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The Casimir effect and its mathematical implications

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Summary. — We describe the Casimir effect in the context of a spectral problem resulting from partial differential equations. Different formulations, namely the local vacuum energy density, Green's functions and functional determinants are used to give formal expressions for the Casimir energy. Regularizations then employed are the zeta function, the frequency-cutoff and point splitting in combination with Green's functions. Examples for single-body Casimir energies are considered. Singularities related to ambiguities are associated with heat kernel coefficients, invariants that describe the small-t asymptotics of the heat kernel. Renormalization is discussed in terms of these, in particular the coefficients are used to elegantly discuss if given configurations lead to unambiguous predictions for the Casimir energy and/or force. An example for the singularity-free situation is the Casimir force between separate bodies and a formalism for its computation is given.

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1. – Introduction

The continuing miniaturization of all kinds of technical devices makes the influence of the precise form of very small systems increasingly more important. This has led to an enormous research activity about the Casimir effect, in which the question how the presence of boundaries modifies the vacuum structure of a quantum field is analyzed [1-3]. The name goes back to the study by Casimir [4], where he computed the vacuum pressure between two perfectly conducting parallel plates due to the ground state of the electromagnetic field. Imposing the relevant boundary condition for the tangential, respectively normal, component of the electric, respectively magnetic, field on the surface $\mathcal{F}$, namely $E_{tan}(t,\vec{r})|_{\mathcal{F}} = 0$ and $B_{nor}(t,\vec{r})|_{\mathcal{F}} = 0$, he found the Casimir interaction energy (per unit area) of two plates a distance a apart to be

$$E_0(a) = -\frac{\pi^2}{720}\frac{\hbar c}{a^3}, \tag{1}$$

which gives the Casimir force (per unit area)

$$F_0(a) = -\frac{\mathrm{d}}{\mathrm{d}a}E(a) = -\frac{\pi^2}{240}\frac{\hbar c}{a^4}\,. \tag{2}$$

Since then, many aspects of the Casimir energy and force have been considered, in particular the influence of the shape of the boundary, the boundary condition imposed, non-trivial topology, and external parameters (background fields) have been analyzed in detail; see, *e.g.*, [5-7]. It is noted that calculations are often plagued by divergencies and of course it is desirable to identify the corresponding situations and to formulate relevant quantities in terms of finite expressions.

Corresponding to the above remarks, this article is organized as follows. In sect. **2** different formulations of the Casimir energy are given. Each formulation naturally leads to a regularization of the Casimir energy, as described in sect. **3**. Examples for Casimir energy calculations are given in sect. **4**. Cases treated are those with an explicit eigenvalue spectrum or a spectrum that is determined by an implicit eigenvalue equation. These examples will show that sometimes divergencies appear, but sometimes they do not. Section **5** will shed light on the origin of divergencies and we will see that the small-t heat equation asymptotics is at the heart of the issue. Particular terms in the heat equation asymptotics will allow us to easily identify configurations with finite Casimir energy or force and this is explained in sect. **6**. Examples for this situation are pistons and the electromagnetic field, furthermore separate bodies and the TGTG representation for the Casimir energy associated with separate bodies is provided in sect. **7**.

2. – Different formulations of the Casimir energy

In order to explain the different formulations of the Casimir energy we will consider a free massive scalar field in four-dimensional spacetime described by the action

$$S[\varphi] = \int \mathrm{d}^4x\mathcal{L}(x) = \int \mathrm{d}^4x \left(\frac{1}{2}\partial^\nu\varphi\partial_\nu\varphi - \frac{m^2}{2}\varphi^2 + \Upsilon\varphi\right),$$

where Υ is some external source. For vanishing external source the corresponding field equations are

$$(\Box + m^2)\varphi(x) = 0. \tag{3}$$

In terms of the field $\varphi(x)$, Noether's theorem gives the canonical energy-momentum tensor as

$$T_{\mu\nu}(x) = \partial_\mu\varphi(x)\partial_\nu\varphi(x) - g_{\mu\nu}\mathcal{L}(x). \tag{4}$$

More explicit representations are obtained by using the mode expansion of $\varphi(x)$. Writing $x = (t, \vec{r}\,)$, using separation of variables, solutions of (3) are written as

$$\varphi_J^{(+)}(t,\vec{r}\,) = \frac{1}{\sqrt{2\omega_J}}e^{-i\omega_J t}\Phi_J(\vec{r}\,), \qquad \varphi_J^{(-)}(t,\vec{r}\,) = \left[\varphi_J^{(+)}(t,\vec{r}\,)\right]^*,$$

where $\Phi_J(\vec{r})$ satisfies the eigenvalue problem

$$-\Delta\Phi_J(\vec{r}) = \Lambda_J\Phi_J(\vec{r}), \quad \Phi_J(\vec{r})|_{\mathcal{F}} = 0, \quad \Lambda_J \equiv \omega_J^2 - m^2. \tag{5}$$

Here, $\mathcal{F}$ denotes a typically smooth boundary at which the field satisfies some boundary condition. We have chosen Dirichlet boundary conditions for simplicity, other conditions could be considered as well.

Introducing as usual annihilation operators a_J and creation operators a_J^+, the field operator is expanded in the form

$$\varphi(x) = \sum_J \left[\varphi_J^{(+)}(x)a_J + \varphi_J^{(-)}(x)a_J^+\right], \tag{6}$$

which, after substitution into (4), gives for the vacuum expectation value of the local vacuum energy density

$$\begin{aligned}\langle 0\,|T_{00}(x)|\,0\rangle &= \frac{1}{2}\left\langle 0\left|\left[\sum_{\mu=0}^{3}\left(\frac{\partial\varphi}{\partial x^\mu}\right)^2 + m^2\varphi^2\right]\right|0\right\rangle, \\ &= \sum_J \frac{1}{4\omega_J}\left[(\omega_J^2+m^2)\Phi_J(\vec{r})\Phi_J^*(\vec{r}) + \sum_{k=1}^{3}\frac{\partial\Phi_J(\vec{r})}{\partial x^k}\frac{\partial\Phi_J^*(\vec{r})}{\partial x^k}\right].\end{aligned} \tag{7}$$

Integrating over the volume V of the system, after an integration by parts, the total vacuum energy is obtained as

$$E_0 = \int_V \mathrm{d}\vec{r}\,\langle 0\,|T_{00}(x)|\,0\rangle = \frac{1}{2}\sum_J \omega_J. \tag{8}$$

Clearly this last expression (8), and also (7), are divergent and later on regularizations will be provided. But first let us describe different representations of the energy that in a natural way will lead to a variety of different regularizations.

Let us next consider the representation of the vacuum energy in terms of Green's functions. For the model considered the relevant partial differential equation is

$$(\Box_x + m^2)G(x,x') = \delta^4(x-x').$$

Clearly, the Green's function is not uniquely defined as solutions of the homogeneous equation can be added. In terms of the modes (5), the causal Green's function reads

$$\begin{aligned}G(x,x') &= \int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\sum_J \frac{\Phi_J(\vec{r})\Phi_J^*(\vec{r}\,')}{-\omega^2+\omega_J^2-i\epsilon}e^{-i\omega(t-t')} \\ &= i\sum_J \frac{1}{2\omega_J}e^{-i\omega_J|t-t'|}\Phi_J(\vec{r})\Phi_J^*(\vec{r}\,'),\end{aligned} \tag{9}$$

where the limit $\epsilon \to 0$ is understood. The second line follows by computing the ω-integral using the residue theorem.

We next would like to relate the energy density (7) with the Green's function. As indicated, (7) is singular as it contains the product of two field operators at coincident points. To establish the connection between (7) and (9), we will take the field operators in (7) at separate points and apply the time-ordered product

$$T\varphi(x)\varphi(x') = \theta(t-t')\varphi(x)\varphi(x') + \theta(t'-t)\varphi(x')\varphi(x).$$

From (6) we easily show

$$\langle 0 \,|\varphi(x)\varphi(x')|\, 0\rangle = \sum_J \frac{1}{2\omega_J} e^{-i\omega_J(t-t')}\Phi_J(\vec r)\Phi_J^*(\vec r\,')$$

and

$$i\,\langle 0\,|T\varphi(x)\varphi(x')|\,0\rangle = G(x,x')$$

can be established. With the interpretation of (7) as described, we therefore arrive at the definition

$$\langle 0\,|T_{00}(x)|\,0\rangle \equiv -\frac{i}{2}\left(\sum_{\mu=0}^{3}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x'^\mu}+m^2\right)G(x,x')\Big|_{x'=x}$$

from which after partial integration, using the equation of motion, the global vacuum energy reads

$$E_0 = i\int_V \mathrm{d}\vec r \frac{\partial^2 G(x,x')}{\partial x_0^2}\Big|_{x'=x}.$$

Let us lastly consider the path-integral formulation of the vacuum energy. In this formulation the generating function is given by

$$Z[\Upsilon] = C\int D\varphi e^{iS[\varphi]}$$

and the vacuum energy, for time-independent boundaries, equals [8]

$$E_0 = \frac{i}{T}\ln Z[0].$$

The integration goes over fields in a suitably defined space and C is an infinite normalization constant. It is independent of external parameters and is irrelevant for the vacuum energy. The generating functional is computed using an analogy with finite-dimensional matrices. For $x, h\in\mathbb{R}^n$ and $\tilde{\mathcal{K}}$ a finite-dimensional matrix with inverse $\tilde{\mathcal{K}}^{-1}$, one computes the integrals

$$\int_{\mathbb{R}^n}\mathrm{d}^n x e^{-\frac12(x,\tilde{\mathcal{K}}x)+(x,h)} = (2\pi)^{n/2}(\det\tilde{\mathcal{K}})^{-1/2}e^{\frac12(h,\tilde{\mathcal{K}}^{-1}h)}. \qquad (10)$$

If $\tilde{\mathcal{K}}$ instead is a partial differential operator, replacing the scalar product in $\mathbb{R}^n$ with a Hilbert space product,

$$(x, \tilde{\mathcal{K}}x) \;\to\; (\varphi, \tilde{\mathcal{K}}\varphi) = \int \mathrm{d}^4x\varphi(x)(\tilde{\mathcal{K}}\varphi)(x),$$

from the action

$$iS[\varphi] = -\frac{i}{2}\int \mathrm{d}^4x\ \varphi(x)(\Box + m^2)\varphi(x) + i\int \mathrm{d}^4x\ \Upsilon\varphi(x) \tag{11}$$

one identifies

$$h = i\Upsilon, \quad \tilde{\mathcal{K}} = i\mathcal{K} \equiv i(\Box + m^2), \quad K(x,x') = \delta(x-x')(\Box_{x'} + m^2), \tag{12}$$

where we introduced the kernel $K(x,x')$ of the operator $\mathcal{K}$. From here, formally

$$\begin{aligned} Z[\Upsilon] &= C(\det \tilde{\mathcal{K}})^{-1/2} e^{\frac{1}{2}(h,\tilde{\mathcal{K}}^{-1}h)} \\ &= C(\det \mathcal{K})^{-1/2} \exp\left[\frac{i}{2}\int \mathrm{d}^4x\mathrm{d}^4x'\Upsilon(x)K^{-1}(x,x')\Upsilon(x')\right]. \end{aligned} \tag{13}$$

Note, that from the definitions

$$K^{-1}(x,x') = G(x,x').$$

The representation of the vacuum energy in this case reads

$$E_0 = \frac{i}{T}\ln(\det \mathcal{K})^{-1/2} = -\frac{i}{2T}\operatorname{Tr}\ln \mathcal{K}.$$

In case boundaries, at which boundary conditions are imposed, are present, one needs to ensure that the integration $D\varphi$ only extends over fields satisfying the correct boundary condition. The method described in the following was developed for quantum electrodynamics with conductor boundary conditions in [9]. For Dirichlet boundary condition the relevant starting point is

$$Z[\Upsilon] = C\int D\varphi \prod_{x\in\mathcal{F}} \delta(\varphi(x)) e^{iS[\varphi]},$$

which guarantees that only fields with $\varphi(x)|_{\mathcal{F}} = 0$ are contributing. The next step involves the rewriting of the δ-functional in such a way that the resulting integration once again is Gaussian and can therefore be done as before. Parameterizing points on the surface using $x_0 = \eta_0$, $\vec{r} = \vec{u}(\eta_1, \eta_2)$, the relevant analogy is

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k e^{ikx} \quad \Longrightarrow \prod_{x\in\mathcal{F}} \delta(\varphi(x)) = C\int Db\ e^{i\int \mathrm{d}\mu(\eta)b(\eta)\varphi(u(\eta))}.$$

Here, C is another irrelevant normalization constant, $\mathrm{d}\mu(\eta)$ is the volume element on $\mathcal{F}$, and $b(\eta)$ is an auxiliary field defined on the surface $\mathcal{F}$ representing the variable of integration; note, that once again a scalar product has been replaced by a Hilbert space product. The integrand then reads

$$\prod_{x\in\mathcal{F}} \delta(\varphi(x))e^{iS[\varphi]} \longrightarrow$$

$$-\frac{i}{2}\int \mathrm{d}^4x\ \varphi(x)(\Box + m^2)\varphi(x) + i\int \mathrm{d}^4x \Upsilon(x)\varphi(x) + i\int \mathrm{d}\mu(\eta)\ b(\eta)\varphi(u(\eta)). \tag{14}$$

In order to identify the relevant pieces in the Gaussian integration (10), we need to rewrite the boundary integral in (14) as an integral over the full space. To this end, we introduce an additional integration by inserting the δ-function $H(\eta, x) = \delta^4(x - u(\eta))$, that is, we write

$$\int \mathrm{d}\mu(\eta)b(\eta)\varphi(u(\eta)) = \int \mathrm{d}^4x \int \mathrm{d}\mu(\eta)\ b(\eta)H(\eta,x)\varphi(x),$$

which turns the integrand (14) into a form that allows to identify the relevant h,

$$-\frac{i}{2}\int \mathrm{d}^4x\ \varphi(x)(\Box + m^2)\varphi(x) + i\int \mathrm{d}^4x \left[\Upsilon(x) + \int \mathrm{d}\mu(\eta)b(\eta)H(\eta,x)\right]\varphi(x)$$
$$\Longrightarrow h = i\left[\Upsilon(x) + \int \mathrm{d}\mu(\eta)b(\eta)H(\eta,x)\right].$$

Performing the φ-integration, using eqs. (11)-(13), the relevant quadratic term for the b-integration is

$$\int \mathrm{d}^4x \int \mathrm{d}^4x' \int \mathrm{d}\mu(\eta)b(\eta)H(\eta,x)K^{-1}(x,x') \int \mathrm{d}\mu(\eta')b(\eta')H(\eta',x')$$
$$= \int \mathrm{d}\mu(\eta) \int \mathrm{d}\mu(\eta')b(\eta)G(u(\eta),u(\eta'))b(\eta'),$$

such that the operator defined by the kernel

$$\tilde{K}(\eta,\eta') = G(u(\eta),u(\eta')) \tag{15}$$

enters the answer. Note that this is the bulk Green's function restricted to the surface $\mathcal{F}$. After performing φ- and b-integrations, the generating functional is seen to have the form

$$Z[\Upsilon] = C(\det\mathcal{K})^{-1/2}(\det\tilde{\mathcal{K}})^{-1/2}\exp\left[\frac{i}{2}\int \mathrm{d}^4x\mathrm{d}^4x'\Upsilon(x)^{\mathcal{F}}G(x,x')\Upsilon(x')\right], \tag{16}$$

where ${}^{\mathcal{F}}G(x,x')$ is the propagator accounting for the boundary conditions; for details see [1]. However, for our purposes ${}^{\mathcal{F}}G(x,x')$ is irrelevant as we only use $Z[0]$. For the vacuum energy, (16) implies

$$E_0 = -\frac{i}{2T}\,\mathrm{Tr}\ln\tilde{\mathcal{K}} = \frac{i}{T}\ln(\det\tilde{\mathcal{K}})^{-1/2}, \tag{17}$$

where we have dropped irrelevant contributions containing only information about the free space configuration.

3. – Regularizations of the Casimir energy

Now that we have different representations of the Casimir energy available, in this section we introduce different regularizations for the formal definitions of the vacuum energy. It must be mentioned that infinitely many regularizations are possible. Here we consider the most frequently used ones.

Let us start with the zeta function regularization, which interprets the divergent mode sum as follows,

$$E_0 = \frac{1}{2}\sum_J \omega_J \quad \to \quad E_0(s) = \frac{\mu^{2s}}{2}\sum_J \omega_J^{1-2s} = \frac{\mu^{2s}}{2}\zeta\left(s-\frac{1}{2}\right), \tag{18}$$

where we introduce the zeta function

$$\zeta(s) = \sum_J \omega_J^{-2s} \tag{19}$$

associated with the spectrum ω_J^2 defined by the eigenvalue problem

$$(-\Delta + m^2)\Phi_J(\vec{r}\,) = \omega_J^2 \Phi_J(\vec{r}\,), \quad \Phi_J(\vec{r}\,)\big|_{\mathcal{F}} = 0. \tag{20}$$

In general, s is complex and the limit of removing the regularization in (18) requires the continuation to $s = 0$. Furthermore, we introduced the length scale μ needed to give $E_0(s)$ the dimension of an energy for all values of s. The factor of μ is arbitrary and represents the ambiguity entering the problem along with the regularization. The series (19) is convergent for $\Re s > d/2$, where d is the dimension of space. As said, for the vacuum energy the relevant value in (18) is $s = 0$ and it is $\zeta(-1/2)$ that needs to be analyzed. This value will sometimes be singular and we will see how to identify (and to avoid) these situations.

Alternatively one can regularize the mode sum by a frequency-cutoff according to

$$E_0 = \frac{1}{2}\sum_J \omega_J \quad \to \quad E_0(\delta) = \frac{1}{2}\sum_J \omega_J e^{-\delta\omega_J^2},$$

where the limit $\delta \to 0$ has to be considered. The regularization factor $e^{-\delta\omega_J^2}$ can be interpreted as to take into account that at high frequencies media become transparent. The ambiguity here results from the freedom to multiply δ by a number: $\delta \to c \cdot \delta$.

Finally, the Green's function calculation presented in the previous section suggests the use of the regularization

$$E_0 = i\int_V \mathrm{d}\vec{r}\frac{\partial^2 G(x,x')}{\partial x_0^2}\bigg|_{x'=x} \to E_0(\epsilon) = i\int_V \mathrm{d}\vec{r}\frac{\partial^2 G(x,x')}{\partial x_0^2}\bigg|_{x'=x+\epsilon},$$

which can be interpreted as a splitting of the point x_0 into two. Typically, the parameter ϵ will indicate a shift in the (imaginary) time and $\epsilon \to 0$ has to be considered.

Later we will discuss the relations between the different regularization schemes, but first let us consider a few example calculations. We will use the zeta function regularization, although the other schemes could have, of course, been chosen as well.

4. – Examples for Casimir energy calculations

In this section we present several examples for computations of Casimir energies. We choose examples where enough knowledge of the eigenvalue spectrum is given such that the Casimir energy can be computed relatively easily. The first class of examples considered are configurations for which the spectrum is known explicitly. The second class comprises examples where an implicit knowledge is given and eigenvalues follow as roots of non-trivial special functions.

4·1. *Explicit eigenvalue spectrum.* – We start with the example of parallel plates a distance a apart. Imposing Dirichlet boundary conditions, the relevant eigenvalue problem reads,

$$-\Delta u_\ell(x,y,z) = \omega_\ell^2 u_\ell(x,y,z), \qquad u_\ell(x,y,0) = u_\ell(x,y,a) = 0,$$

with Δ the Laplacian in $\mathbb{R}^3$. Eigenfunctions clearly are

$$u_{\vec{k},n}(x,y,z) = e^{ik_x x + ik_y y} \sin\left(\frac{n\pi z}{a}\right), \qquad (k_x,k_y) \in \mathbb{R}^2, \quad n \in \mathbb{N},$$

from which the eigenvalues

$$\omega^2_{\vec{k},n} = \vec{k}^2 + \left(\frac{\pi n}{a}\right)^2, \qquad \vec{k} \in \mathbb{R}^2, \quad n \in \mathbb{N},$$

follow. The zeta function density per unit area then becomes

$$\zeta(s) = \int_{-\infty}^{\infty} \frac{\mathrm{d}^2 k}{(2\pi)^2} \sum_{n=1}^{\infty} \left[\vec{k}^2 + \left(\frac{\pi n}{a}\right)^2\right]^{-s} = \frac{1}{4\pi}\frac{1}{s-1}\left(\frac{\pi}{a}\right)^{2-2s} \zeta_R(2s-2) \tag{21}$$

and the Casimir energy

$$E_0 = -\frac{\pi^2}{1440 a^3},$$

and Casimir force,

$$F_0 = -\frac{\mathrm{d}}{\mathrm{d}a} E_0 = -\frac{\pi^2}{480 a^4}, \tag{22}$$

follow. For the continuation of (21) to $s = -1/2$ we used properties of the Riemann zeta function [10].

Note, the above calculation only takes the contributions from in between the two plates into account. The exterior contribution with respect to the plate at a is easily found introducing a plate at $x = L$ and sending L to infinity. The exterior contribution to the force is found by replacing a with $L - a$ in (22) and is seen to vanish,

$$F_{Cas} = \frac{\pi^2}{480(L-a)^4} \quad \stackrel{L\to\infty}{\to} \quad 0. \tag{23}$$

The result (22) therefore gives the correct result for the plate at a.

Other boundary conditions, in particular Neumann boundary conditions, can be treated along the same lines and adding the Dirichlet and Neumann result, the answers (1) and (2) for the electromagnetic field follow.

A natural generalization of the previous problem is to consider higher dimensions. The relevant differential equation in d-dimensions becomes

$$(-\Delta + m^2)\phi_{\vec n}(x_1,\dots,x_d) = \omega_{\vec n}^2 \phi_{\vec n}(x_1,\dots,x_d)$$

with suitable boundary conditions imposed. The relevant zeta function in this context is the zeta function of Epstein [11]. In detail, for periodic boundary conditions eigenfunctions and eigenvalues are

$$\begin{aligned}\phi_{\vec n}(x_1,\dots,x_d) &= A\exp\left\{\frac{2\pi n_1}{L_1}x_1+\dots+\frac{2\pi n_d}{L_d}x_d\right\}\\ \omega_{\vec n}^2 &= \left(\frac{2\pi n_1}{L_1}\right)^2+\dots+\left(\frac{2\pi n_d}{L_d}\right)^2+m^2, \quad n_i\in\mathbb{Z}\end{aligned} \tag{24}$$

leading to the zeta function

$$\zeta(s) = \sum_{\vec n\in\mathbb{Z}^d}\left(r_1n_1^2+\dots+r_dn_d^2+m^2\right)^{-s} = \zeta_E(s,m^2|\vec r),$$

where the series converges for $\Re s > d/2$ and where for the case at hand $r_i = (2\pi/L_i)^2$. The analytical continuation of this type of zeta function can be done by using the integral representation of the Γ-function in the form

$$\frac{1}{\lambda^s} = \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{d}t\; t^{s-1}e^{-\lambda t},$$

where $\lambda = \omega_{\vec n}^2$ is set. Next one applies the Poisson resummation formula [12]

$$\sum_{n=-\infty}^{\infty} e^{-\left(\frac{\pi n}{a}\right)^2 t} = \frac{a}{\sqrt{\pi t}}\sum_{n=-\infty}^{\infty} e^{-\frac{a^2n^2}{t}}$$

to each summation, which leads to an integral representation of the McDonald function [10],

$$\int_0^\infty \mathrm{d}t\; t^{-\nu-1}e^{-ct-\frac{b}{t}} = 2\left(\frac{c}{b}\right)^{\nu/2}K_\nu(2\sqrt{cb}).$$

Applying these formulas to the spectrum (24) at hand, one finds

$$\begin{aligned}\zeta_E(s,m^2|\vec r) &= \frac{\pi^{d/2}}{\sqrt{r_1\dots r_d}}\,\frac{\Gamma\left(s-\frac d2\right)}{\Gamma(s)}\,m^{d-2s} + \frac{2\pi^s m^{\frac{d-2s}{2}}}{\Gamma(s)\sqrt{r_1\dots r_d}}\\ &\times\sum_{\vec n\in\mathbb{Z}^d/\vec 0}\left[\frac{n_1^2}{r_1}+\dots+\frac{n_d^2}{r_d}\right]^{\frac12\left(s-\frac d2\right)} K_{\frac d2 - s}\left(2\pi m\left(\frac{n_1^2}{r_1}+\dots+\frac{n_d^2}{r_d}\right)^{1/2}\right),\end{aligned} \tag{25}$$

which is easily analyzed about $s = -1/2$. Note, that an odd dimension d leads to a singular Casimir energy because of the Γ-function in the first term of (25). A detailed analysis can be found in [13], where a varying number of free and periodic dimensions is considered.

Other boundary conditions can be dealt with along the same lines. For example, considering Dirichlet boundary conditions, and taking for simplicity $d = 2$, the relevant expressions are

$$\begin{aligned}\phi_{n,k}(x,y) &= A \sin\left(\frac{n\pi x}{L_1}\right)\sin\left(\frac{k\pi y}{L_2}\right),\\ \lambda_{n,m} &= \left(\frac{n\pi}{L_1}\right)^2 + \left(\frac{k\pi}{L_2}\right)^2 + m^2, \qquad n,k \in \mathbb{N},\\ \zeta(s) &= \sum_{n=1}^\infty \sum_{k=1}^\infty \left[\left(\frac{n\pi}{L_1}\right)^2 + \left(\frac{k\pi}{L_2}\right)^2 + m^2\right]^{-s},\end{aligned}$$

giving the relevant zeta function

$$\zeta(s) = \frac{1}{4}\zeta_E\left(s, m^2|\vec{r}\right) - \frac{1}{4}\zeta_E(s, m^2|r_1) - \frac{1}{4}\zeta_E(s, m^2|r_2) + \frac{1}{4}m^{-2s}.$$

In fact, Dirichlet or Neumann boundary conditions and any mixture of them can be expressed in terms of periodic boundary conditions [13]. Signs of the energy and force depend in a complicated way on the lengths L_i and no clear pattern evolves.

Another example with explicit eigenvalue knowledge is the sphere. Choosing its radius to be one, the eigenvalue problem reads

$$(-\Delta + \xi R)\mathbf{Y} = \lambda \mathbf{Y}, \qquad R = d(d-1).$$

The simplest situation occurs for conformal coupling $\xi = \frac{1}{4}\,\frac{d-1}{d}$, in which case [14]

$$\lambda = \omega_\ell^2 = \left(\ell + \frac{d-1}{2}\right)^2, \quad \deg(\ell) = (2\ell + d - 1)\frac{(\ell + d - 2)}{\ell!(d-1)!}, \tag{26}$$

where $\deg(\ell)$ is the angular degeneracy. The zeta function associated with this spectrum becomes

$$\zeta(s) = \sum_{\ell=0}^\infty \frac{\deg(\ell)}{\left(\ell + \frac{d-1}{2}\right)^{2s}}.$$

Rewriting the degeneracy using

$$\deg(\ell) = \binom{\ell + d - 1}{d-1} + \binom{\ell + d - 2}{d-1},$$

it is seen that the relevant zeta function for this configuration is the Barnes zeta function [15]

$$\zeta_{\mathcal{B}}(s,b) = \sum_{\vec{m}=0}^\infty (b + m_1 + \ldots + m_d)^{-s} = \sum_{\ell=0}^\infty \binom{\ell + d - 1}{d-1}(\ell + b)^{-s}.$$

In detail, the sphere zeta function in terms of the Barnes zeta function reads [16]

$$\zeta(s) = \zeta_{\mathcal{B}}\left(2s, \frac{d-1}{2}\right) + \zeta_{\mathcal{B}}\left(2s, \frac{d+1}{2}\right).$$

Particular dimensions can easily be worked out by expressing the Barnes zeta function in terms of the Hurwitz zeta function; see [7], Appendix A. For example in $d = 2$ the degeneracy is $\deg(\ell) = 2\ell + 1$ and so

$$\zeta(s) = \sum_{\ell=0}^{\infty} (2\ell+1)\left(\ell + \frac{1}{2}\right)^{2s} = 2\sum_{\ell=0}^{\infty}\left(\ell + \frac{1}{2}\right)^{2s-1} = 2\zeta_H\left(2s-1; \frac{1}{2}\right)$$

with resulting Casimir energy

$$E_0 = 2\zeta_H\left(-2; \frac{1}{2}\right) = 0.$$

In fact one can show that the Casimir energy vanishes for all even dimensions. Furthermore, *e.g.*,

$$E_0^3 = \frac{1}{240}, \quad E_0^5 = -\frac{31}{60480}, \quad E_0^7 = \frac{289}{604800}, \ldots$$

For these results and more, for example that the Casimir energy is singular for other than conformal couplings, see [17], where, however, no use has been made of the Barnes zeta function.

4·2. *Implicit eigenvalue spectrum.* – Often, the eigenvalues are *not* known explicitly, but instead eigenfunctions are known. In that case an eigenfunction expansion of the Green's function can be exploited, as has been done extensively on the ball for different fields, dimensions and boundary conditions; see, *e.g.*, [3]. Alternatively, the zeta function scheme can be used. Then the whole calculation is based upon an implicit eigenvalue equation, by which we mean that eigenvalues are defined implicitly as solutions to transcendental equations. The techniques to deal with this situation have been developed in the context of heat kernel coefficients and subsequently applied to functional determinants and Casimir energies; for a review see [7].

In order to outline the technique in some detail, let us consider a scalar field in a three-dimensional ball of radius R with Dirichlet boundary conditions. In this setting, eigenvalues ω_k^2 are determined through

$$-\Delta\phi_k(\vec{r}) = \omega_k^2\phi_k(\vec{r}), \qquad \phi_k(\vec{r})|_{|\vec{r}|=R} = 0.$$

In terms of spherical coordinates (r, Ω), eigenfunctions have the form

$$\phi_{\ell,m,n}(r,\Omega) = r^{-1/2} J_{\ell+1/2}(\omega_{\ell,n} r) Y_{\ell,m}(\Omega), \tag{27}$$

with $Y_{\ell,m}(\Omega)$ denoting the spherical surface harmonics [14], and where J_ν are Bessel functions of the first kind [10]. The boundary condition forces the eigenvalues to be zeroes of Bessel functions. For a given angular momentum quantum number ℓ, imposing Dirichlet boundary conditions, from (27), eigenvalues $\omega_{\ell,n}^2$ are determined by

$$J_{\ell+1/2}(\omega_{\ell,n} R) = 0, \qquad n = 1, 2, 3, \ldots \tag{28}$$

Choosing an anticlockwise contour γ that encloses all solutions of (28), Cauchy's residue theorem allows the zeta function to be written as

$$\zeta(s) = \sum_{\ell=0}^{\infty} (2\ell+1) \int_\gamma \frac{\mathrm{d}k}{2\pi i} k^{-2s} \frac{\partial}{\partial k} \ln J_{\ell+1/2}(kR). \tag{29}$$

The factor $(2\ell+1)$ following from (26) with $d=3$ represents the degeneracy for each angular momentum ℓ and the summation is over all angular momenta. It can be shown that the representation given is well defined for $\Re s > 3/2$.

The construction of the analytical continuation of $\zeta(s)$ to the relevant point $s=-1/2$ is a non-trivial task. We do it by subtracting and adding back suitable asymptotic terms. In order to explain the basic mechanism at work let us pretend we do not know anything about the following sum related to the Hurwitz zeta function,

$$R(s) = \sum_{\ell=1}^{\infty} \frac{1}{(\ell+a)^s}, \qquad 0 < a < 1, \tag{30}$$

but we do know elementary properties about the zeta function $\zeta_R(s)$ of Riemann,

$$\zeta_R(s) = \sum_{\ell=1}^{\infty} \frac{1}{\ell^s}.$$

Clearly, this series converges for $\Re s > 1$. Let us say we want to know the value of $R(s)$ at $s=0$. One way to proceed is to subtract and add the large-ℓ behavior of the summand and to rewrite (30) as follows:

$$\begin{aligned}
R(0) &= \sum_{\ell=1}^{\infty} \left. \frac{1}{(\ell+a)^s} \right|_{s=0} = \sum_{\ell=1}^{\infty} \left. \frac{1}{\ell^s} \frac{1}{\left(1+\frac{a}{\ell}\right)^s} \right|_{s=0} \\
&= \sum_{\ell=1}^{\infty} \left[\frac{1}{\ell^s} \left(\frac{1}{\left(1+\frac{a}{\ell}\right)^s} - 1 + \frac{as}{\ell} \right) + \frac{1}{\ell^s} - \frac{as}{\ell^{s+1}} \right]_{s=0} \\
&= \sum_{\ell=1}^{\infty} \left[\frac{1}{\ell^s} \left(\frac{1}{\left(1+\frac{a}{\ell}\right)^s} - 1 + \frac{as}{\ell} \right) \right]_{s=0} + \left. \zeta_R(s) \right|_{s=0} - \left. as\zeta_R(s+1) \right|_{s=0} \\
&= 0 + \zeta_R(0) - a \,\mathrm{Res}\,\zeta_R(1) = -\frac{1}{2} - a.
\end{aligned}$$

The relevant features observed here, and that carry through more generally, are that the asymptotic terms can be handled analytically, whereas the original expression with asymptotic terms subtracted is finite and can be evaluated numerically; here it happens to be zero, but that is naturally not so in other cases.

Of course the technicalities in (29) are more involved. For the example of the three-dimensional ball, after deforming the contour in (29) to the imaginary axis, the relevant asymptotics is the uniform asymptotic expansion of the Bessel function $I_\nu(k)$ for $\nu \to \infty$ as $z = k/\nu$ remains fixed [18]. We have

$$I_\nu(\nu z) \sim \frac{1}{\sqrt{2\pi\nu}} \frac{e^{\nu\eta}}{(1+z^2)^{1/4}} \left[1 + \sum_{k=1}^{\infty} \frac{u_k(t)}{\nu^k} \right],$$

where

$$t = \frac{1}{\sqrt{1+z^2}} \qquad \text{and} \qquad \eta = \sqrt{1+z^2} + \ln\left[\frac{z}{1+\sqrt{1+z^2}}\right].$$

The $u_k(t)$ are polynomials in t defined by a recursion relation [18]. The ability to integrate the leading orders in the resulting asymptotic expansion and to perform the summation in closed form, or at least to determine the resulting singularity structure, are the key factors needed to construct the analytical continuation. For the given example this step was done using properties of hypergeometric functions and the Barnes zeta function [19, 20].

Without going into further details, once this is accomplished, it can be used to evaluate, largely analytically, but partly numerically, the zeta-function about $s = -1/2$, thus obtaining the global Casimir energy for this setting [21-24]. Other boundary conditions and fields follow along the same line [7]. For example adding Dirichlet and suitable Robin boundary condition of a massless field, the final answer for the Casimir energy for the electromagnetic field with perfectly conducting boundary conditons is found to be

$$E_0^{el.\ magn.} = \frac{0.04617}{a},$$

thus the resulting Casimir force,

$$F_0^{el.\ magn.} = \frac{0.04617}{a^2},$$

is repulsive. For massive field results are singular and a discussion of that situation is given in sect. **6**.

Similar strategies can be applied to the case of a spherically symmetric potential, when the relevant operator reads

$$P = -\Delta + m^2 + V(r) \tag{31}$$

with m the mass of a scalar field. Although for non-trivial potentials $V(r)$ eigenfunctions are not known, the lack of this knowledge can be replaced by information coming from scattering theory. In particular one can show that

$$\zeta(s) = \frac{\sin(\pi s)}{\pi} \sum_{\ell=0}^{\infty} (2\ell+1) \int_m^\infty \mathrm{d}k (k^2 - m^2)^{-s} \frac{\partial}{\partial k} \ln f_\ell(k),$$

where $f_\ell(k)$ denotes the Jost function associated with the operator (31). The needed asymptotics then follows from the Lippmann-Schwinger equation [25] and the Casimir energy can be found [26]. Other fields can be treated as well [27, 28, 7] and variants of the above approach have been developed [29].

5. – Renormalization of the Casimir energy

As the examples of the previous section have shown, the Casimir energy (and force) sometimes contain singularities and sometimes they do not. It is the aim of this section to systematically study the singularity structure, which will allow us to know at the beginning of a calculation whether or not to expect finite, unambiguous answers. For this study we will use the frequency-cutoff regularization which will naturally lead to a study of the associated zeta functions.

Starting point of the analysis is the Casimir energy representation

$$E_0(\delta) = \frac{1}{2}\sum_J \omega_J e^{-\delta\omega_J^2} \tag{32}$$

and we need to find the $\delta \to 0$ behavior of this expression. An elegant way to approach the problem is the representation of exponentials as

$$e^{-v} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathrm{d}\alpha\ \Gamma(\alpha) v^{-\alpha}, \qquad \Re c > 0, \tag{33}$$

a representation known to be very useful in a variety of contexts [30]. Equation (33) can be easily understood by moving the contour to the left crossing over the singularities coming from the Γ-function. The resulting series expansion appears to be just that of the exponential function.

Using (33) in (32), after interchanging summation and integration, the zeta function of the spectrum results and one finds

$$\begin{aligned} E_0(\delta) &= \frac{1}{2}\sum_J \omega_J \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathrm{d}\alpha \Gamma(\alpha)(\delta\omega_J^2)^{-\alpha} \\ &= \frac{1}{4\pi i}\int_{c-i\infty}^{c+i\infty} \mathrm{d}\alpha\ \Gamma(\alpha)\delta^{-\alpha}\zeta\left(\alpha - \frac{1}{2}\right), \qquad \Re c > \frac{d+1}{2}\,. \end{aligned} \tag{34}$$

The condition on the real part of c restricting the location of the contour is necessary to enable one to interchange sum and integral. It is now clearly seen that the small-δ behavior of $E_0(\delta)$ is determined by the location of the poles of the integrand to the left of the contour and by its residues. The source of singularities are the poles of the Γ-function at $\alpha = -n$, $n \in \mathbb{N}_0$, and the poles of the zeta function. Let us therefore study next the meromorphic structure of the spectral zeta function $\zeta(s)$.

The approach usually employed for this purpose is to relate the zeta function with the associated heat kernel

$$K(t) = \sum_J e^{-t\omega_J^2}.$$

The connection between the two functions is

$$\zeta(s) = \sum_J (\omega_J^2)^{-s} = \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{d}t\ t^{s-1}\sum_J e^{-t\omega_J^2} = \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{d}t\ t^{s-1}K(t). \tag{35}$$

Assuming, as we do, $\omega_J^2 > 0$, the heat kernel is exponentially damped for $t \to \infty$ and singularities in the zeta function can only be generated from the small-t behavior of $K(t)$. The small-t behavior of $K(t)$ is an extremely well studied quantity [7,31-33]. More general than (5), one usually studies it in the context of Laplace-type operators P written in the unified form

$$P = -g^{jk}\nabla_j^V \nabla_k^V - E, \tag{36}$$

where g^{jk} is the metric on a Riemannian manifold M and ∇^V is the connection on M acting on a smooth vector or spinor bundle V over M. Finally, E is an endomorphism of V. In what we have presented so far, $g^{jk} = \delta^{jk}$ was the flat metric, and $E = -m^2$, respectitvely $E = -m^2 - V(r)$ in (31). If the manifold M has a smooth boundary ∂M, we supplement (36) by suitable local boundary conditions represented in the form

$$\mathcal{B}\phi|_{\partial M} = 0.$$

The asymptotic series describing the $t \to 0$ behavior in this context is [31, 34, 35]

$$K(t) \overset{t\to 0}{\sim} \sum_{n=0,1/2,1,\dots}^{\infty} a_n(P,\mathcal{B})\; t^{n-d/2}, \tag{37}$$

with the so-called heat kernel coefficients $a_n(P,\mathcal{B})$ depending, of course, on the operator P considered and on the boundary condition $\mathcal{B}$ imposed. Using the pseudo-differential operator calculus the structure of the heat kernel coefficients is found to be [36, 37]

$$a_n(P,\mathcal{B}) = \int_{\mathcal{M}} \mathrm{d}x\; c_n(x,P) + \int_{\partial\mathcal{M}} \mathrm{d}y\; b_n(y,P,\mathcal{B}), \tag{38}$$

where the coefficients $c_n(x,P)$ and $b_n(y,P,\mathcal{B})$ are built from local (geometric) invariants coming from the operator P and from the geometry defined by M and ∂M. For manifolds without boundary the term involving $b_n(y,P,\mathcal{B})$ is not present and $c_n(x,P)$ vanishes for n a half-integer.

In order to understand how these coefficients are built up from geometric invariants let us use a dimensional argument. From the operator P it is clear that the eigenvalues ω_J^2 carry the dimension *length*$^{-2}$. To make $e^{-t\omega_J^2}$ a well-defined quantity we need to assign a dimension of *length*^{2} to the parameter time t. As a result, the heat kernel $K(t)$ carries no dimension. The expansion (37) therefore implies that $a_n(P,\mathcal{B})$ must have dimension *length*$^{d-2n}$. Taking into account that each integration generates a *length*, this gives

$$c_n(x,P) : length^{-2n}, \qquad b_n(y,P,\mathcal{B}) : length^{1-2n}.$$

The general form of these coefficients is now found by writing down linear combinations of the following building blocks (";" denotes differentiation with respect to the Levi-Civita connection of M),

E, R, R_{ij}, R_{ijkl} : *length*$^{-2}$; $K_{ab} = N_{a;b}$: *length*$^{-1}$; contractions, covariant derivatives,

with unknown universal multipliers such that the correct length dimension results. The universal multipliers in the volume terms involving the $c_n(x,P)$ are determined purely algebraically. These coefficients are known to high orders [38, 39]. The boundary contributions $b_n(y, P, \mathcal{B})$ can be determined very effectively using mainly a mixture of conformal transformation techniques [40, 32] and special case considerations [7]. For these, results up to $b_{5/2}$ are known [41, 42]. As far as needed, explicit results for these coefficients will be given in sect. **6**.

In order to proceed with the analysis of (34), let us relate the heat kernel coefficients to certain zeta function properties. Substituting the asymptotic expansion (37) into (35), we compute

$$\begin{aligned}\zeta(s) &\approx \frac{1}{\Gamma(s)} \sum_{n=0,1/2,1,\ldots}^{\infty} a_n(1, P, \mathcal{B}) \int_0^1 \mathrm{d}t\, t^{s-1+n-\frac{d}{2}} \\ &\approx \frac{1}{\Gamma(s)} \sum_{n=0,1/2,1,\ldots}^{\infty} \frac{a_n(P, \mathcal{B})}{s+n-\frac{d}{2}}.\end{aligned}$$

Here the $\approx$ sign indicates that the zeta function is approximated using the series (37). We neglected an entire function that does not contribute to singular terms as well as to $\zeta(-q)$, $q \in \mathbb{N}_0$. From here we read off

$$\mathrm{Res}(\zeta(s)\Gamma(s))|_{s=\frac{d}{2}-n} = a_n(P, \mathcal{B}),$$

which implies for $z = d/2, (d-1)/2, \ldots, 1/2, -(2n+1)/2$, $n \in \mathbb{N}_0$, that

$$\mathrm{Res}\zeta(z) = \frac{a_{\frac{d}{2}-z}(P, \mathcal{B})}{\Gamma(z)}, \tag{39}$$

and for $q \in \mathbb{N}_0$,

$$\zeta(-q) = (-1)^q q!\, a_{\frac{d}{2}+q}(P, \mathcal{B}).$$

We are now in the position to analyze the singularity structure of the Casimir energy starting from the representation (34). Specifically we consider the case of $d = 3$ dimensions, in which case the rightmost poles of the integrand are located at $\alpha = 2, 3/2, 1, 0$. Shifting the contour in (34) to the left and using the residue theorem shows for the regularized Casimir energy in the frequency-cutoff

$$\begin{aligned}E_0(\delta) = {}& \frac{1}{\sqrt{\pi}} a_0 \delta^{-2} + \frac{\sqrt{\pi}}{4} a_{1/2} \delta^{-3/2} + \frac{1}{2\sqrt{\pi}} a_1 \delta^{-1} \\ & + \frac{1}{4\sqrt{\pi}} a_2 [\gamma + \ln \delta] + \frac{1}{2} \mathrm{FP}\zeta\left(-\frac{1}{2}\right) + \mathcal{O}(\delta),\end{aligned} \tag{40}$$

where FP means the finite part. Note, that the coefficient $a_{3/2}$ does not contribute. Instead, in the zeta function scheme (18), from (39), one finds

$$E_0(s) = -\frac{1}{4\sqrt{\pi}} a_2 \left[\frac{1}{s} + \ln \mu^2\right] + \frac{1}{2} \mathrm{FP}\zeta\left(-\frac{1}{2}\right) + \mathcal{O}(s).$$

The freedom to multiply δ by a number does change the pole and the logarithmic contributions, but not the finite part contribution. While the pole contributions can be identified uniquely, the logarithmic term acquires an additive ambiguous contribution. In the zeta function regularization the parameter μ is arbitrary. Its change gives an additive contribution. In both cases, the additive contributions are proportional to a_2. This is a general feature holding for all regularizations.

This shows that the two regularizations are equivalent in the sense that *finite* ambiguities are the same and in the absence of those, answers agree modulo singularities that are removed in the process of renormalization by introducing suitable counterterms. Note that the absence of finite ambiguities corresponds to $a_2 = 0$, or, if d space dimensions were treated, $a_{(d+1)/2} = 0$. The remarks below eq. (38) then explain why Casimir energies sometimes are singular in odd dimension d, whereas they are finite in even dimension.

From the above results it is clear that the heat kernel coefficients are the mathematical objects at the heart of singular and ambiguous contributions to Casimir energies and forces. These coefficients will now be used to discuss a few configurations that will yield unambiguous Casimir energies or forces.

6. – Configurations with finite Casimir energy/force

In this section we will use heat kernel coefficients to discuss how the *renormalized* Casimir energy should be computed. Furthermore, we use them to argue if configurations will lead to finite answers for the Casimir energy or force. Configurations considered are pistons, the electromagnetic field in the presence of a perfectly conducting surface, and the case of separate bodies.

In order to state the heat kernel coefficients it is convenient to introduce some notation. We will use $G[M] = \int_M \mathrm{d}x G(x)$ and $G[\partial M] = \int_{\partial M} \mathrm{d}y G(y)$. In addition $E_{;m}$ will denote the covariant derivative normal to the boundary, and : denotes covariant differentiation tangentially with respect to the Levi-Civita connection of the boundary. Finally, we sum over repeated upper and lower indices. The needed relevant heat kernel coefficients in three dimensional flat space bounded by some smooth surface ∂M are [7,32,33] (we present $a_{3/2}$ for completeness)

$$a_0^{\pm} = (4\pi)^{-d/2}[\mathcal{M}], \tag{41}$$

$$a_{1/2}^{\pm} = \pm(4\pi)^{(d-1)/2}\frac{1}{4}[\partial\mathcal{M}], \tag{42}$$

$$a_1^{\pm} = (4\pi)^{-d/2}E[\mathcal{M}] + (4\pi)^{-d/2}6^{-1}(2K+12S)[\partial\mathcal{M}], \tag{43}$$

$$a_{3/2}^{\pm} = \pm\frac{1}{384(4\pi)^{(d-1)/2}}\left(96E+\begin{pmatrix}13\\7\end{pmatrix}K^2+\begin{pmatrix}2\\-10\end{pmatrix}K_{ab}K^{ab}+96SK+192S^2\right)[\partial\mathcal{M}], \tag{44}$$

$$\begin{aligned} a_2^{\pm} &= (4\pi)^{-d/2}360^{-1}(60\Delta E + 180E^2)[\mathcal{M}] \\ &\quad +(4\pi)^{-D/2}360^{-1}\Bigg(\begin{pmatrix}-240\\120\end{pmatrix}E_{;m} + 24K^{a}_{:a} + 120EK + \begin{pmatrix}40/3\\40/21\end{pmatrix}K^3 \\ &\quad +\begin{pmatrix}8\\-88/7\end{pmatrix}K_{ab}K^{ab}K + \begin{pmatrix}32/3\\320/21\end{pmatrix}K_{ab}K^{b}_{c}K^{ac} + 720SE + 144SK^2 \\ &\quad +48SK_{ab}K^{ab} + 480S^2K + 480S^3 + 120S^{a}_{:a}\Bigg), \end{aligned} \tag{45}$$

where the upper index + refers to Dirichlet and the − to Robin boundary conditions. These are defined by

$$(\phi_{;m} - S\phi)|_{\partial M} = 0.$$

Let us start by discussing a unique way to define the renormalized Casimir energy of a massive scalar field [21]. The renormalized Casimir energy in this setting is defined by the condition that

$$\lim_{m\to\infty} E_0^{ren} = 0, \tag{46}$$

imposed because an infinitely heavy field should not be able to have quantum fluctuations. To impose condition (46) computationally we need to find the large-m behavior of E_0. Note, that the heat kernel associated with the eigenvalue problem (20) factorizes according to

$$K(t) = e^{-m^2 t} K_0(t), \tag{47}$$

where $K_0(t)$ is the heat kernel for the massless problem. Substituting this form (47) into (35) and performing the integration, after using the small-t asymptotics of $K_0(t)$, gives the large-m expansion of the zeta function

$$\zeta(\alpha) = \frac{1}{\Gamma(\alpha)} \int_0^\infty \mathrm{d}t t^{\alpha-1} e^{-m^2 t} K_0(t) \sim \frac{1}{\Gamma(\alpha)} \sum_{\ell=0,1/2,1,\dots} a_\ell \frac{\Gamma(\alpha+\ell-3/2)}{m^{2(\alpha+\ell-3/2)}},$$

which shows that the heat kernel expansion generates the large mass expansion of the relevant zeta function and of the vacuum energy. To analyze the Casimir energy we expand this about $\alpha = -1/2$, which gives

$$E_0(s) = -\frac{m^4}{8\sqrt{\pi}} a_0 \left(\frac{1}{s} - \frac{1}{2} + \ln\left[\frac{4\mu^2}{m^2}\right]\right) - \frac{2m^3}{6} a_{1/2} + \frac{m^2}{4\sqrt{\pi}} a_1 \left(\frac{1}{s} - 1 + \ln\left[\frac{4\mu^2}{m^2}\right]\right)$$

$$+\frac{m}{2} a_{3/2} - \frac{1}{4\sqrt{\pi}} a_2 \left(\frac{1}{s} - 2 + \ln\left[\frac{4\mu^2}{m^2}\right]\right) + \mathcal{O}\left(\frac{1}{m}\right) + \mathcal{O}(s). \tag{48}$$

This shows that the normalization (46) amounts to

$$E_0^{ren} = E_0 - E_0^{div} \tag{49}$$

with E_0^{div} being given by the explicitly stated terms in (48). It is clearly seen that some of the terms, namely the odd powers in the mass, need renormalization despite being finite. This is specific to the zeta function scheme as we have seen in (40), where terms proportional to $a_0, a_{1/2}, a_1$ and a_2 were seen to need renormalization. Here, in addition $a_{3/2}$ is included as it happens for example in the proper time cutoff [43]. Definition (49) has been applied in a variety of settings [21] and gives a unique answer for renormalized Casimir energies of massive theories.

Let us next discuss the piston configuration with Dirichlet boundary conditions. In their modern form pistons were introduced by Cavalcanti [44] in a two-dimensional setting. In his study, the Casimir piston consists of a rectangular box divided by a movable partition into two compartments. In the meantime, cylindrical boxes with arbitrary cross sections have been studied. For the analysis of the singularities in the Casimir energy and force the relevant heat kernel coefficients consist of a sum of those in (41)–(45), one for each chamber. In this case, as in all other cases, the singularity coming from the volume as represented by a_0^+ is eliminated by subtracting the contribution that would result in the absence of boundaries. That is, we subtract the free space contribution occupying the volume of the piston. The remaining singularities in the Casimir energy cannot as easily be argued away, but the force can be seen to be finite and unambiguously defined. To see this point note that the force on the piston at $x = a$ is

$$F_0 = -\frac{\partial}{\partial a} E_0.$$

So once we know the singularities in E_0 do not depend on a the statement made follows. But clearly as we move around the piston, the area of the boundary of the configuration as well as the extrinsic curvature of the cylinder like shell do not change, which implies that $a_{1/2}^+, \ldots, a_2^+$ do not depend on a. Similarly one can argue for different boundary conditions. Detailed analysis of the force for these types of configurations are available [45-49] and we will not report further on these.

Let us next briefly discuss the electromagnetic field with perfectly conducting boundary conditions. For concreteness let us have the spherical shell in three dimensions as a surface in mind, although the statements to be made hold more generally [50]. In this case the TE modes reduce to Dirichlet boundary conditions and the TM modes result in Robin conditions with the parameter $S = -1$. This fact leads to very subtle cancelations that make even the Casimir energy finite. Namely the singular pieces arising separately from the TE and TM modes cancel because of the opposite signs in (42). This is a consequence of the conformal invariance of the electromagnetic field. Contributions resulting from $a_1^\pm$ cancel for TE and TM separately between the inside and outside of the sphere. The same happens for $a_2^\pm$ such that the Casimir energy is finite. Again, this case is very well known and studied and we refer to [24, 51, 52] for further details.

Finally, for separate bodies the configuration space will be the complement of the bodies. The infinite volume contribution from $a_0^\pm$ is once again removed by subtracting the free space contribution. The remaining singularities contain an integral over the surface of each body; these reflect the divergencies in the self-energy of each body. There are no divergent terms involving any interaction between the bodies. The above implies that once the self-energies of each body are subtracted the remainder will be finite. Furthermore, the Casimir force between the bodies is finite as none of the singularities depends upon the distance between the bodies. Similarly one can argue for a field in the presence of an external potential with two non-overlapping regions of support. How to perform the computation of Casimir energies for these situations has only been understood well relatively recently and we dedicate the remainder of this article to the TGTG-representation of the Casimir energy for separate bodies.

To conclude this section, let us stress that the above described configurations without ambiguities in the Casimir energy are somewhat special and that in general the interpretation of singularities might be difficult pointing to associated physical problems.

7. – Separate bodies and the TGTG representation

Let $\mathcal{F}$ be the boundary surface parameterized by

$$\vec{r} = \vec{u}(\vec{\eta}) = \vec{u}(\eta_1, \eta_2).$$

As derived earlier, see (17),

$$E_0 = -\frac{i}{2T} \operatorname{Tr} \ln \tilde{\mathcal{K}},$$

where $\tilde{K}$ is the Green's function restricted to the surface $\mathcal{F}$, see (15). Taking into account the time independence of the setting, we first Wick rotate to imaginary time x_4 and use the Fourier-transformation for the time variable in $\tilde{K}$, thus writing

$$\begin{aligned}
E_0 &= -\frac{i}{2T} \operatorname{Tr} \ln \tilde{\mathcal{K}} = -\frac{i}{2T} i \int_{-\infty}^{\infty} \mathrm{d}x_4 \; \left\langle x_4 \middle| \operatorname{Tr}_3 \ln \tilde{\mathcal{K}} \middle| x_4 \right\rangle \\
&= -\frac{i}{2T} i \int_{-\infty}^{\infty} \mathrm{d}x_4 \; \int_{-\infty}^{\infty} \frac{\mathrm{d}\xi}{2\pi} \operatorname{Tr}_3 \ln \tilde{\mathcal{K}}_\xi = -\frac{i}{2T} iT \; \int_{-\infty}^{\infty} \frac{\mathrm{d}\xi}{2\pi} \operatorname{Tr}_3 \ln \tilde{\mathcal{K}}_\xi \\
&= \frac{1}{2\pi} \int_0^{\infty} \mathrm{d}\xi \operatorname{Tr}_3 \ln \tilde{\mathcal{K}}_\xi,
\end{aligned}$$

where Tr_3 indicates that only spatial variables are integrated over and it is used that $\tilde{\mathcal{K}}_\xi$ does not depend on the sign of ξ. This follows from the fact that $\tilde{\mathcal{K}}_\xi$ has kernel

$$\tilde{K}_\xi(\vec{\eta}, \vec{\eta}\,') = \int \mathrm{d}\vec{r} \int \mathrm{d}\vec{r}\,' H(\vec{\eta}, \vec{r}) G_\xi(\vec{r}, \vec{r}\,') H(\vec{\eta}\,', \vec{r}\,'),$$

where G_ξ is the Green's function in the Fourier space with respect to the imaginary time variable. If the surface $\mathcal{F}$ describes two disjoint bodies, it can be written as the union of two surfaces with empty intersection, namely $\mathcal{F} = \mathcal{F}_A \cup \mathcal{F}_B$ with $\mathcal{F}_A \cap \mathcal{F}_B = \emptyset$. In that case the kernel $\tilde{K}_\xi$ has the block structure

$$\tilde{K}_\xi = \begin{pmatrix} \tilde{K}_{\xi,AA}(\vec{\eta}_A, \vec{\eta}\,'_A) & \tilde{K}_{\xi,AB}(\vec{\eta}_A, \vec{\eta}\,'_B) \\ \tilde{K}_{\xi,BA}(\vec{\eta}_B, \vec{\eta}\,'_A) & \tilde{K}_{\xi,BB}(\vec{\eta}_B, \vec{\eta}\,'_B) \end{pmatrix}.$$

In order to isolate the interaction between the surfaces and to eliminate the parts corresponding to the self-energy of each body, one needs to rewrite $\tilde{K}_\xi$ as a product in which the factors

$$\begin{pmatrix} \tilde{K}_{\xi,AA} & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & 0 \\ 0 & \tilde{K}_{\xi,BB} \end{pmatrix}$$

appear. This is easily done and the relevant rewriting is

$$\tilde{K}_\xi = \begin{pmatrix} \tilde{K}_{\xi,AA} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \tilde{K}_{\xi,BB} \end{pmatrix} \begin{pmatrix} 1 & \tilde{K}^{-1}_{\xi,AA} \tilde{K}_{\xi,AB} \\ \tilde{K}^{-1}_{\xi,BB} \tilde{K}_{\xi,BA} & 1 \end{pmatrix}.$$

In this representation the self-energy part of the Casimir energy is easily subtracted and the relevant *finite* part of the Casimir energy is found to be

$$E = \frac{1}{2\pi}\int_0^\infty \mathrm{d}\xi \ \ \mathrm{Tr}_3 \ln\left(1 - \tilde{K}^{-1}_{\xi,AA}\tilde{K}_{\xi,AB}\tilde{K}^{-1}_{\xi,BB}\tilde{K}_{\xi,BA}\right).$$

Instead of modeling the presence of bodies by imposing boundary conditions one might think about representing them by some background potential $V(\vec{r})$. Following the same steps that led to (17) the representation of the Casimir energy is

$$E_0 = -\frac{1}{2\pi}\int_0^\infty \mathrm{d}\xi \ \ \mathrm{Tr}_3 \ln \mathcal{G}^{(V)}_\xi$$

with the appropriate Green's function defined through

$$\left[\xi^2 - \vec{\nabla}^2 + V(\vec{r})\right] G^{(V)}_\xi(\vec{r},\vec{r}') = \delta^3(\vec{r}-\vec{r}').$$

As before, the case we really have in mind is where $V(\vec{r})$ consists of two parts with disjoint supports and where we want to subtract those parts in the Casimir energy that only correspond to one support being present. In order to exploit the structure in this case we will reexpress $G^{(V)}_\xi$ in terms of $G^{(0)}_\xi$ and the potential. In some detail, in operator language the defining equations are

$$\left[\xi^2 - \vec{\nabla}^2 + \mathcal{V}\right]\mathcal{G}^{(\mathcal{V})}_\xi = \mathbf{1}, \qquad \left[\xi^2 - \vec{\nabla}^2\right]\mathcal{G}^{(0)}_\xi = \mathbf{1},$$

which allow for different rewriting. First, it is possible to rewrite $\mathcal{G}^{(\mathcal{V})}_\xi$ in terms of $\mathcal{G}^{(0)}_\xi$ as follows,

$$\left[\xi^2 - \vec{\nabla}^2\right]\mathcal{G}^{(V)}_\xi = \mathcal{G}^{(0)-1}_\xi\mathcal{G}^{(V)}_\xi = \mathbf{1} - \mathcal{V}\mathcal{G}^{(V)}_\xi \Longrightarrow \mathcal{G}^{(V)}_\xi = \mathcal{G}^{(0)}_\xi - \mathcal{G}^{(0)}_\xi\mathcal{V}\mathcal{G}^{(V)}_\xi, \tag{50}$$

which corresponds to the Lippmann-Schwinger equation [25]. Alternatively, one can proceed by using the formal solution of the Dyson equation

$$\left(\mathcal{G}^{(0)-1}_\xi + \mathcal{V}\right)\mathcal{G}^{(\mathcal{V})}_\xi = \mathcal{G}^{(0)-1}_\xi\left(\mathbf{1} + \mathcal{G}^{(0)}_\xi\mathcal{V}\right)\mathcal{G}^{(\mathcal{V})}_\xi = \mathbf{1} \Longrightarrow \mathcal{G}^{(\mathcal{V})}_\xi = \left(\mathbf{1} + \mathcal{G}^{(0)}_\xi\mathcal{V}\right)^{-1}\mathcal{G}^{(0)}_\xi,$$

where in a natural way the so-called T-matrix occurs,

$$\mathcal{T} = \mathcal{V}\left(\mathbf{1} + \mathcal{G}^{(0)}_\xi\mathcal{V}\right)^{-1}.$$

This operator is the basic object for expressing the properties of scatterers in the theory of light scattering [53]. Using it in (50) the result is

$$\mathcal{G}^{(\mathcal{V})}_\xi = \mathcal{G}^{(0)}_\xi - \mathcal{G}^{(0)}_\xi\mathcal{T}\mathcal{G}^{(0)}_\xi.$$

Let us next exploit the structure further when

$$V(\vec{r}) = V_A(\vec{r}) + V_B(\vec{r}),$$

where the following manipulations also hold when the supports of V_A and V_B are not disjoint. Clearly the goal of the following manipulations will be the separation of A-only and B-only contributions. To this end note the identity

$$\begin{aligned}\mathbf{1}+\mathcal{G}_\xi^{(0)}(\mathcal{V}_A+\mathcal{V}_B) &= \left(\mathbf{1}+\mathcal{G}_\xi^{(0)}\mathcal{V}_A\right)\left(\mathbf{1}+\mathcal{G}_\xi^{(0)}\mathcal{V}_B\right)-\mathcal{G}_\xi^{(0)}\mathcal{V}_A\mathcal{G}_\xi^{(0)}\mathcal{V}_B\\ &= \left(\mathbf{1}+\mathcal{G}_\xi^{(0)}\mathcal{V}_A\right)(\mathbf{1}-\mathcal{M}_\xi)\left(\mathbf{1}+\mathcal{G}_\xi^{(0)}\mathcal{V}_B\right),\end{aligned}$$

where

$$\mathcal{M}_\xi=\left(\mathbf{1}+\mathcal{G}_\xi^{(0)}\mathcal{V}_A\right)^{-1}\mathcal{G}_\xi^{(0)}\mathcal{V}_A\mathcal{G}_\xi^{(0)}\mathcal{V}_B\left(\mathbf{1}+\mathcal{G}_\xi^{(0)}\mathcal{V}_B\right)^{-1}.$$

Rewriting $\mathcal{M}_\xi$ in terms of the T-matrix, noticing that $(\mathbf{1}+\mathcal{G}_\xi^{(0)}\mathcal{V}_A)^{-1}$ and $\mathcal{G}_\xi^{(0)}\mathcal{V}_A$ commute, gives

$$\mathcal{T}_i=\mathcal{V}_i\left(\mathbf{1}+\mathcal{G}_\xi^{(0)}\mathcal{V}_i\right)^{-1}\Longrightarrow\mathcal{M}_\xi=\mathcal{G}_\xi^{(0)}\ \mathcal{T}_A\ \mathcal{G}_\xi^{(0)}\ \mathcal{T}_B.$$

This allows us to isolate interaction terms in the Casimir energy and to completely express it in terms of the free Green's function and the T-matrix. First we write

$$\begin{aligned}\mathrm{Tr}_3\ln\mathcal{G}_\xi^{V_A+V_B} &= \mathrm{Tr}_3\ln\left(\mathbf{1}+\mathcal{G}_\xi^{(0)}(\mathcal{V}_A+\mathcal{V}_B)\right)^{-1}\mathcal{G}_\xi^{(0)}\\ &= \mathrm{Tr}_3\ln\mathcal{G}_\xi^{(0)}-\mathrm{Tr}_3\ln\left(\mathbf{1}+\mathcal{G}_\xi^{(0)}\mathcal{V}_A\right)-\mathrm{Tr}_3\ln\left(\mathbf{1}+\mathcal{G}_\xi^{(0)}\mathcal{V}_B\right)-\mathrm{Tr}_3\ln(\mathbf{1}-\mathcal{M}_\xi)\\ &= -\mathrm{Tr}_3\ln\mathcal{G}_\xi^{(0)}+\mathrm{Tr}_3\ln\mathcal{G}_\xi^{(V_A)}+\mathrm{Tr}_3\ln\mathcal{G}_\xi^{(V_B)}-\mathrm{Tr}_3\ln(\mathbf{1}-\mathcal{M}_\xi),\end{aligned}$$

such that the finite Casimir interaction energy is

$$E=\frac{1}{2\pi}\int_0^\infty d\xi\ \mathrm{Tr}_3\ln(\mathbf{1}-\mathcal{M}_\xi)=\frac{1}{2\pi}\int_0^\infty d\xi\,\mathrm{Tr}_3\ln\left(1-\mathcal{G}_\xi^{(0)}\mathcal{T}_A\mathcal{G}_\xi^{(0)}\mathcal{T}_B\right),\tag{51}$$

where all separation independent contributions were dropped. The kernel M_ξ more explicitly reads

$$M_\xi(\vec r,\vec r\,')=\int_A dr''\int_B d\tilde r\int_B d\tilde r\,'T^A(\vec r,\vec r\,'')G_\xi^{(0)}(\vec r\,'',\tilde{\vec r})T^B(\tilde{\vec r},\tilde{\vec r}\,')G_\xi^{(0)}(\tilde{\vec r}\,',\vec r\,').$$

The basic statement for this configuration is that E, eq. (51), does not have divergences as a consequence of what was dropped. This can also be checked explicitly by studying its structure. In this way one has a representation of the interaction energy and for the force which is finite at all intermediate steps. This allows for a direct numerical evaluation for arbitrary geometries of interacting bodies. Remarkable results have been obtained in this way; see [1] and references therein. For plane parallel surfaces representation (51) coincides with the famous Lifshitz formula [54].

8. – Conclusions

In this article we have given an introduction into various aspects of the Casimir energy. We started with various representations of the Casimir energy and their regularizations. Example calculations for explicitly or implicitly known spectrum followed emphasizing the basic ideas and strategies. For technical details important references are provided. As the examples make clear, the Casimir energy sometimes is plagued by singularities and we explain in detail how these are related to the small-t asymptotics of the heat kernel. In doing so, different regularization schemes are also related. The insight into the divergence structure of the Casimir energy, and thus force, is then used to discuss unambiguous configurations.

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