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Chowla-Selberg series and other formulas useful in zeta regularization

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Summary. — The role of the Riemann zeta function as a regularization tool is briefly review and a general scheme for the physically relevant quadratic and linear cases is discussed. The use and importance of the Chowla-Selberg series formula, together with its non-trivial extensions, to deal with situations where the spectrum is known explicitly is stressed. The derivation of such formulas is shown to rely on other fundamental expressions of mathematics, as the Poisson summation formula and Jacobi's theta function identity. Their uses in the zeta regularization of infinite quantities in quantum field theory is sketched. The second part of the paper addresses operator zeta functions, regularized traces and residues, and the multiplicative anomaly or defect of the determinant, together with potential applications.

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1. – Introduction

Already in the XIX Century there was the suspicion that one could give sense to divergent series. This has now been proven experimentally (with 15 digit accuracy in some cases) to be true in Physics, but it was the mathematicians —many years before— who first realized such possibility. In fact, Leonard Euler (1707-1783) was convinced of the fact that *"To every series one can assign a number"* [1] (namely, in a reasonable, consistent, and possibly useful way). Euler was unable to prove this statement in full, but he devised a technique (Euler's summation criterion) in order to "sum" a large family of divergent series. His idea was, however, dismissed by some other great mathematicians, as Abel, who proclaimed that *"The divergent series are the invention of the devil, and*

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it is a shame to base on them any demonstration whatsoever" [2]. There is, on those matters, a classical treatise by G. H. Hardy entitled *Divergent series* [3] which can be highly recommended to the reader.

Actually, regularization and renormalization procedures are essential in present-day Physics. Among the different techniques at hand in order to implement these processes, zeta function regularization [4,5] is one of the most elegant. Use of this method yields, for instance, the vacuum energy corresponding to a quantum physical system, which could, *e.g.*, contribute to the cosmic force leading to the observed acceleration in the expansion of our universe. The zeta function method is unchallenged at the one-loop level, where it is rigorously defined and where many calculations of QFT reduce basically, from a mathematical viewpoint, to the computation of determinants of elliptic pseudodifferential operators (ΨDOs) [6]. It is thus no surprise that the preferred definition of determinant for such operators is obtained through the corresponding zeta function (see, *e.g.*, [7, 8]).

For its application in practice [9, 10], the zeta function regularization method relies on the existence of quite simple formulas which yield the analytic continuation of the zeta function, $\zeta(s)$, from the region of the complex plane extending to the right of the abscissa of convergence, $\mathrm{Re}\, s > s_0$, where its series expression is absolutely convergent, to the rest of it [7, 11-13]. These are not only the functional equation of the corresponding zeta function in each case, but also some other, very fundamental expressions, as the Jacobi theta function identity, Poisson's and Plana's resummation formulas, and the Chowla-Selberg series formula. However, some of these powerful expressions are often restricted to specific zeta functions, and their explicit derivation is usually quite involved. For instance, until recently the Chowla-Selberg (CS) formula was only available for the homogeneous, two-dimensional Epstein zeta function. Also, all these formulas make use of the fact that the sum is done over a complete, unbounded lattice in $\mathbb{R}$ or $\mathbb{R}^n$ (extending from $-\infty$ to $+\infty$), and they do not actually stand in the physically important cases of truncated sums (where one can only get *asymptotic* expressions) [11, 12].

A fundamental property shared by all zeta functions is the existence of a functional equation (usually called by physicists reflection formula). For the Riemann zeta function, it reads

$$\Gamma(s/2)\zeta(s) = \pi^{s-1/2}\Gamma(1-s/2)\zeta(1-s). \tag{1}$$

For a generic zeta function, $Z(s)$, we may write it as: $Z(\omega - s) = F(\omega, s)Z(s)$. This expression readily gives the analytic continuation of the zeta function and this is, in simple cases, almost the whole story of the zeta function regularization procedure([1]). But note that the analytically continued expression thus obtained is just another series, which may have again a very slow, power-like convergence behavior [14] (actually the same that the original series had, on the initial domain).

For the Epstein zeta function in two dimensions, S. Chowla and A. Selberg [15] obtained a formula which exhibits *exponentially fast convergence everywhere*, not just in the reflected domain. They were extremely proud of this finding. In ref. [16], a first attempt was done by the author in order to extend such expression to *inhomogeneous* zeta functions (which are important for physical applications, see [17]), but remaining still in *two* dimensions, for this was commonly believed to be a true restriction of the original

([1]) *Almost*, because a finite renormalization contribution which modifies this raw result needs to be taken into account in general.

formula (see, *e.g.*, ref. [18]). More recently, extensions to *arbitrary* dimensions [19, 20], both for the homogeneous (quadratic form) and non-homogeneous (quadratic plus affine form) cases were constructed. Some of the new formulas (remarkably the ones corresponding to the zero-mass case, *e.g.*, the original CS framework) were not so explicit, they involved solving a rather non-trivial recurrence([2]). In [21] all cases have been finally solved by the author in absolute detail.

Aside from the explicit quadratic case, which corresponds to the Epstein zeta function, and generalizations thereof, the linear one is also very important (and quite difficult) for its many physical applications (think just of a system of harmonic oscillators). The most general linear zeta function studied to date is Barnes' one. Here again many explicit expressions are missing, as for its derivative in the general case [22].

Next section will be devoted to some basic considerations on divergent series and to essential ideas of the zeta regularization procedure. The discussion, in another section, of the Chowla-Selberg formula, both in analytic number theory and theoretical physics, will lead us to some fundamental expressions of mathematics (so have been termed by V. Kac), as the Poisson summation formula and Jacobi's theta function identity. Their uses in the zeta function regularization scheme of infinite expressions in quantum field theory (QFT) will be sketched. The second part of the paper addresses operator zeta functions, regularized traces and residues, and the multiplicative anomaly or defect of the determinant, together with some feasible applications.

2. – Basic considerations on divergent series and essentials of zeta regularization

As is usual in modern Mathematics, one starts the attack on divergent series by invoking a number of reasonable axioms, like (see, *e.g.* [3])

1) If $a_0 + a_1 + a_2 + \ldots = s$, then $ka_0 + ka_1 + ka_2 + \ldots = ks$.

2) If $a_0 + a_1 + a_2 + \ldots = s$, and $b_0 + b_1 + b_2 + \ldots = t$, then $(a_0 + b_0) + (a_1 + b_1) + (a_2 + b_2) + \ldots = s + t$.

3) If $a_0 + a_1 + a_2 + \ldots = s$, then $a_1 + a_2 + \ldots = s - a_0$.

A couple of examples are in order.

1) Using the third axiom we see that for the series $s = 1 - 1 + 1 - 1 + \ldots$, we have $s = 1 - s$, and therefore $s = 1/2$. This value is easy to justify, since the series is oscillating between 0 and 1, so that 1/2 is the more "democratic" value for it.

2) Using now the second axiom with the series $t = 1-2+3-4+\ldots$, by subtracting it term by term from the former one it turns out that $s-t = t$, therefore $t = s/2 = 1/4$. Such a result is already quite difficult to swallow.

What about the simplest series $1 + 1 + 1 + \ldots$? This is more difficult to tame, and the given axioms do not serve to this purpose. But there is more to the axioms, which are only intended as a humble starting point. By reading Hardy's book [3] one learns about a number of different methods that have been proposed and is good to know. They are

([2]) What may also explain why the CS formula had not been extended to higher-dimensional Epstein zeta functions before.

due to Abel, Euler, Cesàro, Bernoulli, Dirichlet, Borel and other mathematicians(3). The most powerful of them involve analytic continuation in the complex plane, as is precisely the case of the zeta regularization method.

As advanced, regularization and renormalization procedures are actually essential issues of contemporary physics —without which it would simply not exist, at least in the form we know it [24]. Among the different methods, zeta function regularization —which is obtained by analytic continuation on the complex plane of the zeta function of the relevant physical operator in each case— is most beautiful and the usual procedure adopted in operator and functional analysis for dealing with divergent determinants and traces. Use of it yields, for instance, the vacuum energy corresponding to a quantum physical system (with constraints of some kind). We have sketched the procedure before for an abstract zeta function, but assume now the corresponding Hamiltonian operator of our system, H, has a spectral decomposition of the form (think of a quantum harmonic oscillator): $\{\lambda_i, \varphi_i\}_{i\in I}$, being I some set of indices, which can be discrete, continuous, mixed, or multiple. Then, the quantum vacuum energy [11] is obtained as follows:

$$E/\mu = \sum_{i\in I} \langle \varphi_i, (H/\mu)\varphi_i \rangle = \mathrm{Tr}_\zeta H/\mu = \sum_{i\in I} \lambda_i/\mu = \sum_{i\in I} (\lambda_i/\mu)^{-s}\big|_{s=-1} = \zeta_{H/\mu}(-1), \tag{2}$$

where ζ_A is the zeta function corresponding to the operator A, and the equalities are in the sense of analytic continuation (since, generically, the Hamiltonian operator will not be of the trace class)(4). Note that the formal sum over the eigenvalues is usually ill defined, and that the last step involves analytic continuation, inherent with the definition of the zeta function itself. Also, an unavoidable regularization parameter with dimensions of mass, μ, appears in the process, in order to render the eigenvalues of the resulting operator dimensionless, so that the corresponding zeta function can actually be defined. We shall not discuss further these important details, which are just at the starting stage of the whole renormalization procedure. The mathematically simple-looking relations above involve very deep physical concepts —no wonder that understanding them took several decades in the recent history of QFT.

The method evolved from the consideration of the Riemann zeta function as a "series summation method" [4, 5] (see below). In more general cases, namely corresponding to the Hamiltonians which are relevant to physical applications [11, 12, 26], the situation is in essence quite similar (although in practice it can be rather involved). A mathematical theorem exists, which assures that under very general conditions the zeta function corresponding to a Hamiltonian operator will be also meromorphic, with just a discrete number of possible poles, which are simple and extend to the negative side of the real axis(5).

2·1. *The zeta function as a summation method.* – The above picture already hints towards the use of the zeta function as a summation method. Two examples:

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(3) Padé approximants should in no way be forgotten in this discussion [23, 14].

(4) The reader should be warned that this ζ-trace is actually no trace in the usual sense. In particular, it is highly non-linear, as often explained by the author elsewhere [25]. Some colleagues are unaware of this fact, which has led to important mistakes and erroneous conclusions too often.

(5) There are exceptions to this general behavior, but they correspond to rather twisted situations which lay outside the scope of this brief presentation.

1) We interpret the series $s_1 = 1+1+1+1+\ldots$ as a particular case of the Riemann zeta function, *e.g.* for the value $s=0$. This point is located on the left-hand side of the abscissa of convergence, where the series as such diverges but where its analytic continuation provides a unique, perfectly finite answer: $s_1 = \zeta(0) = -\frac{1}{2}$. This is the value to be given to the series $1+1+1+1+\ldots$.

2) The series $s_2 = 1+2+3+4+\ldots$ corresponds to the exponent $s=-1$, so that: $s_2 = \zeta(-1) = -\frac{1}{12}$.

A couple of comments are in order.

1) In a short period of less than a year, two distinguished physicists, A. Slavnov and the late F. Yndurain, gave seminars in Barcelona, about different subjects. It was remarkable that, in both presentations, at some point the speaker addressed the audience with these words: *"As everybody knows,* $1+1+1+\ldots = -1/2$*"*(6).

2) That positive series, as the ones above, can yield a *negative* result may seem uttermost nonsensical. However, it turns out that the most accurate experiments ever carried out in Physics do confirm such results. More precisely: models of regularization in QED built upon these techniques lead to final numbers which are in full agreement with the experimental values up to the 15th figure [27]. In recent experimental proofs of the Casimir effect [28] the agreement is also quite remarkable (given the difficulties of the experimental setup) [29].

3) The method of zeta regularization is based on the analytic continuation of the zeta function on the complex plane. Now, how easily can this continuation be performed in practice? Will we need to undertake a lengthy complex analytical computation every time? It turns out that this is not so. The needed result immediately follows, in principle, once we know the appropriate functional equation of our zeta function: *e.g.*, in the case of Riemann's zeta $\xi(s) = \xi(1-s)$, $\xi(s) \equiv \pi^{-s/2}\Gamma(s/2)\zeta(s)$ (eq. (1)). In practice these formulas are, however, not useful for actual calculations, since the result (albeit explicit) is given in terms of a very slowly convergent power series expansion (as the Riemann zeta is, too). Fortunately, there are more clever expressions that come to rescue, which converge exponentially fast, as the Chowla-Selberg [15] formula and some other [16,19]. They inject true power to the method of zeta regularization. (More about this point below, where a couple of such expressions will be explicitly discussed.)

4) We have proven (see [30]) that the principal-part prescription in the zeta-function regularization method need *not* be imposed as an additional assumption, since it follows from (and can be replaced by) a more natural and beautiful principle: the corresponding Feynman propagator is obtained as the limit for $\beta \to \infty$ of the thermal propagator.

3. – On the Chowla-Selberg series formula

It is now known that the first appearance of the Chowla-Selberg formula occurred in a work by M. Lerch, *Sur quelques formules relatives du nombre des classes*, published in

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(6) Meaning probably: *If you do not know this it is no use to continue listening.* Recall the *lemma* of the Pythagorean school: *Do not cross this gate if you do not know Geometry.*

1897 [31], while the first paper by Chowla and Selberg on this matter, *On Epstein's Zeta function (I)*, was published in 1949 [15]. No details on the derivation of their famous formula are given there, they promised to give them in a subsequent publication, which in fact was delayed for almost two decades, until 1967 [32]. This last paper, published in Crelle's Journal and sharing the same title with the previous one (although the author's order was reversed), is now recognized as the fundamental reference on the subject. Some other relevant references in the mathematical literature dealing with this matter are the paper by K. Ramachandra [33], the book by A. Weil [34], the *Encyclopedic Dictionary of Mathematics* published by S. Iyanaga and Y. Kawada [18], and the papers by B.H. Gross [35] and by P. Deligne [36].

To give specific details on the history of this discovery, we recall Lerch's pioneering result:

$$\sum_{\lambda=1}^{|D|} \left(\frac{D}{\lambda}\right) \log\Gamma\left(\frac{\lambda}{D}\right) = h\log|D| - \frac{h}{3}\log(2\pi) - \sum_{(a,b,c)} \log a + \frac{2}{3}\sum_{(a,b,c)} \log\left[\theta_1'(0|\alpha)\theta_1'(0|\beta)\right], \tag{3}$$

where D is the discriminant, $\theta_1' \sim \eta^3$, and h is the class number of binary quadratic forms, given by their coefficients (a,b,c).

In mathematics the CS formula is extremely useful in some very involved issues as the so-called *eta evaluations*, namely to give a explicit result for Dedekind's eta function, Im $(\tau) > 0$,

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n), \qquad q := e^{2\pi i\tau}. \tag{4}$$

It follows from this expression that η is a 24th root of the discriminant function $\Delta(\tau)$ of an elliptic curve $\mathbb{C}/L$ from a lattice $L = \{a\tau + b \,|\, a,b \in \mathbb{Z}\}$, with

$$\Delta(\tau) = (2\pi)^{12} q \prod_{n=1}^{\infty}(1-q^n)^{24}. \tag{5}$$

We end this short description with a summary of some properties and recent results.

1) The CS formula gives the value of a product of eta functions.
2) If there is only one form in the class, it yields the value of a *single* eta function in terms of gamma functions.
3) There has been a long series of improvements in more recent years, in particular by Kaneko [37], Nakajima and Taguchi [38], Williams *et al.* [39], and others.
4) In the last years the CS formula has been finally "broken" to *isolate* the eta functions (Williams, van Poorten, Chapman, Hart). For references, see [40] and the PhD Thesis by W.B. Hart [41], and references therein.
5) Other recent work is on analogues of the Chowla-Selberg formula for automorphic L-functions [42], and on its relation with the Colmez conjecture [43].

3˙1. *Basic strategies.* – Start from Jacobi's identity for the theta-function $\theta_3(z,\tau) := 1 + 2\sum_{n=1}^\infty q^{n^2}\cos(2nz)$, $q := e^{i\pi\tau}$, $\tau \in \mathbb{C}$,

$$\theta_3(z,\tau) = \frac{1}{\sqrt{-i\tau}}\, e^{z^2/i\pi\tau}\, \theta_3\left(\frac{z}{\tau}\left|\frac{-1}{\tau}\right.\right), \tag{6}$$

or, equivalently,

$$\sum_{n=-\infty}^{\infty} e^{-(n+z)^2 t} = \sqrt{\frac{\pi}{t}} \sum_{n=0}^{\infty} e^{-\frac{\pi^2 n^2}{t}} \cos(2\pi n z), \quad z,t \in \mathbb{C},\ \Re t > 0. \tag{7}$$

In higher dimensions we use (following Riemann) Poisson's summation formula

$$\sum_{\vec{n}\in\mathbb{Z}^p} f(\vec{n}) = \sum_{\vec{m}\in\mathbb{Z}^p} \tilde{f}(\vec{m}), \tag{8}$$

with $\tilde{f}$ the Fourier transform (see also [44, 45]). All these expressions are no more valid when one has truncated sums. They are replaced by asymptotic series, which are sometimes very well behaved, too [11, 12].

3˙2. *Extended CS series formulas (ECS).* – Consider the zeta function

$$\zeta_{A,\vec{c},q}(s) = \sum_{\vec{n}\in\mathbb{Z}^p}{}' \left[\frac{1}{2}(\vec{n}+\vec{c})^T A(\vec{n}+\vec{c}) + q\right]^{-s} = \sum_{\vec{n}\in\mathbb{Z}^p}{}' \left[Q(\vec{n}+\vec{c}) + q\right]^{-s}, \tag{9}$$

with $\Re s > p/2, A > 0, \Re q > 0$, where the prime means that the point $\vec{n} = \vec{0}$ is to be excluded from the sum (an inescapable condition when $c_1 = \ldots = c_p = q = 0$), and $Q(\vec{n}+\vec{c}) + q = Q(\vec{n}) + L(\vec{n}) + \bar{q}$ (*i.e.*, quadratic plus linear plus constant). Several completely different cases must be considered. Let us just discuss two of them.

(a) Case $q \neq 0$ ($\Re q > 0$). We get

$$\begin{aligned}\zeta_{A,\vec{c},q}(s) = {} & \frac{(2\pi)^{p/2} q^{p/2-s}}{\sqrt{\det A}} \frac{\Gamma(s-p/2)}{\Gamma(s)} + \frac{2^{s/2+p/4+2}\pi^s q^{-s/2+p/4}}{\sqrt{\det A}\ \Gamma(s)} \\ & \times \sum_{\vec{m}\in\mathbb{Z}^p_{1/2}}{}' \cos(2\pi\vec{m}\cdot\vec{c}) \left(\vec{m}^T A^{-1}\vec{m}\right)^{s/2-p/4} K_{p/2-s}\left(2\pi\sqrt{2q\,\vec{m}^T A^{-1}\vec{m}}\right),\end{aligned} \tag{10}$$

tagged as ECS1 in the zeta-literature. K_ν are modified Bessel function of the second kind and the subindex 1/2 in $\mathbb{Z}^p_{1/2}$ means that only *half* of the vectors $\vec{m} \in \mathbb{Z}^p$ go in the sum. *E.g.*, if we take an $\vec{m} \in \mathbb{Z}^p$ we must then exclude $-\vec{m}$ (as simple criterion, just select those vectors in $\mathbb{Z}^p\backslash\{\vec{0}\}$ whose first non-zero component is positive). Note the explicit pole at $s = p/2$ with residue: $\mathrm{Res}_{s=p/2}\,\zeta_{A,\vec{c},q}(s) = \frac{(2\pi)^{p/2}}{\Gamma(p/2)}(\det A)^{-1/2}$.

This very useful expression is a paradigm of all cases, for:

1) It gives the (analytic continuation of) the multidimensional zeta function in terms of an *exponentially convergent* multiseries, valid on the *whole* complex plane

2) It *explicitly* exhibits the singularities (*simple poles*) of the meromorphic continuation, with the corresponding *residua*.

3) The only condition here is that the matrix A must correspond to a non-negative quadratic form, Q. The vector $\vec{c}$ is arbitrary, while q is (to start) a non-negative constant (this last restriction can be somehow relaxed).

(b) Case $c_1 = \ldots = c_p = q = 0$ (the genuine extension of the CS formula to many dimensions), diagonal subcase:

$$(11)\ \zeta_{A_p}(s) = \frac{2^{1+s}}{\Gamma(s)} \sum_{j=0}^{p-1} (\det A_j)^{-1/2} \left[\pi^{j/2} a_{p-j}^{j/2-s}\, \Gamma\left(s - \frac{j}{2}\right) \zeta_R(2s-j) \right.$$
$$\left. + 4\pi^s a_{p-j}^{\frac{j}{4}-\frac{s}{2}} \sum_{n=1}^{\infty} \sum_{\vec{m}_j \in \mathbb{Z}^j}{}' n^{j/2-s} \left(\vec{m}_j^t A_j^{-1} \vec{m}_j\right)^{s/2-j/4} K_{j/2-s}\left(2\pi n \sqrt{a_{p-j} \vec{m}_j^t A_j^{-1} \vec{m}_j}\right) \right],$$

an expression tagged as ECS3d in the literature.

4. – The zeta function of a ΨDO and its associated determinant

The conditions for the existence of the zeta function of a pseudodifferential operator (ΨDO) and the definition of determinant thereby obtained will be here reviewed, as well as the concept of multiplicative anomaly associated with the determinant and its calculation by means of the Wodzicki residue.

4·1. *Pseudodifferential operator*. – A ΨDO, A, of order m on a manifold M_n is defined by its symbol $a(x, \xi)$, which is a function belonging to the space $S^m(\mathbb{R}^n \times \mathbb{R}^n)$ of $\mathbb{R}^\infty$ functions such that for any pair of multi-indices α, β there exists a constant $C_{\alpha,\beta}$ so that $|\partial_\xi^\alpha \partial_x^\beta a(x, \xi)| \leq C_{\alpha,\beta}(1 + |\xi|)^{m-|\alpha|}$. Then the definition of A is given, in the distribution sense, by

$$(12)\qquad Af(x) = (2\pi)^{-n} \int e^{i\langle x,\xi\rangle} a(x, \xi) \hat{f}(\xi)\, \mathrm{d}\xi,$$

where f is a smooth function, $f \in \mathcal{S}$(7), $\mathcal{S}'$ being the space of tempered distributions and $\hat{f}$ the Fourier transform of f. When $a(x, \xi)$ is a polynomial in ξ one gets a differential operator. In general, the order m may be complex. The *symbol* of a ΨDO has the form

$$(13)\qquad a(x, \xi) = a_m(x, \xi) + a_{m-1}(x, \xi) + \ldots + a_{m-j}(x, \xi) + \ldots,$$

being $a_k(x, \xi) = b_k(x)\, \xi^k$.

ΨDOs are useful, both in mathematics and in physics. They were crucial for the proof of the uniqueness of the Cauchy problem [46] and of the Atiyah-Singer index formula [47]. In QFT they appear in any analytical continuation process (as complex powers of differential operators, like the Laplacian) [48]. And they constitute nowadays the basic starting point of any rigorous formulation of QFT through microlocalization, a concept that is considered to be the most important step towards the understanding of linear partial differential equations since the invention of distributions [49].

(7) Remember that $\mathcal{S} = \{f \in C^\infty(\mathbb{R}^n);\ \sup_x |x^\beta \partial^\alpha f(x)| < \infty, \forall \alpha, \beta \in \mathbb{R}^n\}$.

4·2. *The zeta function.* – Let A a positive-definite elliptic ΨDO of positive order $m \in \mathbb{R}$, acting on the space of smooth sections of E, an n-dimensional vector bundle over M, a closed n-dimensional manifold. The *zeta function* ζ_A is defined as

$$\zeta_A(s) = \text{tr } A^{-s} = \sum_j \lambda_j^{-s}, \qquad \text{Re } s > \frac{n}{m} \equiv s_0, \tag{14}$$

where $s_0 =$ dim $M/\text{ord}\, A$ is called the *abscissa of convergence* of $\zeta_A(s)$. Under these conditions, it can be proven that $\zeta_A(s)$ has a meromorphic continuation to the whole complex plane $\mathbb{C}$ (regular at $s = 0$), provided that the principal symbol of A (that is $a_m(x, \xi)$) admits a *spectral cut*: $L_\theta = \{\lambda \in \mathbb{R}; \text{Arg}\, \lambda = \theta, \theta_1 < \theta < \theta_2\}$, $\text{Spec}\, A \cap L_\theta = \emptyset$ (Agmon-Nirenberg condition). The definition of $\zeta_A(s)$ depends on the position of the cut L_θ. The only possible singularities of $\zeta_A(s)$ are *simple poles* at $s_k = (n-k)/m,\ k = 0, 1, 2, \ldots, n-1, n+1, \ldots$. M. Kontsevich and S. Vishik have managed to extend this definition to the case when $m \in \mathbb{R}$ (no spectral cut exists) [50].

4·3. *The zeta determinant.* – Let A a ΨDO operator with a spectral decomposition: $\{\varphi_i, \lambda_i\}_{i \in I}$, where I is some set of indices. The definition of determinant starts by trying to make sense of the product $\prod_{i \in I} \lambda_i$, which can be easily transformed into a "sum": $\ln \prod_{i \in I} \lambda_i = \sum_{i \in I} \ln \lambda_i$. From the definition of the zeta function of A: $\zeta_A(s) = \sum_{i \in I} \lambda_i^{-s}$, by taking the derivative at $s = 0$: $\zeta_A'(0) = -\sum_{i \in I} \ln \lambda_i$, we arrive at the following definition of determinant of A [51]: $\det_\zeta A = \exp[-\zeta_A'(0)]$. An older definition (due to Weierstrass) is obtained by subtracting in the series above the leading behavior of λ_i as a function of i, as $i \to \infty$, until the series $\sum_{i \in I} \ln \lambda_i$ is made to converge. The shortcoming is here —for physical applications— that these additional terms turn out to be *non-local* and, thus, non-admissible in any renormalization procedure.

In algebraic QFT, in order to write down an action in operator language one needs a functional that replaces integration. For the Yang-Mills theory this is the Dixmier trace, which is the *unique* extension of the usual trace to the ideal $\mathcal{L}^{(1,\infty)}$ of the compact operators T such that the partial sums of its spectrum diverge logarithmically as the number of terms in the sum: $\sigma_N(T) \equiv \sum_{j=0}^{N-1} \mu_j = \mathcal{O}(\log N)$, $\mu_0 \geq \mu_1 \geq \ldots$. The definition of the Dixmier trace of T is: $\text{Dtr } T = \lim_{N \to \infty} \frac{1}{\log N} \sigma_N(T)$, provided that the Cesaro means $M(\sigma)(N)$ of the sequence in N are convergent as $N \to \infty$ (remember that: $M(f)(\lambda) = \frac{1}{\ln \lambda} \int_1^\lambda f(u) \frac{\mathrm{d}u}{u}$). Then, the Hardy-Littlewood theorem can be stated in a way that connects the Dixmier trace with the residue of the zeta function of the operator T^{-1} at $s = 1$ (see Connes [52]): $\text{Dtr } T = \lim_{s \to 1^+} (s-1) \zeta_{T^{-1}}(s)$.

The Wodzicki (or non-commutative) residue [53] is the *only* extension of the Dixmier trace to the ΨDOs which are not in $\mathcal{L}^{(1,\infty)}$. It is the *only* trace one can define in the algebra of ΨDOs (up to a multiplicative constant), its definition being: res $A = 2\ \text{Res}_{s=0}\ \text{tr}(A \Delta^{-s})$, with Δ the Laplacian. It satisfies the trace condition: res $(AB) =$ res (BA). A very important property is that it can be expressed as an integral (local form) res $A = \int_{S^*M} \text{tr } a_{-n}(x, \xi)\, \mathrm{d}\xi$, with $S^*M \subset T^*M$ the co-sphere bundle on M (some authors put a coefficient in front of the integral: Adler-Manin residue).

If dim $M = n = -$ ord A (M compact Riemann, A elliptic, $n \in \mathbb{N}$) it coincides with the Dixmier trace, and $\text{Res}_{s=1} \zeta_A(s) = \frac{1}{n} \text{res } A^{-1}$. The Wodzicki residue also makes sense for ΨDOs of arbitrary order and, even if the symbols $a_j(x, \xi)$, $j < m$, are not invariant under coordinate choice, their integral is, and defines a trace. All residua at poles of the zeta function of a ΨDO can be obtained from the Wodzciki residue [54].

4·4. *The multiplicative anomaly and its implications.* – Given A, B and AB ΨDOs, even if ζ_A, ζ_B and ζ_{AB} exist, it turns out that, in general, $\det_\zeta(AB) \neq \det_\zeta A \det_\zeta B$. The multiplicative (or non-commutative) anomaly (or defect of the determinant) is defined as

$$\delta(A,B) = \ln\left[\frac{\det_\zeta(AB)}{\det_\zeta A \ \det_\zeta B}\right] = -\zeta'_{AB}(0) + \zeta'_A(0) + \zeta'_B(0). \tag{15}$$

Wodzicki's formula for the multiplicative anomaly [53, 55] reads

$$\delta(A,B) = \frac{\mathrm{res}\left\{[\ln \sigma(A,B)]^2\right\}}{2 \,\mathrm{ord}\, A \,\mathrm{ord}\, B \,(\mathrm{ord}\, A + \mathrm{ord}\, B)}, \quad \sigma(A,B) := A^{\mathrm{ord}\, B} B^{-\mathrm{ord}\, A}. \tag{16}$$

We now explain how this anomaly may appear in physics. At the level of Quantum Mechanics (QM), where it was originally introduced by Feynman, the path-integral approach is just an alternative formulation of the theory but in QFT it is much more than this, being in many occasions *the* actual formulation of QFT [6]. Consider then the Gaussian functional integration

$$\int [\mathrm{d}\Phi] \ \exp\left\{-\int \mathrm{d}^D x \left[\Phi^\dagger(x) (\quad) \Phi(x) + \ldots\right]\right\} \quad \longrightarrow \quad \det(\quad)^{\pm 1} \tag{17}$$

(the $\pm$ depends on the spin of the fields), and assume that the operator matrix is reducible to the more simple structure (each A_i is an operator on its own)

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \tag{18}$$

the last expression being the result of diagonalization. The question arises: what is the determinant of the operator matrix? Is it $\det(AB)$ or $\det A \cdot \det B$ [56]. We may agree on that: i) In a situation where a superselection rule exists, AB has no sense (much less its determinant), and then the answer should be $\det A \cdot \det B$. ii) If the diagonal form is obtained after a change of basis (diagonalization process), then the quantity that is preserved by such transformations is the value of $\det(AB)$ and *not* the product of the individual determinants (there are examples supporting this viewpoint [57]). For more detailed information on the multiplicative anomaly see the seminal references [58].

* * *

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