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Algebraic theory of entanglement

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Summary. — We have developed a novel approach to entanglement, suitable to be used in general quantum systems and specially in systems of identical particles. The approach is based on the GNS construction of representation of C^*-algebra of observables. In particular, the notion of partial trace is replaced by the more general notion of restriction of a state to a subalgebra. We here recollect some simple examples of the application of this novel approach after reviewing the GNS construction.

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1. – Introduction

In spite of the numerous efforts to achieve a satisfactory understanding of entanglement for systems of identical particles, there is no general agreement on the appropriate generalization of concepts valid for non-identical constituents [1-9]. That is because many concepts are usually only discussed in the context of quantum systems for which the Hilbert space $\mathcal{H}$ is a simple tensor product with no additional structure. An example

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is the Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ of two non-identical particles. In this case, the partial trace $\rho_A = \mathrm{Tr}_B |\psi\rangle\langle\psi|$ for $|\psi\rangle \in \mathcal{H}$ to obtain the reduced density matrix has a good physical meaning: it corresponds to observations only on the subsystem A.

Now, the Hilbert space of a system of N identical bosons (fermions) is given by the symmetric (antisymmetric) N-fold tensor product of the single-particle spaces. Therefore any multi-particle state is endowed with *intrinsic* correlations between subsystems due to quantum indistinguishability. Hence the use of singular value decomposition (SVD), Schmidt rank or entanglement entropy to study subsystem correlations seems no longer available or in need of a complete reassessment.

We recollect here an approach proposed by us in [10-12] to the study of entanglement, based on the foundational results of Gel'fand, Naimark and Segal on the representation theory of C^*-algebras, dubbed the *GNS construction* [13]. A crucial novel point of our approach is that the notion of *partial trace* is replaced by the more general notion of *restriction of a state to a subalgebra* [14, 15]. This allows us to treat entanglement of identical and non-identical particles on an equal footing.

We provide here simple examples([1]) of the advertised approach. In particular we obtain by our approach a zero von Neumann entropy for fermionic or bosonic states containing the least possible amount of correlations.

This seems to settle some confusion raised by the straightforward use of partial trace in computing the von Neumann entropy as a measure of entanglement for systems with identical particles [16, 17]. For a comprehensive review of previous works on this topic, see [8].

2. – The GNS construction

A vector state of a quantum system is usually described by a vector $|\psi\rangle$ in a Hilbert space $\mathcal{H}$ (pure case). More generally, a state is a density matrix $\rho : \mathcal{H} \to \mathcal{H}$, a linear map satisfying $\mathrm{Tr}\,\rho = 1$ (normalization), $\rho^\dagger = \rho$ (self-adjointness) and $\rho \geq 0$ (positivity). For *pure* states the additional condition $\rho^2 = \rho$ is required, so that ρ is of the form $|\psi\rangle\langle\psi|$ for some normalized vector $|\psi\rangle \in \mathcal{H}$.

Recall that the expectation value of an observable $\mathcal{O}$ is defined by $\langle\mathcal{O}\rangle = \mathrm{Tr}(\rho\,\mathcal{O})$. Now, we can equivalently regard a state as a *linear functional* $\omega_\rho : \mathcal{A} \to \mathbb{C}$ on a (C^*-)algebra $\mathcal{A}$ of observables with unity $\mathbb{1}_A$. The normalization and positivity conditions take the form $\|\omega_\rho\| := \omega_\rho(\mathbb{1}_A) = 1$ and $\omega_\rho(\mathcal{O}^\dagger\mathcal{O}) \geq 0$ (for any $\mathcal{O} \in \mathcal{A}$). Such a positive linear functional with unit norm is referred to as a *state on the algebra $\mathcal{A}$*.

In the bipartite case $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, the definition of ρ_A above involves a partial trace operation. Instead, we can consider the subalgebra $\mathcal{A}_0$ of operators of the form $K \otimes \mathbb{1}_B$, for K an observable on $\mathcal{H}_A$. We can then define a state $\omega_{\rho,0} : \mathcal{A}_0 \to \mathbb{C}$ which is the *restriction* $\omega_\rho|_{\mathcal{A}_0}$ of ω_ρ to $\mathcal{A}_0$ defined by $\omega_{\rho,0}(\alpha) = \omega_\rho(\alpha)$ if $\alpha \in \mathcal{A}_0$. Since $\omega_{\rho,0}(K \otimes \mathbb{1}_B) \equiv \mathrm{Tr}_A(\rho_A\, K)$, partial trace and restriction give the same answer in this case.

When $\mathcal{H}$ is not of the form of a "simple tensor product", partial trace is not a suitable operation. For these cases, the description of the quantum system in terms of a state ω_ρ on an algebra $\mathcal{A}$ and its restriction $\omega_{\rho,0}$ to a subalgebra are still valid. The GNS theory is effective in the determination of $\omega_{\rho,0}$.

([1]) For a more complete treatment [10, 11].

The idea of the GNS construction [18] is that given an algebra $\mathcal{A}$ of observables and a state ω on this algebra, we can construct a Hilbert space on which $\mathcal{A}$ acts. The key steps are: a) the state ω endows $\mathcal{A}$ with an inner product; it then becomes an "inner product" vector space $\hat{\mathcal{A}}$; b) this inner product may be degenerate, that is, the norm of some non-null elements of $\hat{\mathcal{A}}$ may be zero; c) the quotient $\hat{\mathcal{A}}/\widehat{\mathcal{N}}$ of $\hat{\mathcal{A}}$ by the null space $\widehat{\mathcal{N}}$ removes the null vectors and gives a well-defined Hilbert space (after completion); d) The algebra $\mathcal{A}$ acts on this Hilbert space in a simple manner.

We now make this set of ideas more precise [18].

Given a state ω on a C^*-algebra $\mathcal{A}$, we obtain a representation π_ω of $\mathcal{A}$ on a Hilbert space $\mathcal{H}_\omega$ as follows. Since $\mathcal{A}$ is an algebra, it is in particular a vector space denoted as $\hat{\mathcal{A}}$. Elements $\alpha \in \mathcal{A}$ regarded as elements of the vector space $\hat{\mathcal{A}}$ are written as $|\alpha\rangle$. We then set $\langle\beta|\alpha\rangle = \omega(\beta^*\alpha)$. This is an inner product, $\langle\alpha|\alpha\rangle \geq 0$, but there could be a null space $\widehat{\mathcal{N}}_\omega$ of zero norm vectors: $\widehat{\mathcal{N}}_\omega = \{|\alpha\rangle \in \mathcal{A} \mid \omega(\alpha^*\alpha) = 0\}$. By Schwarz inequality, one shows that $\langle a|\alpha\rangle = 0$, for any $a \in \mathcal{A}$ and $\alpha \in \mathcal{N}_\omega$. One also shows that $a\mathcal{N}_\omega \subseteq \mathcal{N}_\omega$, for any $a \in \mathcal{A}$, that is, $\mathcal{N}_\omega$ is a left ideal.

The space $\hat{\mathcal{A}}/\widehat{\mathcal{N}}_\omega$ with elements $|[\alpha]\rangle$, where $[\alpha] = \alpha + \widehat{\mathcal{N}}_\omega$, for any $\alpha \in \mathcal{A}$, has a well-defined scalar product

$$\langle[\alpha]|[\beta]\rangle = \omega(\alpha^*\beta) \tag{1}$$

independent of the choice of α from $[\alpha]$ and with no non-zero vectors of zero norm. Its closure is the GNS Hilbert space $\mathcal{H}_\omega$.

The representation π_ω of $\mathcal{A}$ on $\mathcal{H}_\omega : \pi_\omega(\alpha)|[\beta]\rangle = |[\alpha\beta]\rangle$ is in general reducible. So $\mathcal{H}_\omega$ can be decomposed into a direct sum of irreducible spaces: $\mathcal{H}_\omega = \bigoplus_i \mathcal{H}_i$. Let $P_i : \mathcal{H}_\omega \to \mathcal{H}_i$ be the corresponding orthogonal projectors. We have $\mu_i = \|P_i\,|[\mathbb{1}_\mathcal{A}]\rangle\|$. Since $\omega(\alpha) = \langle[\mathbb{1}_\mathcal{A}]|\pi_\omega(\alpha)|[\mathbb{1}_\mathcal{A}]\rangle$ and $|[\mathbb{1}_\mathcal{A}]\rangle = \sum_i P_i|[\mathbb{1}_\mathcal{A}]\rangle$, one obtains $\omega(\alpha) = \mathrm{Tr}_{\mathcal{H}_\omega}(\rho_\omega\,\pi_\omega(\alpha))$, where $\rho_\omega = \sum_i P_i\,|[\mathbb{1}_\mathcal{A}]\rangle\langle[\mathbb{1}_\mathcal{A}]|P_i$. The von Neumann entropy of ρ_ω is $S(\rho_\omega) = -\sum_i \mu_i^2 \log_2 \mu_i^2$.

The crucial fact is that ω is *pure* if and only if the representation π_ω is *irreducible.* In particular, the von Neumann entropy of ω, $S(\omega) \equiv S(\rho_\omega)$, is zero if and only if $\mathcal{H}_\omega$ is irreducible. *This property depends on both the algebra* $\mathcal{A}$ *and the state* ω.

Consider now a subalgebra $\mathcal{A}_0 \subset \mathcal{A}$ of $\mathcal{A}$. Let ω_0 denote the *restriction* to $\mathcal{A}_0$ of a pure state ω on $\mathcal{A}$ [14]. We can apply the GNS construction to the pair $(\mathcal{A}_0, \omega_0)$ and use the von Neumann entropy of ω_0 to study the entanglement emergent from restriction.

2˙1. *Bell state*. – In [18], G. Landi shows a simple instructive example of the GNS construction. We now recollect his example in order to illustrate how to apply the GNS construction to entanglement.

Consider the Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \equiv \mathbb{C}^2 \otimes \mathbb{C}^2$ acted on by the algebra $\mathcal{A} = M_4(\mathbb{C})$ of 4×4 complex matrices generated by elements of the form $\sigma_\mu \otimes \sigma_\nu$ $(\mu, \nu = 0, 1, 2, 3)$, with $\sigma_0 = \mathbb{1}_2$ and $\{\sigma_1, \sigma_2, \sigma_3\}$ the Pauli matrices. Let us consider the normalized vector $|\psi\rangle = (1/\sqrt{2})(|+1\rangle \otimes |-1\rangle - |-1\rangle \otimes |+1\rangle)$ (± 1 denoting the eigenvalues of σ_3) with corresponding state $\omega = |\psi\rangle\langle\psi|$ on the algebra $\mathcal{A}$.

Entanglement of $|\psi\rangle$ is to be understood in terms of correlations between "local" measurements performed separately on subsystems A and B. Measurements performed on A correspond to the restriction $\omega_A = \omega\,|_{\mathcal{A}_A}$ of ω to the subalgebra $\mathcal{A}_A \subset \mathcal{A}$ generated by elements of the form $\sigma_\mu \otimes \mathbb{1}_2$.

In this case there are no non-trivial null states $[\omega((\sigma_\mu \otimes \mathbb{1})^*(\sigma_\nu \otimes \mathbb{1})) = \langle\psi|\sigma_\mu^*\sigma_\nu \otimes \mathbb{1}|\psi\rangle = \delta_{\mu\nu}]$, so $\mathcal{N}_{\omega_A} = \{0\}$. Therefore the GNS-space is simply $\mathcal{H}_{\omega_A} = \hat{\mathcal{A}}_A/\widehat{\mathcal{N}}_{\omega_A} \cong \mathbb{C}^4$ with basis vectors $|[\sigma_\nu]\rangle \equiv |[\sigma_\nu \otimes \mathbb{1}_2]\rangle$ and inner product $\langle[\sigma_\mu]|[\sigma_\nu]\rangle = \delta_{\mu\nu}$. An element $\alpha \in \mathcal{A}_A$ acts on $\mathcal{H}_{\omega_A}$ as $\pi_{\omega_A}(\alpha)|[\beta]\rangle = |[\alpha\beta]\rangle$, where the RHS can be explicitly computed using $\sigma_i\sigma_j = \delta_{ij}\mathbb{1}_2 + i\varepsilon_{ijk}\sigma_k$.

This representation is reducible. The GNS-space splits as $\mathcal{H}_{\omega_A} = \mathbb{C}^2 \oplus \mathbb{C}^2$. One invariant subspace is spanned by $|[\sigma_+ \otimes \mathbb{1}_2]\rangle$, $|[(1/2)(1-\sigma_3)\otimes \mathbb{1}_2]\rangle$ and the other by $|[(1/2)(1+\sigma_3)\otimes \mathbb{1}_2]\rangle$, $|[\sigma_- \otimes \mathbb{1}_2]\rangle$, where $\sigma_\pm = \sigma_1 \pm i\sigma_2$. The corresponding projections are

$$P_i = \frac{1}{2}\pi_{\omega_A}(\mathbb{1}_\mathcal{A} + (-1)^i\sigma_3 \otimes \mathbb{1}_2), \quad \text{with} \quad i = 1, 2, \tag{2}$$

so that $\mu_i^2 = \|P_i|[\mathbb{1}_\mathcal{A}]\rangle\|^2 = 1/2$. We may then compute the corresponding von Neumann entropy as $S(\omega_A) = \log_2 2$. Therefore ω_A is not pure. It is maximally entangled. This is a standard result.

3. – Systems of identical particles

Let $\mathcal{H}^{(1)} = \mathbb{C}^d$ be the Hilbert space of a one-particle system. The group $U(d) = \{g\}$ acts on $\mathbb{C}^d$ by the representation $g \mapsto U^{(1)}(g)$. The algebra of observables is given by a $*$-representation of the *group algebra* $\mathbb{C}U(d)$ on $\mathcal{H}^{(1)}$. Its elements are of the form

$$\widehat{\alpha}^{(1)} = \int_{U(d)} \mathrm{d}\mu(g)\alpha(g)U^{(1)}(g), \tag{3}$$

where α is a complex function on $U(d)$ and μ the Haar measure [19]. The elements $\hat{\alpha}$ span the matrix algebra $M_d(\mathbb{C})$.

The k-particle Hilbert space $\mathcal{H}^{(k)}$ for bosons (fermions) is the symmetrized (anti-symmetrized) k-fold tensor product of $\mathcal{H}^{(1)}$. We can associate the operator $A^{(k)} := (A^{(1)} \otimes \mathbb{1}_d \ldots \otimes \mathbb{1}_d) + (\mathbb{1}_d \otimes A^{(1)} \otimes \mathbb{1}_d \otimes \ldots \otimes \mathbb{1}_d) + \ldots + (\mathbb{1}_d \otimes \ldots \otimes \mathbb{1}_d \otimes A^{(1)})$ on $\mathcal{H}^{(k)}$ with a one-particle observable $A^{(1)}$ on $\mathcal{H}^{(1)}$. The operator $A^{(k)}$ preserves the symmetries of $\mathcal{H}^{(k)}$. The map $A^{(1)} \longrightarrow A^{(k)}$ is a Lie algebra homomorphism. By considering $e^{iA^{(1)}}$, we can also get a group homomorphism and accordingly associate

$$\widehat{\alpha}^{(k)} = \int_{U(d)} \mathrm{d}\mu(g)\alpha(g)U^{(1)}(g) \otimes \ldots \otimes U^{(1)}(g), \tag{4}$$

with the one-particle operator $\widehat{\alpha}^{(1)}$.

These constructions are most conveniently expressed in terms of a *coproduct* Δ [19]. In fact, an approach based on Hopf algebras [19] has the advantage that para- and braid-statistics can be *automatically* included. In what follows we use the simple coproduct $\Delta(g) = g \otimes g$, $g \in U(d)$, linearly extended to all of $\mathbb{C}U(d)$. This choice fixes the form (4). Physically, the existence of such a coproduct is very important. It allows us to homomorphically represent one-particle observables in the k-particle sector.

We may now consider two main choices:

1) In a many particle system, observations may be restricted to the homomorphic image of the one-particle observable algebra obtained with the coproduct.

2) We may perform only partial one-particle observations such as only its spin degrees of freedom or only its position. The one-particle algebra at the k-particle level has to be further restricted accordingly.

3˙1. *Two fermions,* $\mathcal{H}^{(1)} = \mathbb{C}^3$. – We now consider the two-fermion space $\mathcal{H}^{(2)} = \bigwedge^2 \mathbb{C}^3 \subset \mathcal{H}^{(1)} \otimes \mathcal{H}^{(1)}$ ($\bigwedge$ denotes antisymmetrization), with basis $\{|f^k\rangle = \varepsilon^{ijk}|e_i \wedge e_j\rangle\}$, with $i,j = 1,2,3$, where $\{|e_1\rangle, |e_2\rangle, |e_3\rangle\}$ is an orthonormal basis for $\mathcal{H}^{(1)}$. The algebra $\mathcal{A}^{(2)}$ of observables for the two-fermion system is the matrix algebra generated by $|f^i\rangle\langle f^j|$ $(i,j = 1,2,3)$. Therefore $\mathcal{A}^{(2)} \cong M_3(\mathbb{C})$.

Now, $U(3)$ acts on $\mathcal{H}^{(1)}$ through the defining representation $U^{(1)}(g) = g$, so that one-particle observables are given at the two-fermion level by the action of $\mathbb{C}U(3)$ on $\mathcal{H}^{(2)}$. This action is given by the restriction of the operators $\widehat{\alpha}^{(2)} \in \mathcal{A}^{(2)}$ to the space of antisymmetric vectors. Let 3 be the defining (or fundamental) $SU(3)$ representation on $\mathcal{H}^{(1)}$. Then the restriction can be obtained from the decomposition $3 \otimes 3 = 6 \oplus \bar{3}$ of the $SU(3)$ representation. The $|f^i\rangle$ span this $\bar{3}$ representation.

We first consider the first choice above so that $\mathcal{A}_0$ is the full algebra of one-particle observables acting on $\mathcal{H}^{(2)}$. The GNS representation corresponding to $(\mathcal{A}_0, \omega_\psi = |\psi\rangle\langle\psi| : \psi \in \mathcal{H}^{(2)})$ is irreducible. The state remains unchanged upon restriction. This is just the fact that the $\bar{3}$ representation of $SU(3)$ is irreducible. This also corresponds to the fact that for $d = 3$, all two-fermion vector states have Slater rank 1. The von Neumann entropy is thus zero.

Notice, however, that the *von Neumann entropy computed by partial trace is equal to* $\log_2 2$ *for all choices of* $|\psi\rangle$ *(cf.* [5]*), in disagreement with the GNS-approach.*

We next consider the second choice. Let $\mathcal{A}_0$ be the image under Δ of those one-particle observables pertaining *only* to the particles 1 and 2. This algebra contains the projector $|e_1\rangle\langle e_1| + |e_2\rangle\langle e_2|$. But observables of 1 and 2 may give null answer as the particle occupies $|e_3\rangle$. This corresponds to observing $|e_3\rangle\langle e_3|$. Thus the one-particle algebra contains also $\sum_i |e_i\rangle\langle e_i| = \mathbb{1}_{3\times3}$. So in this case, $\mathcal{A}_0$ is the five-dimensional algebra generated by $M^{ij} := |f^i\rangle\langle f^j|$ $(i,j = 1,2)$ and $\mathbb{1}_{\mathcal{A}^{(2)}} = \mathbb{1}_{3\times3}$.

For the particular choice $|\psi_\theta\rangle = \cos\theta|f^1\rangle + \sin\theta|f^3\rangle$, the corresponding state is $\omega_\theta = |\psi_\theta\rangle\langle\psi_\theta|$ with $\omega_\theta(\alpha) = \langle\psi_\theta|\alpha|\psi_\theta\rangle$ for any $\alpha \in \mathcal{A}^{(2)}$. For the restriction $\omega_{\theta,0} = \omega_\theta \mid_{\mathcal{A}_0}$ we find that the null vectors contain the span of M^{12} and M^{22}.

For $0 < \theta < \pi/2$, there are no more linearly independent null vectors. Therefore, the GNS-space $\mathcal{H}_\theta = \hat{\mathcal{A}}_0/\widehat{\mathcal{N}}_{\theta,0}$ is three-dimensional with basis $\{|[M^{11}]\rangle, |[M^{21}]\rangle, |[E^3]\rangle\}$, where $E^3 := \mathbb{1}_\mathcal{A} - M^{11} - M^{22}$.

Since $\alpha_0\, E^3 = 0$ for any $\alpha_0 \in \mathcal{A}_0$, we immediately recognize that, in terms of irreducibles, $\mathcal{H}_\theta = \mathbb{C}^2 \oplus \mathbb{C}^1$. Noting that $[M^{11} + M^{22}] = [\mathbb{1}_2]$, we obtain $P_1|[\mathbb{1}_\mathcal{A}]\rangle = |[M^{11}]\rangle$ and $P_2|[\mathbb{1}_\mathcal{A}]\rangle = |[E^3]\rangle$ for the projectors. The corresponding "weights" are $|\mu_1|^2 = \cos^2\theta$ and $|\mu_2|^2 = \sin^2\theta$. Hence, the entropy as a function of θ is $S(\theta) = -\cos^2\theta\log_2\cos^2\theta - \sin^2\theta\log_2\sin^2\theta$.

For $\theta = 0$ there are additional null vectors. The null space is spanned by $|M^{12}\rangle$, $|M^{22}\rangle$ and $|E^3\rangle$. The GNS-space $\mathcal{H}_0 = \hat{\mathcal{A}}_0/\widehat{\mathcal{N}}_{0,0}$ is two-dimensional and irreducible. It is spanned by $|[M^{11}]\rangle$ and $|[M^{21}]\rangle$. Since $\pi_\omega(\mathcal{A}_0)$ acts non-trivially on this space, and the smallest non-trivial representation of $\mathcal{A}_0$ is its two-dimensional IRR, this representation is irreducible. Hence $\omega_{0,0}$ is pure with zero entropy.

For $\theta = \frac{\pi}{2}$ instead, *all* of $|M^{ij}\rangle$ are null vectors. So $\mathcal{H}_{\frac{\pi}{2}}$ is one-dimensional and spanned by $|[E^3]\rangle$. Clearly $\omega_{\frac{\pi}{2},0}$ is pure with zero entropy.

3˙2. *Two bosons,* $\mathcal{H}^{(1)} = \mathbb{C}^3$. – We start with a one-particle space $\mathcal{H}^{(1)} = \mathbb{C}^3$ with an orthonormal basis $\{|e_1\rangle, |e_2\rangle, |e_3\rangle\}$. A two-boson space $\mathcal{H}^{(2)}$ is the space of symmetric vectors in $\mathcal{H}^{(1)} \otimes \mathcal{H}^{(1)}$. It is equivalent to the six-dimensional space coming from the decomposition $3 \otimes 3 = 6 \oplus \bar{3}$ of $SU(3)$. An orthonormal basis for $\mathcal{H}^{(2)}$ is given by vectors

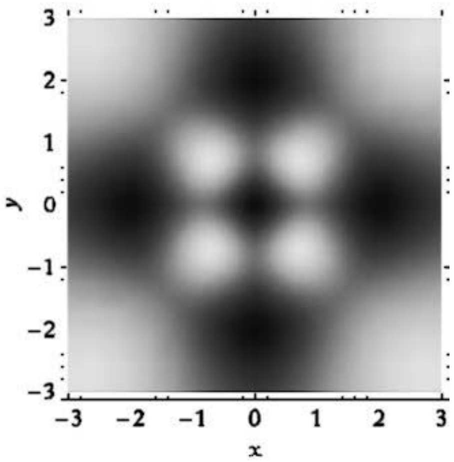

Fig. 1. – The two-boson entropy as a function of x and y which represent the (θ,ϕ)-sphere via stereographic projection. The expression of the entropy is $S(\theta,\phi) = -\sin^2\theta[\cos^2\phi\log_2(\sin\theta\cos\phi)^2 + \sin^2\phi\log_2(\sin\theta\sin\phi)^2] - \cos^2\theta\log_2(\cos\theta)^2$. Darker regions correspond to lower values of the entropy. Five of the six vanishing points of the entropy can be seen on the picture (black spots). The sixth one, corresponding to the north-pole of the sphere, lies "at infinity". The entropy vanishes whenever $|\psi_{\theta,\phi}\rangle$ lies in a single irreducible component.

$\{|e_i \vee e_j\rangle\}_{i,j\in\{1,2,3\}}$ where $\vee$ denotes symmetrization (and the vectors are normalized). The two-boson algebra of observables $\mathcal{A}^{(2)}$ is isomorphic to $M_6(\mathbb{C})$.

We now choose the vector $|\psi_{\theta,\phi}\rangle = \sin\theta\cos\phi|e_1\vee e_2\rangle + \sin\theta\sin\phi|e_1\vee e_3\rangle + \cos\theta|e_3\vee e_3\rangle$. The corresponding state is $\omega_{\theta,\phi}$ defined by $\omega_{\theta,\phi}(\alpha) = \langle\psi_{\theta,\phi}|\alpha|\psi_{\theta,\phi}\rangle$ for any $\alpha \in \mathcal{A}$. We fix the subalgebra $\mathcal{A}_0$ consisting of those one-particle observables pertaining *only* to the one-particle states $|e_1\rangle$ and $|e_2\rangle$.

The restriction of the state to the subalgebra is provided by $\omega_{\theta,\phi}\ |_{\mathcal{A}_0}$. The 6-representation under the $SU(2)$ action on $|e_1\rangle$ and $|e_2\rangle$ splits as $6 = 3\oplus 2\oplus 1$. The subalgebra $\mathcal{A}_0$ consists of block-diagonal matrices. Each block is one of the irreducible components in the decomposition $6 = 3\oplus 2\oplus 1$. The dimension of $\mathcal{A}_0$ is therefore $3^2 + 2^2 + 1^2 = 14$.

The construction of the GNS-representation corresponding to each particular value of the parameters θ and ϕ follows the same procedure as in the previous example. The von Neumann entropy as a function of the parameters is depicted in fig. 1.

4. – Conclusions

We have presented a new approach to the study of quantum entanglement based on restriction of states to subalgebras. The GNS construction allows us to obtain a representation space for the subalgebra such that its decomposition into irreducible subspaces can be used to study quantum correlations. We showed that, when applied to bipartite systems for which the Hilbert space is a "simple" tensor product, our method reproduces the standard results on entanglement. We furthermore showed, with explicit examples, how the formalism can be applied to systems of identical particles. The main result is that the von Neumann entropy remains the suitable entanglement measure, when understood in terms of states on algebras of observables.

Our formalism can be easily generalized to more sophisticated situations involving para- and braid-statistics. It can even be extended to study for instance a k-particle subsystem in an N-particle Hilbert space.

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