Colloquia: QCDN12

# The nucleon spin sum rule

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**Summary.** — Definitions of orbital angular momentum based on Wigner distributions are used as a framework to discuss the connection between the Ji definition of the quark orbital angular momentum and that of Jaffe and Manohar. We find that the difference between these two definitions can be interpreted as the change in the quark orbital angular momentum due to final state interactions as it leaves the target in a DIS experiment.

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## 1. – Angular momentum decompositions

Since the famous EMC experiments revealed that only a small fraction of the nucleon spin is due to quark spins [1], there has been a great interest in "solving the spin puzzle", *i.e.* in decomposing the nucleon spin into contributions from quark/gluon spin and orbital degrees of freedom. In this effort, the Ji decomposition [2]

$$(1) \qquad \frac{1}{2} = \frac{1}{2} \sum_q \Delta q + \sum_q L_q^z + J_g^z$$

appears to be very useful: through GPDs, not only the quark spin contributions $\Delta q$ but also the quark total angular momenta $J_q \equiv \frac{1}{2}\Delta q + L_q^z$ (and by subtracting the spin piece also the quark orbital angular momenta $L_q^z$) entering this decomposition can be accessed experimentally. The terms in (1) are defined as expectation values of the corresponding terms in the angular momentum tensor

$$(2) \qquad M^{0xy} = \sum_q \frac{1}{2} q^\dagger \Sigma^z q + \sum_q q^\dagger (\vec{r} \times i\vec{D})^z q + \left[ \vec{r} \times (\vec{E} \times \vec{B}) \right]^z$$

in a nucleon state with zero momentum. Here $i\vec{D} = i\vec{\partial} - g\vec{A}$ is the gauge-covariant derivative. The main advantages of this decomposition are that each term can be expressed as the expectation value of a manifestly gauge invariant local operator and that the quark total angular momentum $J^q = \frac{1}{2}\Delta q + L^q$ can be related to GPDs [2] and is thus accessible

in deeply virtual Compton scattering and deeply virtual meson production and can also be calculated in lattice gauge theory.

Jaffe and Manohar have proposed an alternative decomposition of the nucleon spin, which does have a partonic interpretation [3], and in which also two terms, $\frac{1}{2}\Delta q$ and $\Delta G$, are experimentally accessible

$$
(3) \qquad \frac{1}{2} = \frac{1}{2}\sum_q \Delta q + \sum_q \mathcal{L}^q + \Delta G + \mathcal{L}^g.
$$

The individual terms in (3) can be defined as matrix elements of the corresponding terms in the +12 component of the angular momentum tensor

$$
(4) \qquad M^{+12} = \frac{1}{2}\sum_q q_+^\dagger \gamma_5 q_+ + \sum_q q_+^\dagger (\vec{r}\times i\vec{\partial})^z q_+
$$
$$
+\varepsilon^{+-ij}\,\mathrm{Tr}\,F^{+i}A^j + 2\,\mathrm{Tr}\,F^{+j}(\vec{r}\times i\vec{\partial})^z A^j
$$

for a nucleon polarized in the $+\hat{z}$ direction. The first and third term in (3), (4) are the "intrinsic" contributions (no factor of $\vec{r}\times$) to the nucleon's angular momentum $J^z = +\frac{1}{2}$ and have a physical interpretation as quark and gluon spin respectively, while the second and fourth term can be identified with the quark/gluon OAM. Here $q_+ \equiv \frac{1}{2}\gamma^-\gamma^+ q$ is the dynamical component of the quark field operators, and light-cone gauge $A^+ \equiv A^0 + A^z = 0$ is implied. The residual gauge invariance can be fixed by imposing anti-periodic boundary conditions $\vec{A}_\perp(\mathbf{x}_\perp, \infty) = -\vec{A}_\perp(\mathbf{x}_\perp, -\infty)$ on the transverse components of the vector potential. $\mathcal{L}$ also naturally arises in a light-cone wave function description of hadron states, where $\frac{1}{2} = \frac{1}{2}\sum_q \Delta q + \Delta G + \mathcal{L}$, in the sense of an eigenvalue equation, is manifestly satisfied for each Fock component individually [4].

A variation of (1) has been suggested in ref. [5], where part of $L_q^z$ is attributed to the glue as "potential angular momentum". As we will discuss in the following, the potential angular momentum also has a more physical interpretation as the effect from final state interactions. Other decompositions, in which only one term is experimentally accessible, will not be discussed in this brief note.

## 2. – Orbital angular momentum from Wigner distributions

Wigner distributions can be defined as defined as off forward matrix elements of non-local correlation functions [6-8]

$$
(5) \qquad W^{\mathcal{U}}(x, \vec{b}_\perp, \vec{k}_\perp) \equiv \int \frac{\mathrm{d}^2\vec{q}_\perp}{(2\pi)^2} \int \frac{\mathrm{d}^2\xi_\perp \mathrm{d}\xi^-}{(2\pi)^3} e^{-i\vec{q}_\perp \cdot \vec{b}_\perp} e^{i(xP^+\xi^- - \vec{k}_\perp \cdot \vec{\xi}_\perp)}
$$
$$
\times \langle P'S'|\bar{q}(0)\Gamma \mathcal{U}_{0\xi} q(\xi)|PS\rangle
$$

with $P^+ = P^{+\prime}$, $P_\perp = -P'_\perp = \frac{q_\perp}{2}$. Throughout this paper, we will chose $\vec{S} = \vec{S}' = \hat{z}$. Furthermore, we will focus on the "good" component by selecting $\Gamma = \gamma^+$. In order to ensure manifest gauge invariance, a Wilson line gauge link $\mathcal{U}_{0\xi}$ connecting the quark field operators at position 0 and $\xi$ must be included [9,10]. The issue of choice of path for the Wilson line will be addressed below.
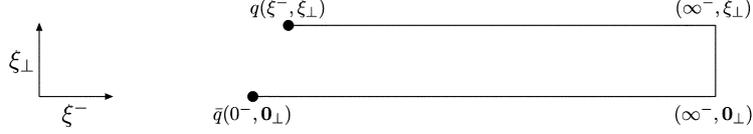
Fig. 1. – Illustration of the path for the Wilson line gauge link used to define the Wigner distribution $W^{+LC}$ (5).

In terms of Wigner distributions, quark OAM can be defined as [11]

$$(6) \qquad L_{\mathcal{U}} = \int \mathrm{d}x \mathrm{d}^2\vec{b}_\perp d^2\vec{k}_\perp (\vec{b}_\perp \times \vec{k}_\perp)_z W^{\mathcal{U}}(x, \vec{b}_\perp, \vec{k}_\perp).$$

No issues with Heisenberg's uncertainty principle arise as only perpendicular combinations of position $\vec{b}_\perp$ and momentum $\vec{k}_\perp$ are needed simultaneously in eq. (6).

A straight line for the Wilson line in $\mathcal{U}_{0\xi}$ is often the most natural choice, yielding [7]

$$(7) \qquad L^q_{straight} \equiv \int \mathrm{d}x \mathrm{d}^2\vec{b}_\perp d^2\vec{k}_\perp \left(\vec{b}_\perp \times \vec{k}_\perp\right)_z W^{straight}(x, \vec{b}_\perp, \vec{k}_\perp)$$
$$= \frac{\int \mathrm{d}^3\vec{r}\langle PS|q^\dagger(\vec{r})(\vec{r} \times i\vec{D})q(\vec{r})|PS\rangle}{\langle PS|PS\rangle} = L^q_{Ji}$$

for a nucleon polarized in the $+\hat{z}$ direction, where $i\vec{D} = i\vec{\partial} + g\vec{A}(\vec{r})$ is the usual gauge-covariant derivative. This is also the OAM that appears in the Ji decomposition (1).

However, depending on the context, other choices for the path in the Wilson link $\mathcal{U}$ should be made. Indeed, in the context of Transverse Momentum dependent parton Distributions (TMDs) probed in Semi-Inclusive Deep-Inelastic Scattering (SIDIS) [12-14] the path should be taken to be a straight line to $x^- = \infty$ along (or very close to) the light-cone. This particular choice ensures proper inclusion of the Final State Interactions (FSI) experienced by the struck quark as it leaves the nucleon along a nearly light-like trajectory in the Bjorken limit. However, a Wilson line to $\xi^- = \infty$, for fixed $\vec{\xi}_\perp$ is not yet sufficient to render Wigner distributions manifestly gauge invariant, but a link at $x^- = \infty$ must be included to ensure manifest gauge invariance. While the latter may be unimportant in some gauges, it is crucial in light-cone gauge for the description of TMDs relevant for SIDIS [15].

Let $\mathcal{U}_{0\xi}^{+LC}$ be the Wilson path ordered exponential obtained by first taking a Wilson line from $(0^-, \vec{0}_\perp)$ to $(\infty, \vec{0}_\perp)$, then to $(\infty, \vec{\xi}_\perp)$, and then to $(\xi^-, \vec{\xi}_\perp)$, with each segment being a straight line (fig. 1) [9]. The shape of the segment at $\infty$ is irrelevant as the gauge field is pure gauge there, but it is still necessary to include a connection at $\infty$ and for simplicity we pick a straight line. A a similar "staple" to $-\infty$ is used to define the Wilson path ordered exponential $\mathcal{U}_{0\xi}^{-LC}$. Using those light-like gauge links we define

$$(8) \qquad W^{\pm LC}(x, \vec{b}_\perp, \vec{k}_\perp) \equiv \int \frac{\mathrm{d}^2\vec{q}_\perp}{(2\pi)^2} \int \frac{\mathrm{d}^2\xi_\perp \mathrm{d}\xi^-}{(2\pi)^3} e^{i(xP^+\xi^- - \vec{k}_\perp \cdot \vec{\xi}_\perp)}$$
$$\times \langle P'S'|\bar{q}(0)\Gamma\mathcal{U}_{0\xi}^{\pm LC}q(\xi)|PS\rangle.$$

This definition for $W^{+LC}$ the same as that in [9] and similar to that of $W_{LC}$ in ref. [7], except that the link segment at $x^- = \infty$ was not included in the definition of $W_{LC}$ [7]. The Wilson-like gauge link used to guarantee manifest gauge invariance is defined using a light-like staple, *i.e.* it is constructed using three straight line gauge links $\mathcal{U}_{0\xi}^{+LC} = W_{0-0_\perp,\infty-0_\perp} W_{\infty-0_\perp,\infty-\xi_\perp} W_{\infty-\xi_\perp,\xi-\xi_\perp}$ and similarly for $\mathcal{U}_{0\xi}^{-LC}$.

In light-cone gauge $A^+ = 0$ the Wilson lines to $x^- = \infty$ become trivial and only the piece at $x^- = \infty$ remains. While the gauge field at light-cone infinity $\vec{A}_\perp(\pm\infty, \vec{r}_\perp)$ cannot be neglected or set equal to zero in light-cone gauge, it can be chosen to satisfy anti-symmetric boundary conditions

$$(9) \qquad \vec{\alpha}_\perp(\vec{r}_\perp) \equiv \vec{A}_\perp(\infty, \vec{r}_\perp) = -\vec{A}_\perp(-\infty, \vec{r}_\perp).$$

This choice maintains manifest PT (sometimes called "light-cone parity") invariance.

Using these Wigner distributions, one can now proceed to introduce orbital angular momentum as

$$(10) \qquad \mathcal{L}_\pm^q \equiv \int \mathrm{d}x \mathrm{d}^2\vec{b}_\perp d^2\vec{k}_\perp (\vec{b}_\perp \times \vec{k}_\perp)^z W^{\pm LC}(x, \vec{b}_\perp, \vec{k}_\perp)$$

$$= \frac{\int \mathrm{d}^3\vec{r}\langle PS|\bar{q}(\vec{r})\gamma^+[\vec{r} \times (i\vec{\partial} \pm g\vec{\alpha}_\perp(\vec{r}_\perp)]^z q(\vec{r})|PS\rangle}{\langle PS|PS\rangle},$$

and similar for the glue. Equation (10) differs from

$$(11) \qquad \mathcal{L}^q = \frac{\int \mathrm{d}^3\vec{r}\langle PS|\bar{q}(\vec{r})\gamma^+(\vec{r} \times i\vec{\partial})^z q(\vec{r})|PS\rangle}{\langle PS|PS\rangle}$$

(denoted $\tilde{L}^q$ in ref. [7]) by the contribution from the gauge field $\pm\vec{\alpha}_\perp$ at $\pm\infty$. $\mathcal{L}^q$ is also identical to the quark OAM appearing in the Jaffe-Manohar decomposition (3).

## 3. – Connections between different definitions for OAM

First of all from PT invariance one finds that $\mathcal{L}_+^q = \mathcal{L}_-^q$ [9]. As a corollary, since the piece at $\pm\infty$ cancels in the average both must thus be identical to the OAM appearing in the Jaffe-Manohar decomposition (with antiperiodic boundary conditions for $A_\perp$)

$$(12) \qquad \mathcal{L}^q = \frac{1}{2}\left(\mathcal{L}_+^q + \mathcal{L}_-^q\right) = \mathcal{L}_+^q = \mathcal{L}_-^q.$$

Therefore, even though the gauge link at $x^- = \pm\infty$ is essential for the description of TMDs [15], it does not contribute to the OAM provided anti-periodic boundary conditions (9) in light-cone gauge are implied [10].

To establish the connection with the quark OAM entering the Ji decomposition, we consider (for simplicity in light-cone gauge)

$$(13) \quad \mathcal{L}^q - L^q = \mathcal{L}_+^q - L^q = \frac{\int \mathrm{d}^3\vec{r}\langle PS|\bar{q}(\vec{r})\gamma^+[\vec{r}_\perp \times (g\vec{\alpha}_\perp(\vec{r}_\perp) - g\vec{A}_\perp(\vec{r}))]_z q(\vec{r})|PS\rangle}{\langle PS|PS\rangle}.$$

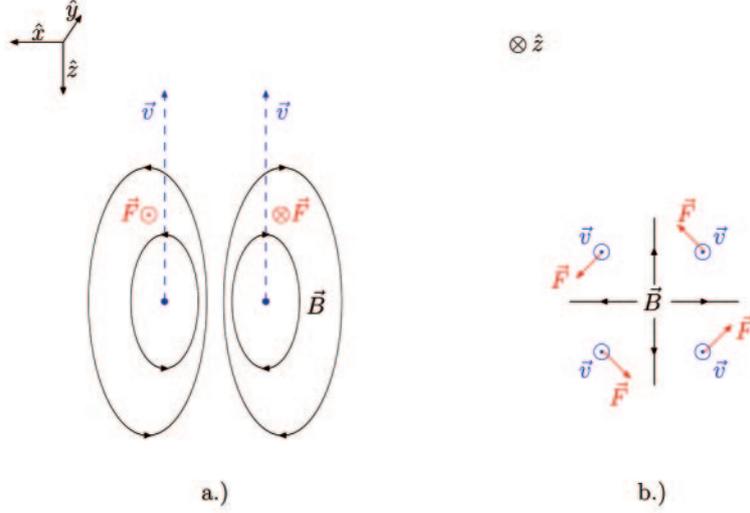Here we replaced $\gamma^0 \to \gamma^+$ for a nucleon at rest in the definition for $L^q$ [16].

Fig. 2. – Illustration of the torque acting on the struck quark in the $-\hat{z}$ direction through a color-magnetic dipole field caused by the spectators. a) side view; b) top view. In this example the $\hat{z}$ component of the torque is negative as the quark leaves the nucleon.

Using (in light-cone gauge $A^+ = 0$ and hence $G^{+i} = \partial_- A^i$)

$$(14) \qquad \alpha^i(\vec{r}_\perp) - A^i(\vec{r}) = \int_{r^-}^{\infty} \mathrm{d}r^- \partial_- A^i_\perp(\vec{r}) = \int_{r^-}^{\infty} \mathrm{d}x^- G^{+i}(\vec{r})$$

and noting that

$$(15) \qquad T^z(\vec{r}) \equiv g\left( x G^{+y}(\vec{r}) - y G^{+x}(\vec{r}) \right)$$

represents the $\hat{z}$ component of the torque that acts on a particle moving with (nearly) the velocity of light in the $-\hat{z}$ direction (see fig. 2) — the direction in which the ejected quark moves. Thus the difference between the (forward) light-cone and the local definitions of the OAM is the change in OAM due the color force from the spectators [17] on the active quark

$$(16) \qquad \mathcal{L}^q - L^q = \frac{\int \mathrm{d}^3\vec{r}\langle PS|\bar{q}(\vec{r})\gamma^+ \int_{r^-}^{\infty} dy^- T^z(y^-,\vec{r}_\perp)q(\vec{r})|PS\rangle}{\langle PS|PS\rangle} \ .$$

Therefore, while $L^q$ represents the local and manifestly gauge invariant OAM of the quark *before* it has been struck by the $\gamma^*$, $\mathcal{L}^q$ represents the gauge invariant OAM *after* it has left the nucleon and moved to $\infty$. This physical interpretation of the difference between the TMD based (*i.e.* Jaffe-Manohar) definition of quark OAM with a light-cone staple represents our main result [18].

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APPENDIX A.

**Gauge invariance**

Although we discussed the difference between $\mathcal{L}^q$ and $L^q$ in $A^+ = 0$ gauge in order to keep the equations simple, the interpretation of their difference, as the change in orbital angular momentum, is manifestly gauge invariant. To see this, we first consider (for simplicity, we only write down the expression in the abelian case—in the non-Abelian case there are additional gauge links connecting the quark and gluon operators)

$$\mathcal{L}^q_+ = \frac{\int \mathrm{d}^3\vec{r}\langle PS|\bar{q}(\vec{r})\gamma^+[\vec{r} \times (i\vec{\partial} + g\mathcal{A}_\perp(\vec{r}))]^z q(\vec{r})|PS\rangle}{\langle PS|PS\rangle},$$

in an arbitrary gauge, which also involves a contribution from the gauge links to $x^- = \infty$

$$(A.1) \qquad \mathcal{A}_\perp(\vec{r}) \equiv A_\perp(\vec{r}_\perp, \infty) - \int_{r^-}^{\infty} \mathrm{d}y^- \partial_\perp A^+(\vec{r}_\perp, y^-)$$

$\mathcal{L}^q_+ - L^q$ thus contains the matrix element of $\bar{q}(\vec{r})\gamma^+[\vec{r} \times (g\mathcal{A}_\perp(\vec{r}) - gA_\perp(\vec{r}))]^z q(\vec{r})$. Using

$$(A.2) \qquad \mathcal{A}_\perp(\vec{r}) - A_\perp(\vec{r}) = A_\perp(\vec{r}_\perp, \infty) - A_\perp(\vec{r}) - \int_{r^-}^{\infty} \mathrm{d}y^- \partial_\perp A^+(\vec{r}_\perp, y^-)$$

$$= \int_{r^-}^{\infty} \mathrm{d}y^- G^{+\perp}(\vec{r}_\perp, r^-),$$

which is identical to the terms entering eq. (16).

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