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Introduction to tomography, classical and quantum

M. A. Man'ko(1)(*), V. I. Man'ko(1), G. Marmo(2),
A. Simoni(2) and F. Ventriglia(2)

(1) *P. N. Lebedev Physical Institute - Leninskii Prospect 53, Moscow 119991, Russia*
(2) *Dipartimento di Fisica dell'Università "Federico II" e Sezione INFN di Napoli Complesso Universitario di Monte S. Angelo - via Cintia, 80126 Naples, Italy*

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Summary. — The tomographic approach to identify quantum states with fair probability distributions as alternatives to wave functions or density operators is reviewed. The tomographic-probability representation is shown also for classical states. The star-product formalism of quantizers and dequantizers associated with the tomographic picture of classical and quantum mechanics is presented and some kernels of star products are given in explicit forms. The inequalities for Shannon and Rényi entropies determined by tomographic-probability distributions are discussed.

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1. – Introduction

There exist different formulations of both classical mechanics (see, *e.g.*, [1, 2]) and quantum mechanics [3]. In quantum mechanics, there are also different pictures like the standard Schrödinger picture [4] with the wave function evolution equation, the Heisenberg picture [5], the Feynmann path integral formulation, as well as the Moyal formulation [6] of quantum mechanics in a classical-like form. In the second part of the last century, attempts were made to find formulations of both classical and quantum mechanics which are similar and provide the possibility to see in clear form the quantum-classical transition. The mathematical basis of such attempts, called the tomographic picture of quantum (also classical) mechanics, is the application of integral Radon transform [7]. Generalizations of the Radon transform were suggested in [8-11].

In the tomographic picture, we replace a quantum state by a collection of probability distributions. By quantum tomography we usually mean the process of reconstruction

(*) E-mail: mmanko@sci.lebedev.ru

of the state by means of the collection of probability distributions. The tomographic approach was also applied to cosmological problems [12-15]. In fact, in quantum cosmology the notion of the Universe state can be given in terms of probability distributions which replace the wave function of the Universe. Within the probability framework, the classical picture of the cosmological processes and quantum picture of the cosmological processes can be treated in a unified manner.

We point out that the tomographic-probability representation of quantum mechanics makes more clear the phase-space representation of quantum states. In fact, since the measurable tomographic-probability distribution is considered as a primary description of quantum states (containing its complete information) [16], the experiment to measure photon states [17, 18], for instance, do not need the procedure of quasidistribution reconstruction [19, 20] since all physical characteristics are extracted from the optical tomograms.

The aim of this paper is to present a review of this formalism for classical and mainly quantum mechanics where probability distributions play a primary role in the description of states.

2. – Classical mechanics within the tomographic framework

Before introducing the probability representation in quantum mechanics, first we show how the tomographic representation can be introduced in classical statistical mechanics [21, 22]. Let us consider the Radon transform of the probability distribution function $f(q,p)$ on the phase space of a classical particle. We denote the transform as $w(X,\mu,\nu)$ where the arguments are X,μ,ν and take real values. The association called symplectic tomogram: mapping a probability distribution onto a classical tomogram is

$$w(X,\mu,\nu) = \int f(q,p)\delta\left(X-\mu q-\nu p\right)\mathrm{d}q\,\mathrm{d}p. \tag{1}$$

The tomogram is normalized $\int w(X,\mu,\nu)\,\mathrm{d}X = 1$. The tomogram can be rewritten in the form

$$w(X,\mu,\nu) = \frac{1}{2\,\pi}\int f(q,p)e^{ik(X-\mu q-\nu p)}\,\mathrm{d}k\,\mathrm{d}q\,\mathrm{d}p. \tag{2}$$

From the homogeneity of the delta-function $\delta(\lambda y) = |\lambda|^{-1}\delta(y)$, we derive the homogeneity of the symplectic tomogram, $w(\lambda X,\lambda\mu,\lambda\nu) = |\lambda|^{-1}w(X,\mu,\nu)$.

Being the probability distribution of a random variable X, the tomogram determines the probability distribution $f(q,p)$ by means of the inverse Radon transform

$$f(q,p) = \frac{1}{4\,\pi^2}\int w(X,\mu,\nu)e^{i(X-\mu q-\nu p)}\,\mathrm{d}X\,\mathrm{d}\mu\,\mathrm{d}\nu, \tag{3}$$

which is here presented as a Fourier transform of the symplectic tomogram.

Formulae (1) and (3) provide bijective map between the probability density $f(q,p)$ and the tomographic probability $w(X,\mu,\nu)$. Thus, all mean values of physical observables $F = F(q,p)$ can be evaluated using the probability density $f(q,p)$,

$$\langle F\rangle = \langle F(q,p)\rangle = \int f(q,p)F(q,p)\,\mathrm{d}q\,\mathrm{d}p. \tag{4}$$

In the tomographic-probability representation of the classical state, this formula for mean value $\langle F\rangle$ can be written as follows:

$$\langle F\rangle = \langle w_F^d(X,\mu,\nu)\rangle = \int w_F^d(X,\mu,\nu) w(X,\mu,\nu)\,\mathrm{d}X\,\mathrm{d}\mu\,\mathrm{d}\nu, \tag{5}$$

where

$$w_F^d(X,\mu,\nu) = \frac{1}{4\,\pi^2}\int F(q,p) e^{i(X-\mu q-\nu p)}\,\mathrm{d}q\,\mathrm{d}p. \tag{6}$$

All the highest moments of the observable can be expressed in terms of the tomographic-probability distribution, using the characteristic function $\xi(k) = \langle e^{ikF}\rangle = \int e^{ikF(q,p)} \times f(q,p)\mathrm{d}q\,\mathrm{d}p$, which can be given in the tomographic representation as follows:

$$\xi(k) = \langle e^{ikF}\rangle = \frac{1}{4\pi^2}\int w(X,\mu,\nu)\left[\int e^{ikF(q,p)+i(X-\mu q-\nu p)}\,\mathrm{d}q\,\mathrm{d}p\right]\mathrm{d}X\,\mathrm{d}\mu\,\mathrm{d}\nu. \tag{7}$$

Classical observables form an associative and commutative algebra, with multiplication given by the standard point-wise product $C(q,p) = A(q,p)B(q,p)$. In the tomogra–phic-probability representation, the functions $w_A^d(X,\mu,\nu)$ and $w_B^d(X,\mu,\nu)$, which provide the mean value of the product, $\langle C\rangle$, using

$$w_C^d(X,\mu,\nu) = \frac{1}{4\,\pi^2}\int A(q,p)B(q,p)e^{i(X-\mu q-\nu p)}\,\mathrm{d}q\,\mathrm{d}p, \tag{8}$$

are multiplied according to the formulae

$$\begin{aligned} A(q,p) &= \int w_A^d(X_1,\mu_1,\nu_1)\delta\left(X_1-\mu_1 q-\nu_1 p\right)\mathrm{d}X_1\,\mathrm{d}\mu_1\,\mathrm{d}\nu_1,\\ B(q,p) &= \int w_B^d(X_2,\mu_2,\nu_2)\delta\left(X_2-\mu_2 q-\nu_2 p\right)\mathrm{d}X_2\,\mathrm{d}\mu_2\,\mathrm{d}\nu_2. \end{aligned} \tag{9}$$

These formulae provide the following relationship:

$$\begin{aligned} w_C^d(X,\mu,\nu) = \int & w_A^d(X_1,\mu_1,\nu_1) w_B^d(X_2,\mu_2,\nu_2)\\ &\times K\left(X_1,\mu_1,\nu_1,X_2,\mu_2,\nu_2,X,\mu,\nu\right)\mathrm{d}X_1\,\mathrm{d}\mu_1\,\mathrm{d}\nu_1\,\mathrm{d}X_2\,\mathrm{d}\mu_2\,\mathrm{d}\nu_2, \end{aligned} \tag{10}$$

where the kernel of this nonlocal commutative and associative product reads

$$\begin{aligned} &K\left(X_1,\mu_1,\nu_1,X_2,\mu_2,\nu_2,X,\mu,\nu\right) =\\ &\frac{1}{4\,\pi^2}\int \delta\left(X_1-\mu_1 q-\nu_1 p\right)\delta\left(X_2-\mu_2 q-\nu_2 p\right)e^{i(X-\mu q-\nu p)}\,\mathrm{d}q\,\mathrm{d}p =\\ &\frac{1}{4\,\pi^2}\frac{1}{|\nu_2\mu_1-\nu_1\mu_2|}\exp\left[i\left(X-\mu\,\frac{\nu_1 X_2-\nu_2 X_1}{\mu_2\nu_1-\mu_1\nu_2}+\nu\,\frac{\mu_1 X_2-\mu_2 X_1}{\mu_2\nu_1-\mu_1\nu_2}\right)\right]. \end{aligned} \tag{11}$$

Thus, in this tomographic picture, the product of two observables retains commutativity but becomes nonlocal!

As for states in the tomographic picture of classical statistical mechanics, they are associated with tomographic-probability distributions $w(X,\mu,\nu)$ and the observables F —functions $F(q,p)$ in the standard phase-space picture— are associated with the functions $w_F^d(X,\mu,\nu)$. The product of observables is a commutative star-product with the kernel given by (11).

The evolution equation of the classical probability distribution $f(q,p,t)$ on the phase space is given by the Liouville equation

$$\frac{\partial f(q,p,t)}{\partial t}+p\frac{\partial f(q,p,t)}{\partial q}-\frac{\partial U(q)}{\partial q}\frac{\partial f(q,p,t)}{\partial p}=0, \tag{12}$$

where we use the Hamiltonian function

$$H=(p^2/2)+U(q), \tag{13}$$

with the particle mass $m=1$ and potential energy $U(q)$. In the tomographic picture, it is transformed into

$$\begin{aligned}\frac{\partial w(X,\mu,\nu,t)}{\partial t}&-\mu\frac{\partial w(X,\mu,\nu,t)}{\partial \nu}\\ -\frac{\partial U}{\partial q}&\left(q\to-\left(\frac{\partial}{\partial X}\right)^{-1}\frac{\partial}{\partial \mu}\right)\nu\frac{\partial w(X,\mu,\nu,t)}{\partial X}=0.\end{aligned} \tag{14}$$

Like in the Heisenberg picture of quantum mechanics, one can consider the evolution equation for the observables by considering the state probability distributions, either $f(q,p)$ or $w(X,\mu,\nu)$ as being independent of time but ascribing the time dependence to the phase-space observables $F(q,p,t)$ or the tomographic observables $w_F^d(X,\mu,\nu,t)$. The resulting equation for the phase-space observables reads

$$\frac{\partial F(q,p,t)}{\partial t}-p\frac{\partial F(q,p,t)}{\partial q}+\frac{\partial U}{\partial q}\frac{\partial F(q,p,t)}{\partial p}=0. \tag{15}$$

For the tomographic observables, the evolution equation is

$$\begin{aligned}\frac{\partial w_F^d(X,\mu,\nu,t)}{\partial t}&+\mu\frac{\partial w_F^d(X,\mu,\nu,t)}{\partial \nu}\\ +\frac{\partial U}{\partial q}&\left(q\to-\left(\frac{\partial}{\partial X}\right)^{-1}\frac{\partial}{\partial \mu}\right)\nu\frac{\partial w_F^d(X,\mu,\nu,t)}{\partial X}=0.\end{aligned} \tag{16}$$

We conclude that in classical statistical mechanics the state can be associated either with the probability distribution on the phase space or with tomographic-probability distribution $w(X,\mu,\nu)$. In classical statistical mechanics, the observables can be associated either with functions $F(q,p)$ on the phase space and point-wise product multiplication rule or with the functions $w_F^d(X,\mu,\nu)$ which are related to the functions $F(q,p)$ by inverse Radon transform and the star-product of these functions is commutative but not point-wise with the kernel given by (11). Also the evolution equation of the states and observables in both formulations of classical statistical mechanics can be given in the form of evolution equation either for the $f(q,p,t)$ distribution or for the tomogram

$w(X,\mu,\nu,t)$ which is the Radon transform of the phase-space distribution density. Alternatively, the evolution of a classical system can be associated with the evolution of observables $F(q,p,t)$ and $w^d_F(X,\mu,\nu,t)$. The observables are connected by the Radon transform too with its inverse (dual kernel) [23].

The tomographic description of classical statistical mechanics described is appropriate for introducing the tomographic-probability representation of quantum mechanics.

3. – States in quantum mechanics

The states in quantum mechanics are associated with the wave function $\psi(x)$ or density matrix $\rho(x,x')$. This notion can be also replaced in a more geometrical picture by vectors $|\psi\rangle$ and density operators $\hat\rho$ —we call them density states $\hat\rho$— in a Hilbert space [24]. Then the wave function is the scalar product $\psi(x)=\langle x|\psi\rangle$ and the density matrix is the matrix element of the density operator $\rho(x,x')=\langle x|\hat\rho|x'\rangle$. Here we understand the vector $|x\rangle$ as improper eigenvector of the position operator $\hat q$ which acts on the wave function $(\hat q\psi)(x)=x\psi(x)$. As we see, the quantum notion of state and observables like the position is very different with respect to the one discussed in classical statistical mechanics. As we show below, one can use a different description of states to make both, classical and quantum, very close to each other.

Let us start now not from the standard definition of the states by means of the density operator but use the point of view that the quantum state is identified with the probability distribution function $w(X,\mu,\nu)$ which has the properties of nonnegativity and normalization, as well as homogeneity, which exactly coincide with the properties of the classical tomographic-probability distribution. Then the question arises: where is the density operator $\hat\rho$ in this picture? To answer this question, we must go back to the classical picture. Also we will show that the density operators $\hat\rho$ and the vectors in the Hilbert space $|\psi\rangle$ can be easily introduced in classical statistical mechanics following the spirit of the old Koopman paper [25] but from the tomographic point of view [26]. The idea is simply to use the standard formulae of Weyl symbols [27] in the phase-space representation. The first one provides the operator $\hat\rho_{\rm cl}$ from the probability density $f(q,p)$ as follows:

$$\hat\rho_{\rm cl}=\int f(q,p)\,|q-u/2\rangle\langle q+u/2|\,e^{-ipu}{\rm d}u. \tag{17}$$

For normalized nonnegative probability density, this operator is Hermitian and satisfies the normalization condition ${\rm Tr}\,\hat\rho_{\rm cl}=1$. The state $|q+u/2\rangle$ in (17) is improper eigenvector of the operator $\hat q$ acting in the Hilbert space as the position operator. Thus, in classical mechanics the distribution function $f(q,p)$ is mapped onto the density operator $\hat\rho_{\rm cl}$. It is easy to see that this formula can be inverted and, as a result, the state distribution function $f(q,p)$ is written in terms of the density operator $\hat\rho_{\rm cl}$ as follows:

$$f(q,p)=\frac{1}{2\pi}\int {\rm Tr}\left(\hat\rho_{\rm cl}\,|q-u/2\rangle\langle q+u/2|\,e^{-ipu}\,{\rm d}u\right). \tag{18}$$

Also for the observable $F(q,p)$ in classical statistical mechanics, one can introduce the corresponding operator

$$\hat F_{\rm cl}=\int F(q,p)\,|q-u/2\rangle\langle q+u/2|\,e^{-ipu}\,{\rm d}u. \tag{19}$$

The formula for the mean value of the observable in classical statistical mechanics reads

$$\langle F\rangle = \int F(q,p)f(q,p)\,\mathrm{d}q\,\mathrm{d}p = \mathrm{Tr}\,\hat\rho_{\mathrm{cl}}\hat F_{\mathrm{cl}}. \tag{20}$$

Analogously, we could also start from the classical tomographic-probability distribution $w(X,\mu,\nu)$ and introduce the density operator in classical statistical mechanics as

$$\hat\rho_{\mathrm{cl}} = \frac{1}{2\pi}\int w(X,\mu,\nu)e^{i(X-\mu\hat q-\nu\hat p)}\,\mathrm{d}X\,\mathrm{d}\mu\,\mathrm{d}\nu. \tag{21}$$

In order to introduce the observable $\hat F_{\mathrm{cl}}$, which provides the formula for the classical mean value $\langle F\rangle$ (20), one needs to introduce the operator using dual expression, *i.e.*,

$$\hat F_{\mathrm{cl}} = \int w_F^d(X,\mu,\nu)\delta(X-\mu\hat q-\nu\hat p)\,\mathrm{d}X\,\mathrm{d}\mu\,\mathrm{d}\nu. \tag{22}$$

In this case,

$$\mathrm{Tr}\,\hat\rho_{\mathrm{cl}}\hat F_{\mathrm{cl}} = \int w_F^d(X,\mu,\nu)w(X,\mu,\nu)\,\mathrm{d}X\,\mathrm{d}\mu\,\mathrm{d}\nu. \tag{23}$$

In classical statistical mechanics, the state operators and the observable operators are introduced in different ways in the phase space and in the tomographic picture, and this is related to the fact that the star-product schemes in the both pictures are different. The star-product in the phase-space picture is based on formulae in terms of Weyl symbols and is self-dual, but the tomographic star-product formula is not self-dual (we explain the details in the following section). The operators obtained in view of this procedure do not contain all the operators but only the ones which have symmetrized form in the position and momentum.

Now we are starting to introduce state and observables in quantum mechanics using the same procedure. We take the quantum tomogram of a state, *i.e.*, the probability distribution $w(X,\mu,\nu)$ which is nonnegative, normalized and homogeneous. We define the state density operator $\hat\rho$ as follows:

$$\hat\rho = \frac{1}{2\pi}\int w(X,\mu,\nu)e^{i(X-\mu\hat q-\nu\hat p)}\,\mathrm{d}X\,\mathrm{d}\mu\,\mathrm{d}\nu. \tag{24}$$

We impose an extra condition which was not used for the state density operator in classical statistical mechanics, namely, the nonnegativity condition, *i.e.*, we consider as a state only such tomographic-probability distribution for which

$$\left\langle\psi\left|\int w(X,\mu,\nu)e^{i(X-\mu\hat q-\nu\hat p)}\,\mathrm{d}X\,\mathrm{d}\mu\,\mathrm{d}\nu\right|\psi\right\rangle \geq 0 \tag{25}$$

for any vector in the Hilbert space. This is a difference between the quantum and classical states expressed in terms of tomogram $w(X,\mu,\nu)$.

Let us point out that in classical statistical mechanics the tomograms are such that for some of them one has inequality

$$\left\langle \psi \left| \int w(X,\mu,\nu) e^{i(X-\mu\hat{q}-\nu\hat{p})} \, \mathrm{d}X \, \mathrm{d}\mu \, \mathrm{d}\nu \right| \psi \right\rangle < 0, \tag{26}$$

i.e., we have inequality (26) for some vectors $|\psi\rangle$ in the Hilbert space.

On the other hand, classical tomograms must satisfy the condition of nonnegativity of Fourier integral

$$\int w(X,\mu,\nu) e^{i(X-\mu q-\nu p)} \, \mathrm{d}X \, \mathrm{d}\mu \, \mathrm{d}\nu \geq 0. \tag{27}$$

For quantum-state tomograms satisfying (25), we can relax condition (27). Thus, introducing the classical and quantum states starting from the tomographic-probability distributions $w(X,\mu,\nu)$, we can introduce density operators for classical states and density operators for quantum states using the same formula. Nevertheless, we impose different constraints on these operators. In the classical case, the density operator being Hermitian can be either positive or negative. In the quantum case, the density operator being Hermitian is mandatorily nonnegative. These conditions provide different constraints on classical and quantum tomograms. If the quantum state is determined by a nonnegative density operator $\hat{\rho}$, its tomogram $w(X,\mu,\nu)$ reads

$$w(X,\mu,\nu) = \mathrm{Tr}\, \hat{\rho}\delta(X-\mu\hat{q}-\nu\hat{p}). \tag{28}$$

In the Hilbert space, other Hermitian operators may act which are not given in the form of series of symmetrized polynomials in position and momentum. For these nonclassical observables $\hat{F}$, one has the dual tomographic symbols

$$w_F^d(X,\mu,\nu) = \frac{1}{2\pi} \mathrm{Tr}\, \hat{F} e^{i(X-\mu\hat{q}-\nu\hat{p})}. \tag{29}$$

The quantum product of observables is not commutative and this fact reflects the noncommutativity of Weyl symbols of quantum observables which is given by a twisted classical kernel (Grönewold kernel)

$$K(q_1,p_1,q_2,p_2,q_3,p_3) = \frac{1}{4\pi^2} \exp\left[2i(q_1p_2 - q_2p_1 + q_2p_3 - q_3p_2 + q_3p_1 - q_1p_3)\right], \tag{30}$$

where for the exponent one has a symplectic area of the triangle associated with three points in phase space.

4. – Star product of functions and operators

In order to explain rules of multiplications of operators which provide the operator form of classical mechanics, in this section we discuss the star product of functions or the rules of multiplications of the functions satisfying the associativity condition.

Given function $F(\vec{X})$ where $\vec{X} = (X_1, X_2, \ldots, X_N)$ contains components which may be either continuous variables X_j or discrete variables. Also one can consider the case where a set of variables is continuous and another set contains discrete variables. By definition, the product $(F_1 \star F_2)(\vec{X}\,)$ of two functions $F_1(\vec{X}\,)$ and $F_2(\vec{X}\,)$ is associative if it satisfies the condition

$$\big(F_1 \star (F_2 \star F_3)\big)(\vec{X}\,) = \big((F_1 \star F_2) \star F_3\big)(\vec{X}\,). \tag{31}$$

This condition written in the form of constraints for the kernel, giving the product of two functions

$$(F_1 \star F_2)(\vec{X}\,) = \int K\left((\vec{X}_1, \vec{X}_2, \vec{X}\,)F_1(\vec{X}_1)F_2(\vec{X}_2)\right) \mathrm{d}\vec{X}_1\, \mathrm{d}\vec{X}_2, \tag{32}$$

provides one with the nonlinear equation for the kernel (see, *e.g.*, [28,29]). We point out that the integral over $\vec{X}_{1,2}$ in (32) means the integration over continuous components and the summation over discrete components of argument $\vec{X}_{1,2}$. The product of the functions is commutative if the kernel is a symmetric function with respect to the permutation $\vec{X}_1 \leftrightarrow \vec{X}_2$. The standard point-wise product has the kernel

$$K_{\mathrm{pw}}(\vec{X}_1, \vec{X}_2, \vec{X}\,) = \delta(\vec{X}_1 - \vec{X}\,)\delta(\vec{X}_2 - \vec{X}\,). \tag{33}$$

We make two comments.

Any vector can be considered as a function of one variable. Also any matrix element can be considered as a function of two variables, and the matrix itself can be considered as a column vector. From these observations follows the understanding that the star-product can also be introduced for vectors and operators which, in a chosen basis, are mapped onto matrices. The matrix elements are functions of column and row indices, and one can introduce any kind of star product for these functions which induces the star product for the operators. The star product for the operators can differ from the standard operator product which corresponds to the standard product of matrices given by rule row-by-column product. We employ this freedom for choosing and constructing different products of operators, in particular, to construct the product-of-operators classical observables. We use the following notation for two operators. The first operator which we call dequantizer reads

$$\hat{U}(\vec{X}\,) \equiv \hat{U}(q,p) = 2\hat{\mathcal{D}}(2\alpha)\hat{\mathcal{P}}, \qquad \vec{X} = (q,p) \in R, \quad \alpha = (q+ip)/\sqrt{2}, \tag{34}$$

where

$$\hat{\mathcal{D}}(\gamma) = \exp\left(\gamma\hat{a}^\dagger - \gamma^*\hat{a}\right), \qquad \hat{a} = (\hat{q} + i\hat{p})/\sqrt{2}, \tag{35}$$

and $\hat{\mathcal{P}}$ is the parity operator. In another form, operator $\hat{U}(q,p)$ used in eqs. (18) and (19) is

$$\hat{U}(q,p) = \int |q + u/2\rangle\langle q - u/2|\, e^{ipu}\, \mathrm{d}u. \tag{36}$$

The second operator, called quantizer, reads

$$\hat{D}(\vec{X}) = \hat{D}(q,p) = \frac{1}{2\pi}\hat{U}(q,p). \tag{37}$$

One can check that

(38) $\operatorname{Tr}\hat{D}(\vec{X})\hat{U}(\vec{X}') = \delta(X - X')$, *i.e.*, $\operatorname{Tr}\hat{D}(q,p)\hat{U}(q',p') = \delta(q-q')\delta(p-p')$.

These properties provide the following relationships for any given function $F(q,p)$, namely,

$$\hat{F} = \int F(q,p)\hat{D}(q,p)\,\mathrm{d}q\,\mathrm{d}p = \int F(\vec{X})\hat{D}(\vec{X})\,\mathrm{d}\vec{X}, \tag{39}$$

$$F(q,p) = \operatorname{Tr}\hat{U}(q,p)\hat{F}. \tag{40}$$

Thus, given any two functions $F_1(q,p)$ and $F_2(q,p)$, one has two operators, given in view of eq. (39), as follows:

$$\hat{F}_1 = \int F_1(q,p)\hat{D}(q,p)\,\mathrm{d}q\,\mathrm{d}p, \qquad \hat{F}_2 = \int F_2(q,p)\hat{D}(q,p)\,\mathrm{d}q\,\mathrm{d}p. \tag{41}$$

A question arises.

If the product of functions $F_1(\vec{X})$ and $F_2(\vec{X})$ is defined as a point-wise product, which corresponds to multiplication rule of classical observables, what kind of product is induced by this multiplication rule of functions $F_1(q,p)$ and $F_2(q,p)$ for the constructed operators?

By answering this question, we arrive at the result which we first formulate within the general framework, namely, given a pair of operators, quantizer $\hat{D}(\vec{X})$ and dequantizer $\hat{U}(\vec{X})$, satisfying (38), two functions, $F_1(\vec{X})$ and $F_2(\vec{X})$, and their star product with the kernel provided by (32), let us construct two operators

$$\hat{F}_j = \int F_j(\vec{X})\hat{D}(\vec{X})\,\mathrm{d}\vec{X}, \qquad j = 1,2. \tag{42}$$

What is the product rule for operators $\hat{F}_j$ (we call star product) such that

$$\hat{F}_1 \star \hat{F}_2 \leftrightarrow (F_1 \star F_2)(\vec{X})? \tag{43}$$

In fact, we must construct the kernel for multiplication of matrix elements of the operators $\hat{F}_1$ and $\hat{F}_2$, if the kernel for multiplication of the functions $F_1(\vec{X})$ and $F_2(\vec{X})$ is given.

Let us have a basis $|n\rangle$ in the Hilbert space where $\hat{D}(\vec{X})$ and $\hat{U}(\vec{X})$ act. In this basis, which we consider as a complete and orthonormal set of vectors in the Hilbert space, our operators have the matrix elements

$$\hat{D}(\vec{X})_{nm} = \langle n|\hat{D}(\vec{X})|m\rangle = \operatorname{Tr}\hat{D}(\vec{X})|m\rangle\langle n|,$$
$$\hat{U}(\vec{X})_{nm} = \langle n|\hat{U}(\vec{X})|m\rangle = \operatorname{Tr}\hat{U}(\vec{X})|m\rangle\langle n|,$$

i.e.,

$$\hat{D}(\vec{X}\,) = \sum_{nm}\hat{D}(\vec{X}\,)_{nm}|m\rangle\langle n|, \qquad \hat{U}(\vec{X}\,) = \sum_{nm}\hat{U}(\vec{X}\,)_{nm}|m\rangle\langle n|. \tag{44}$$

The star product of operators $\hat{F}_1$ and $\hat{F}_2$ reads

$$\hat{F}_1 \star \hat{F}_2 = \int \mathrm{d}\vec{X}_1\,\mathrm{d}\vec{X}_2\,\mathrm{d}\vec{X} \sum_{abcdnm} K(\vec{X}_1,\vec{X}_2,\vec{X})\langle b|\hat{U}(\vec{X}_1)|a\rangle \times\langle \mathrm{d}|\hat{U}(\vec{X}_2)|c\rangle\langle m|\hat{D}(\vec{X}_2)|n\rangle\langle a|\hat{F}_1|b\rangle\langle c|\hat{F}_2|d\rangle|n\rangle\langle m|. \tag{45}$$

This formula means that the kernel of the star product of functions $F_1(\vec{X})$ and $F_2(\vec{X})$ induces a star product of the matrix elements of the corresponding operators $(\hat{F}_1)_{ab}$ and $(\hat{F}_2)_{cd}$, providing the star product of the operators. It is given by the kernel

$$k(a,b,c,d,m,n) = \int \mathrm{d}\vec{X}_1\,\mathrm{d}\vec{X}_2\,\mathrm{d}\vec{X}\,K(\vec{X}_1,\vec{X}_2,\vec{X})\hat{U}(X_1)_{ba}\hat{U}(X_2)_{dc}\hat{D}(X)_{nm}. \tag{46}$$

Thus, the star product of the matrix elements of operators $\hat{F}_1$ and $\hat{F}_2$ reads

$$(\hat{F}_1 \times \hat{F}_2)_{nm} = \sum_{abcd} k(a,b,c,d,m,n)(\hat{F}_1)_{ab}(\hat{F}_2)_{cd}. \tag{47}$$

If the product of functions is point-wise and given by the kernel (33), the kernel of the product of matrix elements reads

$$k_{\mathrm{pw}}(a,b,c,d,m,n) = \int \mathrm{d}\vec{X}\,\hat{U}(\vec{X})_{ba}\hat{U}(\vec{X})_{dc}\hat{D}(\vec{X})_{mn}. \tag{48}$$

Also in the case where

$$K(\vec{X}_1,\vec{X}_2,\vec{X}) = \mathrm{Tr}\left(\hat{D}(\vec{X}_1)\hat{D}(\vec{X}_2)\hat{U}(\vec{X})\right), \tag{49}$$

the star-product of the operators is the usual operator product, *i.e.*, the kernel of the product of matrices gives a standard row-column rule of the multiplication of matrices. If the product of functions is commutative, *i.e.*, the kernel $K(\vec{X}_1,\vec{X}_2,\vec{X})$ is symmetric with respect to permutation $1 \leftrightarrow 2$, the star-product of the operators is also commutative, *i.e.*, $\hat{F}_1 \star \hat{F}_2 = \hat{F}_2 \star \hat{F}_1$, that follows from the corresponding permutation symmetry of kernel (43).

Thus, the Grönewold kernel of the star product of Weyl symbols just satisfies the condition which is obtained using dequantizer (36) and quantizer (37) in view of formula (45). This means that the product-of-operators observables corresponding to the functions on the phase space which are Weyl symbols of the operators is just the standard product of the operators, but the commutative kernel for the product of functions on the phase space induces the kernel for the star product of the operators observables in the formalism of Hilbert space and operators for classical statistical mechanics.

5. – The evolution equation for quantum tomograms

The Schrödinger equation for the state vector $|\psi,t\rangle$ for the system with the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2} + U(\hat{q}) \tag{50}$$

reads

$$i\frac{\partial}{\partial t}\left|\psi,t\right\rangle = \hat{H}\left|\psi,t\right\rangle \qquad (\hbar=1). \tag{51}$$

In the coordinate representation, the equation has the form of a differential equation for the wave function $\psi(x,t)$, *i.e.*,

$$i\frac{\partial\psi(x,t)}{\partial t} = -\frac{1}{2}\frac{\partial^2\psi(x,t)}{\partial x^2} + U(x)\psi(x,t). \tag{52}$$

The von Neumann equation for the density matrix of pure state $\rho(x,x',t)=\psi(x,t)\psi^*(x',t)$ can be easily derived from eq. (52) and it has the form

$$i\frac{\partial\rho(x,x',t)}{\partial t} = -\frac{1}{2}\left(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial x'^2}\right)\rho(x,x',t) + \left(U(x)-U(x')\right)\psi(x,x',t). \tag{53}$$

This equation is also valid for any convex sum of density matrices of pure states, *i.e.*, for mixed states.

The evolution equation can be transformed into the Moyal equation for the Wigner function $W(q,p,t)$ using the change of variables induced by Fourier transform of the density matrix providing the Wigner function. The Moyal equation reads

$$\frac{\partial W(q,p,t)}{\partial t} + p\frac{\partial W(q,p,t)}{\partial q} + \frac{1}{i}\left[U\left(q-\frac{i}{2}\frac{\partial}{\partial p}\right) - \text{c.c.}\right]W(q,p,t) = 0. \tag{54}$$

In operator form, this equation for the quantum state associated with the density operator $\hat{\rho}(t)$ is

$$\frac{\partial\hat{\rho}(t)}{\partial t} + i\left[\hat{H},\hat{\rho}(t)\right] = 0. \tag{55}$$

This means that the density operator is an integral of the motion.

Thus, we have the quantum evolution equation for the system's state written in the three different forms (53)–(55). The tomographic form of the evolution equation can easily be obtained applying the Radon integral transform to the Moyal equation, and the result is written in [16, 30, 31] as follows:

$$\begin{aligned}\frac{\partial w(X,\mu,\nu,t)}{\partial t} &- \mu\frac{\partial w(X,\mu,\nu,t)}{\partial \nu}\\ &-2\,\text{Im}\left[U\left(-\left(\frac{\partial}{\partial X}\right)^{-1}\frac{\partial}{\partial\mu} + \frac{i\nu}{2}\frac{\partial}{\partial X}\right)\right]w(X,\mu,\nu,t) = 0.\end{aligned} \tag{56}$$

Making change of variables $\mu=\cos\theta$ and $\nu=\sin\theta$ which provides the optical tomogram $w(X,\mu,\nu,t)\to w(X,\theta,t)$ yields the evolution equation for the optical tomogram [32]

$$\begin{aligned}\frac{\partial}{\partial t}w(X,\theta,t) = &\left[\cos^2\theta\frac{\partial}{\partial\theta} - \frac{1}{2}\sin 2\theta\left\{1+X\frac{\partial}{\partial X}\right\}\right]w(X,\theta,t)\\ &+2\left[\text{Im}\,U\left\{\sin\theta\frac{\partial}{\partial\theta}\left[\frac{\partial}{\partial X}\right]^{-1} + X\cos\theta + i\frac{\sin\theta}{2}\frac{\partial}{\partial X}\right\}\right]w(X,\theta,t).\end{aligned} \tag{57}$$

6. – Energy level equations for tomograms

For stationary states, the energy level equations are obtained by solving the Schrödinger equation for the wave function

$$\hat{H}\psi_E(x) = -\frac{1}{2}\frac{\partial^2}{\partial x^2}\psi_E(x) + U(x)\psi_E(x) = E\psi_E(x). \tag{58}$$

This equation can also be given in a tomographic form as well as in a Moyal form. The Moyal equation for the energy levels is

$$\begin{aligned} EW_E(q,p) = &-\frac{1}{4}\left[\left(\frac{1}{2}\frac{\partial}{\partial q}+ip\right)^2+\left(\frac{1}{2}\frac{\partial}{\partial q}-ip\right)^2\right]W_E(q,p) \\ &+\frac{1}{2}\left[U\left(q+\frac{i}{2}\frac{\partial}{\partial p}\right)+U\left(q-\frac{i}{2}\frac{\partial}{\partial p}\right)\right]W_E(q,p). \end{aligned} \tag{59}$$

For the symplectic tomogram, the energy-level equation reads

$$\begin{aligned} Ew_E(X,\mu,\nu) = &-\frac{1}{4}\left[\left(\frac{1}{2}\mu\frac{\partial}{\partial X}-i\frac{\partial}{\partial \nu}\left(\frac{\partial}{\partial X}\right)^{-1}\right)^2+\text{c.c.}\right]w_E(X,\mu,\nu) \\ &+\frac{1}{2}\left[U\left(-\frac{\partial}{\partial \mu}\left(\frac{\partial}{\partial X}\right)^{-1}+\frac{i}{2}\nu\frac{\partial}{\partial X}\right)+\text{c.c.}\right]w_E(X,\mu,\nu). \end{aligned} \tag{60}$$

For the optical tomogram, the energy-level equation has the form

$$\begin{aligned} Ew_E(X,\theta) = &\left[\left\{\frac{\cos^2\theta}{2}\left[\frac{\partial}{\partial X}\right]^{-2}\left(\frac{\partial^2}{\partial\theta^2}+1\right)-\frac{X}{2}\left[\frac{\partial}{\partial X}\right]^{-1}\right.\right. \\ &\times\left.\left.\left(\cos^2\theta+\sin 2\theta\frac{\partial}{\partial\theta}\right)+\frac{X^2}{2}\sin^2\theta-\frac{\cos^2\theta}{8}\frac{\partial^2}{\partial X^2}\right\}\right]w_E(\vec{X},\vec{\theta}) \\ &+\left[\operatorname{Re}U\left\{\sin\theta\frac{\partial}{\partial\theta}\left[\frac{\partial}{\partial X}\right]^{-1}+X\cos\theta+i\,\frac{\sin\theta}{2}\frac{\partial}{\partial X}\right\}\right]w_E(\vec{X},\vec{\theta}). \end{aligned} \tag{61}$$

The solutions of the energy level equation in the symplectic form for the harmonic oscillator reads

$$w_n(X,\mu,\nu) = \frac{e^{-X^2/(\mu^2+\nu^2)}}{\sqrt{\pi(\mu^2+\nu^2)}}\frac{1}{n!\,2^n}H_n^2\left(\frac{X}{\sqrt{\mu^2+\nu^2}}\right). \tag{62}$$

The solutions of the energy level equation in the optical form for the harmonic oscillator is

$$w_n(X,\theta) = \frac{e^{-X^2}}{\sqrt{\pi}}\frac{1}{n!\,2^n}H_n^2(X). \tag{63}$$

One can see that the optical tomogram of the Fock state $|n\rangle$ does not depend on the local oscillator phase θ.

7. – Quantum inequalities for continuous variables

For continuous variables, the wave function $\psi(x)$ provides the probability distribution density

$$P(x) = |\psi(x)|^2. \tag{64}$$

The corresponding Shannon entropy reads (see, *e.g.*, [33])

$$S_x = -\int |\psi(x)|^2 \ln |\psi(x)|^2 \, \mathrm{d}x. \tag{65}$$

In the momentum representation, one has the wave function

$$\widetilde{\psi}(p) = \frac{1}{\sqrt{2\pi}} \int \psi(x) e^{-ipx} \, \mathrm{d}x \qquad (\hbar = 1). \tag{66}$$

The corresponding Shannon entropy in terms of the momentum probability density $|\widetilde{\psi}(p)|^2$ is

$$S_p = -\int |\widetilde{\psi}(p)|^2 \ln |\widetilde{\psi}(p)|^2 \, \mathrm{d}p. \tag{67}$$

There exists a correlation for the entropies S_x and S_p, since the function $\psi(x)$ determines the Fourier component $\widetilde{\psi}(p)$. This means that the entropies S_x and S_p have to obey some constrains. These constrains are entropic uncertainty relations.

For the one-mode system, the inequality reads (see [33], p. 28)

$$S_x + S_p \geq \ln(\pi e). \tag{68}$$

One has the optical-tomogram expression in terms of the wave function

$$w(X, \theta) = \left| \int \psi(y) \exp\left[\frac{i}{2} \left(\cot\theta (y^2 + X^2) - \frac{2X}{\sin\theta}\, y \right) \right] \frac{\mathrm{d}y}{\sqrt{2\pi i \sin\theta}} \right|^2. \tag{69}$$

On the other hand, this tomogram is formally equal to

$$w(X, \theta) = |\psi(X, \theta)|^2, \tag{70}$$

where the wave function reads

$$\psi(X, \theta) = \frac{1}{\sqrt{2\pi i \sin\theta}} \int \exp\left[\frac{i}{2} \left(\cot\theta (y^2 + X^2) - \frac{2X}{\sin\theta}\, y \right) \right] \psi(y) \mathrm{d}y, \tag{71}$$

being the fractional Fourier transform of the wave function $\psi(y)$. This wave function corresponds to the wave function of a harmonic oscillator with $\hbar = m = \omega = 1$ taken at the time moment θ provided the wave function at the initial time moment $\theta = 0$ equals

$\psi(y)$. In view of expressions of tomogram in terms of the wave function (70) and (71), one has the entropic uncertainty relation in the form

$$S(\theta)+S(\theta+\pi/2)\geq \ln \pi e. \tag{72}$$

Here $S(\theta)$ is the tomographic Shannon entropy associated with optical tomogram (69) which is measured by homodyne detector. We illustrate the entropic inequality (72) by the example of the harmonic oscillator's ground state with the wave function

$$\psi_0(x)=\pi^{-1/4}e^{-x^2/2} \tag{73}$$

written in dimensionless variables. Using (71), we obtain the tomogram

$$w(X,\theta)=\pi^{-1/2}e^{-X^2}. \tag{74}$$

The ground-state tomogram does not depend on the angle θ. In view of this, the entropy is $S(\theta)=S(\theta+\pi/2)=(\ln \pi e)/2$. The sum of these two entropies saturates inequality (72).

In a recent paper [34], the new uncertainty relation was obtained for Rényi entropy [35] related to the probability distributions for position and momentum of quantum state with density operator $\hat{\rho}$. The uncertainty relation reads

$$\begin{aligned}&\frac{1}{1-\alpha}\ln\left(\int_{-\infty}^{\infty}\mathrm{d}p\,[\rho(p,p)]^{\alpha}\right)+\frac{1}{1-\beta}\ln\left(\int_{-\infty}^{\infty}\mathrm{d}x\,[\rho(x,x)]^{\beta}\right)\\&\geq-\frac{1}{2(1-\alpha)}\ln\frac{\alpha}{\pi}-\frac{1}{2(1-\beta)}\ln\frac{\beta}{\pi}\,,\end{aligned} \tag{75}$$

where positive parameters α and β satisfy the constraint

$$(1/\alpha)+(1/\beta)=2. \tag{76}$$

Rényi entropies R_α and R_β related to the momentum and position distributions, respectively, are just two terms on the left-hand side of (75). For $\alpha,\beta\longrightarrow 1$, these entropies become Shannon entropies S_p and S_x.

We illustrate this inequality by the example of the harmonic oscillator's ground state. In this case, one has the Rényi entropies

$$R_\alpha=\frac{\ln\pi}{2}-\frac{1}{2}\frac{\ln\alpha}{1-\alpha}\,,\qquad R_\beta=\frac{\ln\pi}{2}-\frac{1}{2}\frac{\ln\beta}{1-\beta}\,,$$

which, in the limit $\alpha\to 1$ and $\beta\to 1$, go to $(\ln\pi)/2$. Also the sum of the entropies reads

$$R_\alpha+R_\beta=\ln\pi-\frac{1}{2}\frac{\ln\alpha}{1-\alpha}-\frac{1}{2}\frac{\ln\beta}{1-\beta}\,.$$

In view of (76), this sum equals the right-hand side of inequality (75). Thus, the harmonic oscillator's ground state saturates this inequality.

Using the same argument that we employed to obtain inequality (72) for Shannon entropies, we arrive at the condition for optical tomogram [36]

$$(q-1)\ln\left\{\int_{-\infty}^{\infty}\mathrm{d}X\,[w\,(X,\theta+\pi/2)]^{1/(1-q)}\right\} \\ +(q+1)\ln\left\{\int_{-\infty}^{\infty}\mathrm{d}X[w(X,\theta)]^{1/(1+q)}\right\} \\ \geq(1/2)\Big\{(q-1)\ln[\pi(1-q)]+(q+1)\ln[\pi(1+q)]\Big\}, \tag{77}$$

where the parameter q is defined by $\alpha=(1-q)^{-1}$. This inequality has been checked experimentally [37].

8. – Checking the position-momentum uncertainty relations

In view of the physical meaning of optical tomogram, one can calculate higher moments of the probability distribution

$$\langle X^n\rangle(\mu,\nu)=\int X^u M(X,\mu,\nu)\,\mathrm{d}X,\qquad n=1,2,\dots, \tag{78}$$

for any value of the parameters μ and ν; in particular, for any given phase of the local oscillator θ. This provides the possibility to check the inequalities for the quantum uncertainty relations [19].

The Heisenberg uncertainty relation connects position and momentum variances σ_{QQ} and σ_{PP} by means of an inequality. In the tomographic-probability representation, the Heisenberg relation reads (see, *e.g.*, [21]):

$$\sigma_{PP}\sigma_{QQ}=\left(\int X^2M(X,0,1)\,\mathrm{d}X-\left[\int XM(X,0,1)\,\mathrm{d}X\right]^2\right) \\ \times\left(\int X^2M(X,1,0)\,\mathrm{d}X-\left[\int XM(X,1,0)\,\mathrm{d}X\right]^2\right)\geq 1/4. \tag{79}$$

The Schrödinger-Robertson uncertainty relation contains the contribution of the position-momentum covariance σ_{QP} and reads

$$\sigma_{QQ}\sigma_{PP}-\sigma_{QP}^2\geq 1/4. \tag{80}$$

In view of eq. (78), the variance σ_{XX} of the homodyne quadrature X in terms of the parameters μ, ν, and the quadratures variances and covariance is

$$\sigma_{XX}(\mu,\nu)=\mu^2\sigma_{QQ}+\nu^2\sigma_{PP}+2\,\mu\nu\sigma_{QP}. \tag{81}$$

The above formula is obtained using the definition of homodyne-quadrature-component operator

$$\hat{X}=\mu\hat{q}+\nu\hat{p}. \tag{82}$$

Thus, one has

$$\hat{X}^2 = \mu^2 \hat{q}^2 + \nu^2 \hat{p}^2 + 2\mu\nu(\hat{q}\hat{p} + \hat{p}\hat{q})/2. \tag{83}$$

Taking the average for any state in (82), one has the equality for the mean value

$$\langle \hat{X} \rangle = \mu \langle \hat{q} \rangle + \nu \langle \hat{p} \rangle. \tag{84}$$

Averaging (83), we obtain

$$\langle \hat{X}^2 \rangle = \mu^2 \langle \hat{q}^2 \rangle + \nu^2 \langle \hat{p}^2 \rangle + 2\mu\nu \left\langle (\hat{q}\hat{p} + \hat{p}\hat{q})/2 \right\rangle . \tag{85}$$

Thus,

$$\begin{aligned} \sigma_{XX}(\mu,\nu) = \langle \hat{X}^2 \rangle - \langle \hat{X} \rangle^2 &= \mu^2 \left(\langle \hat{q}^2 \rangle - \langle \hat{q} \rangle^2 \right) + \nu^2 \left(\langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2 \right) \\ &+ 2\mu\nu \left(\left\langle \frac{\hat{q}\hat{p} + \hat{p}\hat{q}}{2} \right\rangle - \langle \hat{q} \rangle \langle \hat{p} \rangle \right). \end{aligned} \tag{86}$$

To derive (86), we used $\langle \hat{X} \rangle^2 = \mu^2 \langle \hat{q} \rangle^2 + \nu^2 \langle \hat{p} \rangle^2 + 2\mu\nu \langle \hat{q} \rangle \langle \hat{p} \rangle$. Since $\sigma_{QP} = \langle (\hat{q}\hat{p} + \hat{p}\hat{q})/2 \rangle - \langle \hat{q} \rangle \langle \hat{p} \rangle$, one obtains the expression of the covariance σ_{QP} in terms of the tomographic characteristics of the state. Taking $\mu = \nu = \sqrt{2}/2$ corresponding to the local oscillator phase $\theta = \pi/4$, one has

$$\sigma_{QP} = \sigma_{XX} \left(\theta = \pi/4 \right) - \left(\sigma_{QQ} + \sigma_{PP} \right)/2, \tag{87}$$

where σ_{PP} and σ_{QQ} are the factors appearing on the left-hand side of eq. (79), respectively. The term $\sigma_{XX}(\theta = \pi/4)$ is given by eq. (78) as

$$\sigma_{XX} \left(\theta = \pi/4 \right) = \left\langle X^2 \right\rangle \left(\sqrt{2}/2, \sqrt{2}/2 \right) - \left[\langle X \rangle \left(\sqrt{2}/2, \sqrt{2}/2 \right) \right]^2 . \tag{88}$$

The check of Schrödinger-Robertson uncertainty relations requires extra elaboration to use the available optical tomogram of the photon quantum state experimentally obtained. We express this procedure as the following inequality for optical tomogram. Let us calculate the function $F(\theta)$ which we call the tomographic uncertainty function

$$\begin{aligned} F(\theta) = &\left(\int X^2 w(X,\theta) \mathrm{d}X - \left[\int X w(X,\theta)\, \mathrm{d}X \right]^2 \right) \\ &\times \left(\int X^2 w\left(X, \theta + \frac{\pi}{2}\right) \mathrm{d}X - \left[\int X w\left(X, \theta + \frac{\pi}{2}\right) \mathrm{d}X \right]^2 \right) \\ &- \left\{ \int X^2 w\left(X, \theta + \frac{\pi}{4}\right) \mathrm{d}X - \left[\int X w\left(X, \theta + \frac{\pi}{4}\right) \mathrm{d}X \right]^2 \right. \\ &- \frac{1}{2} \left[\int X^2 w(X,\theta) \mathrm{d}X - \left[\int X w(X,\theta) \mathrm{d}X \right]^2 \right. \\ &\left. \left. + \int X^2 w\left(X, \theta + \frac{\pi}{2}\right) \mathrm{d}X - \left[\int X w\left(X, \theta + \frac{\pi}{2}\right) \mathrm{d}X \right]^2 \right] \right\}^2 - \frac{1}{4} . \end{aligned} \tag{89}$$

The tomographic uncertainty function must be nonnegative $F(\theta) \geq 0$ for all the values of the local oscillator phase angle $0 \leq \theta \leq 2\pi$ [19]. The previous eq. (89) for $\theta = 0$ yields eq. (80). This inequality has been checked experimentally in [20].

9. – Thick quantum tomography

We now turn our attention to "thick" tomographic maps [9], which is a more realistic approach for practical applications, because instead of marginals defined over lines, as in the classical Radon transform [7] or in the transform on quadratic curves [8, 11], it involves a "thick" window function Ξ. This is convoluted with the tomographic map and concentrates the marginals around some given background curves (that can be lines or quadrics), without resorting to a singular delta function. For example, if the weight function Ξ is a step function, it defines marginals along thick lines or thick quadratic curves. In the quantum case, this amounts to replacing in the definition of the dequantizer $\hat{U}(x)$ the Dirac delta function by the weight function Ξ.

For the symplectic quantum tomography, one has the dequantizer

$$\hat{U}(X, \mu, \nu) = \Xi\,(X - \mu\hat{q} - \nu\hat{p})\,. \tag{90}$$

The new tomogram reads

$$w_\Xi(X, \mu, \nu) = \mathrm{Tr}\,\hat{\rho}\,\Xi\,(X - \mu\hat{q} - \nu\hat{p})\,. \tag{91}$$

Using the Weyl map one obtains a thick tomogram for the Wigner function

$$w_\Xi(X, \mu, \nu) = (2\pi)^{-1} \int W(p, q)\,\Xi\,(X - \mu q - \nu p)\;\mathrm{d}p\,\mathrm{d}q. \tag{92}$$

The interesting property of the above formula (92) is that it can be inverted in completely analogy with the classical thick tomography introduced in [11]. The thick tomogram can be expressed in terms of standard symplectic tomograms via a convolution formula

$$w_\Xi(X, \mu, \nu) = \int w(Y, \mu, \nu)\,\Xi\,(X - Y)\,\mathrm{d}Y, \tag{93}$$

which leads to the explicit construction of the inverse transform. Indeed, the inverse transform is obtained by means of a Fourier transform of the convolution integral

$$W(p, q) = \frac{\mathcal{N}_\Xi}{2\pi} \int w_\Xi(Y, \mu, \nu)\mathrm{e}^{i(X - \mu q - \nu p)}\;\mathrm{d}X\;\mathrm{d}\mu\;\mathrm{d}\nu, \tag{94}$$

where

$$\mathcal{N}_\Xi = \frac{1}{\tilde{\Xi}(-1)}, \qquad \tilde{\Xi}(-1) = \int \Xi(z)\mathrm{e}^{iz}\,\mathrm{d}z. \tag{95}$$

In invariant form, the state reconstruction is achieved by

$$\hat{\rho} = \frac{\mathcal{N}_\Xi}{2\pi} \int w_\Xi(X, \mu, \nu)\mathrm{e}^{i(X - \mu\hat{q} - \nu\hat{p})}\;\mathrm{d}X\;\mathrm{d}\mu\;\mathrm{d}\nu.$$

The quantizer operator in thick symplectic tomography is

$$\hat{D}(X,\mu,\nu) = \frac{\mathcal{N}_\Xi}{2\pi}\, \mathrm{e}^{i(X-\mu\hat{q}-\nu\hat{p})}. \tag{96}$$

Now we consider a particular example of thick tomogram to illustrate the potentialities of the new method. If the weight function is a Gaussian function

$$\Xi(z) = \frac{1}{\sqrt{2\pi\sigma^2}}\, \mathrm{e}^{-z^2/2\sigma^2}, \tag{97}$$

which tends to the delta distribution in the $\sigma \to 0$ limit $[\lim_{\sigma\to 0} \Xi(z) = \delta(z)]$, the thick tomogram of the coherent states $|\alpha\rangle\langle\alpha|$ reads

$$w_\sigma^\alpha(X,\mu,\nu) = \frac{1}{\sqrt{\pi(\mu^2+\nu^2+\sigma^2)}}\, \mathrm{e}^{-(X-\bar{X})^2/(\mu^2+\nu^2+\sigma^2)}, \tag{98}$$

where $\bar{X} = \sqrt{2}\,\mu\,\mathrm{Re}\,\alpha + \sqrt{2}\,\nu\,\mathrm{Im}\,\alpha$.

For the vacuum state $|0\rangle\langle 0|$, the tomogram reads

$$w_\sigma^{\mathrm{vac}}(X,\mu,\nu) = \frac{1}{\sqrt{\pi(\mu^2+\nu^2+\sigma^2)}} \mathrm{e}^{-X^2/(\mu^2+\nu^2+\sigma^2)}, \tag{99}$$

The quantizer is

$$\hat{D}_\sigma(X,\mu,\nu) = \frac{1}{2\pi}\, \mathrm{e}^{(\sigma^2/2)+i(X-\mu\hat{q}-\nu\hat{p})}$$

and the dequantizer is given by

$$\hat{U}_\sigma(X,\mu,\nu) = \frac{1}{\sqrt{2\pi\sigma^2}}\, \mathrm{e}^{-[(X-\mu\hat{q}-\nu\hat{p})^2]/2\sigma^2}.$$

One interesting property, that is preserved by the smoothing of the tomogram, is that the marginals $w_\Xi(X,\mu,\nu)$ are also probability distributions. In the limit $\sigma \to 0$, $\Xi(z) \to \delta(z)$, $\widetilde{\Xi}(-1) = 1$ and $\mathcal{N}_\Xi = 1$.

10. – Conclusions and outlooks

To conclude, we summarize the main results of our work.

The symplectic tomographic-probability distribution, considered as the primary concept of a particle quantum state alternative to the wave function or density matrix, was shown to be associated with a unitary representation of the Weyl-Heisenberg group. It can be shown how to deal with the tomographic picture for general Lie groups and for finite groups [38,39]. In this connection, the C^*-algebraic approach to quantum mechanics and its counterpart in terms of tomograms can be elaborated [40].

We have shown that quantum mechanics can be formulated using a fair probability distribution as a replacement of the quantum state expressed as a wave function or a density operator. This provides the possibility to obtain the quantum evolution and energy-level equations for the fair probability distributions like the evolution equations in classical statistical mechanics.

* * *

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