instability of relativistic hydrogenoid atoms with $Z > 137$. Gribov raised the possibility of a QCD vacuum instability due to the very large values that the effective α_s coupling constant can reach at the infrared regime. The instability generates a vacuum decay on light quark-antiquark pairs [13, 14]. In fact, the Gribov picture can be derived from first principles for heavy quarks [27]. In presence of a static heavy quark the Yang-Mills action is given by

$$S^{YM}(A) = -\frac{1}{2g^2}\int \mathrm{d}^4x\, Tr(F^{\mu\nu}F_{\mu\nu}) + Q\int \mathrm{d}x^0\, A_0^3(0), \tag{8}$$

where the quark color has been chosen along the third component of Gell-Mann matrices for simplicity. The Euclidean functional integral is dominated by the static Coulomb solutions of Euclidean Yang-Mills equations

$$\vec{A} = 0, \quad A_0^3(x) = i\frac{g^2 Q}{4\pi|\vec{x}|} = i\frac{\alpha}{|\vec{x}|}, \quad \alpha = \frac{g^2 Q}{4\pi}\,. \tag{9}$$

The Gaussian approximation around these Coulomb backgrounds is given by the second-order variation of the Euclidean action

$$\delta^2 S = -\int \mathrm{d}^4x\, Tr\ \tau^\mu(-\delta_{\mu\nu}D^2 + D_\mu D_\nu - 2[F_{\mu\nu}\cdot\,])\tau^\nu. \tag{10}$$

If the second-order differential operator involved in (10) is positive the functional integral reduces to the inverse square root of its determinant. However, if the operator is non-positive their negative eigenvectors will give rise to vacuum instabilities. Having in mind the dual superconductor picture of QCD vacuum the search for vacuum instabilities can be restricted to pure static magnetic gauge field perturbations

$$\vec{\tau}(x) = \frac{\vec{x}\times\vec{\rm n}}{|\vec{x}|}\,\phi(\vec{x})\,{\rm T}_{12}, \quad \tau_0 = 0,$$

with negative eigenvalues of the second-order variation operator

$$(-\delta_{\mu\nu}D^2 + D_\mu D_\nu - 2[F_{\mu\nu},\cdot])\tau^\nu = -\lambda^2\tau_\mu, \tag{11}$$

where $\vec{\rm n}$ is any unit vector and ${\rm T}_{12}$ is any normalized linear combination of the first two components of Gell-Mann matrices (${\rm T}_{12}^2 = -1/4$).

In spherical coordinates if we assume that $\phi(\vec{x}) = \phi(r)$ with $r = |\vec{x}|$ the eigenvalue equation (11) becomes

$$\left(\frac{{\rm d}^2}{{\rm d}r^2} + \frac{2}{r}\frac{\rm d}{{\rm d}r} - \frac{2-\alpha^2}{r^2}\right)\phi(r) = \lambda^2\phi(r). \tag{12}$$

This equation is similar to the eigenvalue equation of the Hamiltonian (1) and also presents three different regimes depending on the strength of the coupling constant α [1]. If $\alpha^2 < \frac{5}{4}$ there is no solution of (12) vanishing on the quark and in the absence of negative eigenvalues the Gaussian integral is convergent and the system is stable. If $\frac{5}{4} < \alpha^2 < \frac{9}{4}$ there is one solution of (12) with $\lambda_0 = 2\Lambda(\frac{\Gamma(1+\nu)}{\Gamma(1-\nu)})^{\frac{1}{2\nu}}$ and satisfying the boundary condition

$$\lim_{r\to 0} 2r\phi'(r) = \lim_{r\to 0}\left(-1 + 2\nu\coth[\nu\log(\Lambda\, r)]\right)\phi(r),$$

where $\nu = \sqrt{9/4-\alpha^2}$. The arbitrary parameter Λ introduced by the boundary condition (13) in order to guarantee the Hermiticity of the second-order variation operator of the Euclidean action breaks conformal invariance and is crucial for the existence of the negative eigenvalue

$$-\lambda_0^2 = -4\Lambda^2\left(\frac{\Gamma(1+\nu)}{\Gamma(1-\nu)}\right)^{\frac{1}{\nu}}. \tag{13}$$

If $\alpha^2 > \frac{9}{4}$ there is an infinity of solutions of (12) with negative eigenvalues

$$-\lambda_n^2 = -4\Lambda^2\exp\left(\frac{2\pi n i}{\nu} + \frac{1}{\nu}\log\frac{\Gamma(1+\nu)}{\Gamma(1-\nu)}\right),$$

satisfying the boundary condition

$$\lim_{r\to 0}\left(-1 + 2i\nu\cot[i\nu\log(\Lambda\, r)]\right)\phi_n(r) = \lim_{r\to 0} 2r\phi_n'(r). \tag{14}$$

This is a case of extreme instability of the Coulomb solution for the heavy-quark background and again the parameter Λ introduced by the boundary condition (14) breaks conformal invariance, but not completely since a discrete conformal symmetry is preserved. The boundary condition and the spectrum are invariant under the discrete rescaling of $\Lambda \to \Lambda\, e^{2\pi i/\nu}$ [8].

The instability of the Coulomb phase is intrinsically associated to the breaking of conformal symmetry. In perturbation theory the conformal anomaly emerges from the renormalization of the coupling constant α. In this picture it arises from the need of fixing the boundary conditions of the singularity of quark potentials. The novelty is that in this case it implies the instability of the Coulomb vacuum background for large enough coupling constant. The existence a suprecritical value $\alpha_c = \sqrt{5}/2$ of the coupling constant for heavy quarks has been anticipated by earlier analyses [15-26]. However, the connection of the picture with real confinement is not yet clear because one quark background alone does not match the global gauge invariance conditions of Gauss law. For this reason is convenient to analyze what happens with several quarks and their interactions. This has been studied in ref. [27] and the result is that in the weak-coupling regime $\sqrt{2} < \alpha < \frac{3}{2}$ there is a critical quark-antiquark distance L_c such that heavy quark-antiquark pair is unstable for $L > L_c$ while asymptotic freedom is preserved a shorter distances $L < L_c$. In the first case the theory is asymptotically free and in the second one expects a confining behavior.

4. – Pólya-Hilbert conjecture on Riemann hypothesis

One of the interesting properties of the Riemann zeta function $\zeta(s)$ is the connection with the product of prime numbers given by Euler's formula [28]

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{\substack{p \\ \text{prime}}} \left(1 - \frac{1}{p^s}\right)^{-1}, \qquad s > 1. \tag{15}$$

The analytic continuation of the Riemann function $\zeta(s)$ to the full complex s plane is a meromorphic function has a only a simple pole at $s = 1$ and satisfies the Riemann functional equation

$$\pi^{\frac{s-1}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s) = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s). \tag{16}$$

This relation permits to identify an infinity of (trivial) zeros of the $\zeta(s)$-function: those sitting at the real negative integers. The rest of the zeros called non-trivial can be identified by the zeros of the function

$$\xi(s) = \binom{s}{2} \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s).$$

The Riemann hypothesis (conjecture) states that any non-trivial zero of $\zeta(s)$ lies on the critical line Re $s = \frac{1}{2}$; *i.e.* non-trivial zeros, are of the form

$$\rho = \frac{1}{2} + i\gamma$$

The conjecture is based on explicit calculations of the lowest zeros but a full proof is still missing.

Pólya and Hilbert formulated another conjecture which if true will provide a proof of Riemann hypothesis. The conjecture states that all non-trivial zeros of $\zeta(s)$ are eigenvalues of an operator of the form

$$\frac{1}{2}\mathbb{I} + i\mathbb{H}$$

with $\mathbb{H}$ self-adjoint.

A softer approach to the problem is based on the statistical properties of the one-dimensional distribution of non-trivial zeros of $\zeta(s)$ which is connected with the distribution of prime number on the real line by the Riemann-Mangoldt formula [28].

The counting function of Riemann zeros $N(E)$ defined by the number of non-trivial zeros with $0 < \operatorname{Im} s < E$ and $0 < \operatorname{Re} s < 1$ can be split into a smooth (semiclassical) and an oscillatory part (quantum),

$$N(E) = \langle N(E) \rangle + N_{\rm osc}(E), \tag{17}$$

where

$$\langle N(E) \rangle = \frac{1}{\pi} \arg\ \Gamma\left(\frac{1}{4} + i\frac{E}{2}\right) - \frac{E}{2\pi}\log \pi + 1, \tag{18}$$

and

$$N_{\rm osc}(E) = \frac{1}{\pi} \arg\ \zeta\left(\frac{1}{2} + iE\right). \tag{19}$$

In the limit $E \gg 1$ the asymptotic behaviour of both parts is

$$\langle N(E) \rangle = \frac{E}{2\pi}\left(\log \frac{E}{2\pi} - 1\right) + \frac{7}{8} + \mathcal{O}(1/E), \qquad N_{\rm osc}(E) = \mathcal{O}(\log E). \tag{20}$$

More statistical information on the distribution of the zeros of $\zeta(s)$ can be obtained from their pair correlators. For short-range correlators the behaviour is identical to the similar correlators of eigenvalues of a random matrix in a Gaussian Unitary Ensemble (GUE). This fact suggests that the Pólya-Hilbert Hamiltonian has to break time reversal symmetry. The distribution of Riemann zeros can also be related to that of energy eigenvalues of the quantum Hamiltonian of a classically chaotic system which not time-reversal invariant. Inspired by these hints Berry and Keating proposed a very simple quantum system with Hamiltonian [29]

$$H = xp + px$$

to approach the Pólya-Hilbert problem. The counting of energy levels of Berry-Keating model does asymptotically coincide with the semiclassical smooth terms (20) of $N(E)$.

TABLE I. – *Altland-Zirnbauer classification of quantum Hamiltonians.*

Class	Time reversal	Spin rotation	Symmetric space
D	−	−	$SO(4N)$
C	−	+	$Sp(2N)$
DIII	+	−	$SO(4N)/U(2N)$
CI	+	+	$Sp(2N)/U(N)$

Connes obtained a similar result from an adelic approach [30] (see [31] for a recent review). Following this approach Sierra [32, 33] considered a slightly different Hamiltonian

$$H = \sqrt{x}\left(p + \frac{1}{p}\right)\sqrt{x}, \tag{21}$$

which is equivalent to

$$H = p + \frac{1}{p}\,. \tag{22}$$

This new type of singular Hamiltonian can be made self-adjoint with the choice of a non-local boundary condition

$$\mathrm{e}^{i\theta}\psi(0) = \int_0^\infty \psi(x)\mathrm{d}x$$

parametrized by a $U(1)$ phase $\mathrm{e}^{i\theta}$.

The spectrum of this Hamiltonians can be found by composing the Hamiltonian with the operator p. The corresponding stationary Schrödinger equation of (21) becomes

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{E}{x^2} + \frac{3}{4}\right)\psi(x) = 0, \tag{23}$$

which is similar to the Schrödinger equation with singular potentials analyzed in the previous sections.

The interesting result is that the corresponding counting of energy levels now depends on the parameter θ of the boundary conditions

$$\langle N(E)\rangle = \frac{E}{2\pi}\left(\log\frac{E}{2\pi} - 1\right) - \frac{\theta}{2\pi} - \frac{1}{2} + \mathcal{O}(1/E). \tag{24}$$

However as remarked by Sredniski, according the Altland-Zirnbauer classification of quantum random Hamiltonians [34] in terms of their behavior under time reversal symmetry and spin rotation symmetry (see table I), the only class which seems adequate to describe the statistics of the Riemann zeros is class C, *i.e.* the system has to carry spin and to violate time reversal symmetry. All the models considered so far belong to class D. Thus it is natural to extend the family of models to systems with spin which can be achieved with slight modification of the Hamiltonian [35]

$$H = A + \vec{\sigma}\vec{B}. \tag{25}$$

One interesting model of this type is under study [36]. A similar but time-reversal–symmetric model has been analyzed recently [37]. All these models involve singular potentials of the type $1/x^2$ in the supercritical regime.

5. – AdS/CFT correspondence

Singular Hamiltonians also appear in the AdS/CFT correspondence. In Poincaré coordinates the anti-de Sitter metric $g_{\alpha\beta}$ is given by

$$g_{\alpha\beta} = \frac{1}{z^2}\begin{pmatrix} -1 & 0 \\ 0 & \mathbb{I} \end{pmatrix}$$

where z denotes the anti-de Sitter radial coordinate.

The motion equations of a free complex scalar field in anti-de Sitter AdS_{4+1} space-time with action

$$S(\phi) = \frac{1}{2}\int \frac{\mathrm{d}t\,\mathrm{d}z}{z^3}\int \mathrm{d}^3x\,\left[|\dot{\phi}|^2 - |\partial_z\phi|^2 - |\nabla\phi|^2 - \frac{m^2}{z^2}|\phi|^2\right]$$

are

$$\left(\partial_t^2 - \partial_z^2 + \frac{3}{z^2}\partial_z - \nabla^2 - \frac{m^2}{z^2}\right)\phi = 0.$$

In the non-relativistic limit after the transformation $\phi = z^{\frac{3}{2}}\psi$ they can be formulated in terms of an effective Hamiltonian [38]

$$H = -\partial_z^2 + \frac{m^2 + \frac{15}{4}}{z^2}$$

in the half line $z \in (0,\infty)$, which can be identified with (1) provided that $g = m^2 + \frac{15}{4}$. Thus, H is symmetric but the extension to a self-adjoint operator has three different regimes. For larger masses $m^2 > -3$ there is a unique self-adjoint extension and the theory is stable. In the range of masses $-4 < m^2 < -3$ there is a family of boundary conditions (13) parametrized by $\Lambda \in \mathbb{R}$, and for masses beyond that range, *i.e.* $m^2 < -4$, there is also a family of boundary conditions (14) parametrized also by $\Lambda \in \mathbb{R}$ but with an infinity of negative-energy levels. The difference between the last two regimes is that in the intermediate regime it is always possible to chose $\Lambda = 0$ or $\Lambda = \infty$ and preserve conformal invariance whereas in the supercritical regime this is never possible for any value of Λ. Therefore the theory is only stable if the mass is larger than the Breitenlohner-Freedman mass bound $m_{BF}^2 = -4$.

6. – Efimov effect and graphene

In nuclear physics singular potentials emerge in the effective description of the three-body problem at low energies in the regime where the scattering length is very large compared with the short range of nuclear interactions. In the extreme unitarity limit of infinite correlation length an infinity of bound states appears with energies following the scaling rule of a geometrical sequence. This phenomenon is known as Efimov effect [3]. It is a kind of nuclear Borromeo's ring where three particles can be bound into an infinite number of states whereas it is not possible for any pair of its components. The effective Hamiltonian of the three spinless particles problem is given by

$$H = -\frac{1}{2}\partial_R^2 - \frac{s_0^2 + \frac{1}{4}}{2R^2}\,,$$

where $R^2 = \frac{1}{3}(|x_1 - x_2|^2 + |x_1 - x_3|^2 + |x_2 - x_3|^2)$ is the symmetric radius of the three particles and s_0 a universal constant $s_0 \approx 1.00624$. This Hamiltonian is in the supercritical regime and has an infinite number of bound states whose energy spectrum exhibits the characteristic geometric scaling [3, 39, 40]

$$E_{n+1} = E_n \mathrm{e}^{-2\pi/s_0}.$$

The universal constant s_0 might change but depends only on the statistics, spin and parity behavior of the three particles. The Efimov effect is not exclusive of nuclear physics but it can appear in any compound of three particles with short-range interactions, *e.g.* it has has also been detected in cold atom physics.

Another field where singular potentials play a relevant role is graphene physics. Due to the special geometry of the material the effective dynamics is governed by a two-dimensional Dirac equation. An interesting property of this material is that charge impurities remain unscreened. In the presence of an impurity of charge Ze the solutions of the graphene massless Dirac equation satisfy the following equations:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{E^2}{v^2} - \frac{2Z\alpha' + i}{vr}E - \frac{\gamma(\gamma \pm 1)}{r^2}\right)\phi_\pm(r) = 0, \tag{26}$$

where $\gamma = -\frac{1}{2} + \sqrt{j^2 - Z^2\alpha'^2}$, j is the total angular momentum of the electron, α' is the effective fine structure constant of graphene and v is the speed of light on graphene which is 300 times smaller than in vacuum. The Coulomb potential becomes supercritical at $Z\alpha' = 1/2$ [41-47]. Although the Dirac equation (26) differs from the Schrödinger equations analyzed in previous sections they share the similar nature of singularities and give rise to similar instabilities. Beyond the supercritical charge $Z\alpha' > 1/2$ the energy of bound states becomes complex which is the smoking gun for instability. As in the massive case of Dirac equation the Hamiltonian ceases to be self-adjoint and the effective field-theoretical description of the system becomes inconsistent, which, in fact, reflects a dramatic change on the behavior of the microscopical graphene in the presence of so strong impurities.

All the above applications associated to the presence of singular potentials have in common the existence of critical coupling where there is a radical change of the behavior of the systems. In some cases this becomes the source of new physical effects as the appearance of an infinity of bound states with geometric recurrent relations like in the

Efimov effect. In others they are pathological and impose severe constraints on the physical systems, *e.g.* on the value of heavy charges in relativistic models like QCD or charged impurities in graphene. The analysis of how these facts can be reconciled within a consistent QFT is an interesting problem which deserves further study. In particular, for a consistent theory like QCD the results imply that the Coulomb phase becomes unstable at large distances for a quark-antiquark pair, whereas it is stable at short distances, in perfect agreement with asymptotic freedom [27].

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We thank J. G. Esteve and G. Sierra for collaboration in different stages of this work. We have been supported by the Spanish DGIID-DGA grant 2009-E24/2, the Spanish MICINN grants FPA2009-09638 and CPAN-CSD2007-00042. A. S. thanks DGIID-DGAfor a Ph. D. grant.

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